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The Diophantine Equation $y^2 + k = x^3$

W. J. ELLISON*, F. ELLISON, J. PESEK, C. E. STAHL AND D. S. STALL

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

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The effective method of Baker is applied to the equation $y^2 = x^3 - 28$ and all integral solutions are found.

1. In a recent paper Baker [1] proved that all the integral solutions of the equation $y^2 + k = x^3$ satisfy the following inequality

$$\max\{|x|, |y|\} \leqslant \exp\{10^{10} |k|^{10^4}\}. \tag{1}$$

Thus, in principle a "constructive" algorithm for finding all the integral solutions for a given value of k would be: "Try all possible values of x, y less than the bound given by (1)." Needless to say, this could never be done in practice for any given equation. However, it is the purpose of this paper to demonstrate that given a specific equation and following Baker's method of proof it is a perfectly reasonable proposition to find all the integral solutions of the equation. For each value of $|k|$ less than say 1,000 it would take about five minutes computation time on a modern calculating machine.

Our intention is not to compute a table of solutions of the equation $y^2 + k = x^3$ for numerous values of k, but to show that if the occasion arises in a number-theoretic investigation when all the integral solutions must be found, then it is a fairly routine matter to find them. Idle curiosity ought to be satisfied by the book [4].

Ljunggren [5] gave a list of all the unsolved equations with $|k| \leqslant 100$. The complete solutions for $k = 18, 25, 100$ are claimed by London and Finkelstein [6]. Presumably these equations were solved using Skolem's method.[1] The smallest unsolved equation now seems to be $y^2 + 28 = x^3$, so we will take this as our example and find all the integral solutions using Baker's method.

* Département de Mathématiques, Université de Bordeaux, 33-Talence, France.

[1] Note added October 1970: Finkelstein and London have subsequently published their proof for $k = 18$. See [3].

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2. As a first step we reduce the equation $y^2 + 28 = x^3$ to a finite number of Thue equations. The field $\mathbf{Q}(\sqrt{-7})$ has unique factorisation, the only units are ± 1, an integral basis is

$$\left\{1, \frac{1+\sqrt{-7}}{2}\right\} \quad \text{and} \quad 2 = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right).$$

Writing $y^2 + 28$ as $(y + 2\sqrt{-7})(y - 2\sqrt{-7})$ we have

$$(y + 2\sqrt{-7})(y - 2\sqrt{-7}) = x^3.$$

Let $y + 2\sqrt{-7} = aX^3$ and $y - 2\sqrt{-7} = bY^3$, where a, b are cube free integers in $\mathbf{Q}(\sqrt{-7})$, ab is a cube in $\mathbf{Q}(\sqrt{-7})$ and $\bar{a} = b$. If p is a prime in $\mathbf{Q}(\sqrt{-7})$ and $p \mid a$ then $p^3 \mid ab$; since a is cube free this implies $p \mid b$. Hence p divides both $y + 2\sqrt{-7}$ and $y - 2\sqrt{-7}$ and so must divide $4\sqrt{-7}$. Thus, p must be one of

$$\left\{p_1 = \sqrt{-7}, \quad p_2 = \frac{1+\sqrt{-7}}{2}, \quad p_3 = \frac{1-\sqrt{-7}}{2}\right\}.$$

Let $a = \pm p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, $b = \pm p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3}$ where $0 \leqslant \alpha_i \leqslant 2$, $0 \leqslant \beta_i \leqslant 2$ and $\alpha_i + \beta_i \equiv 0 \pmod 3$ for $1 \leqslant i \leqslant 3$. On noting that $p_2 = \bar{p}_3$, $\bar{a} = b$ we see that the only possibilities are $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ or $(0, 1, 2)$ or $(0, 2, 1)$. Putting

$$X = u + v\left(\frac{1+\sqrt{-7}}{2}\right) \quad \text{with} \quad u, v \in \mathbf{Z}$$

three cases result:

Case 1

$$y + 2\sqrt{-7} = \pm\left(u + v\,\frac{1+\sqrt{-7}}{2}\right)^3,$$

which yields

$$\begin{aligned} 2y &= \pm\{2u^3 + 3u^2v - 9uv^2 - 5v^3\}, \\ 4 &= \pm\{3u^2v + 3uv^2 - v^3\}. \end{aligned} \tag{2}$$

Case 2

$$y + 2\sqrt{-7} = \pm\left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right)^2\left(u + v\,\frac{1+\sqrt{-7}}{2}\right)^3,$$

which yields

$$\begin{aligned} y &= \pm\{u^3 + 12u^2v + 6uv^2 - 6v^3\}, \\ 2 &= \pm(u^3 - 6uv^2 - 2v^3). \end{aligned} \tag{3}$$

Case 3

$$y + 2\sqrt{-7} = \pm\left(\frac{1+\sqrt{-7}}{2}\right)^2\left(\frac{1-\sqrt{-7}}{2}\right)\left(u + v\,\frac{1+\sqrt{-7}}{2}\right)^3,$$

which yields

$$\begin{aligned} y &= \pm\{u^3 - 6u^2v - 15uv^2 + v^3\}, \\ 2 &= \pm\{u^3 + 3u^2v - 3uv^2 - 3v^3\}. \end{aligned} \tag{4}$$

All the integral solutions of Eq. (2) are easily found and they are

$$(u, v) = (1, 4),\ (-1, -4),\ (-5, 4),\ (5, -4).$$

These give the two solutions $x = 37$, $y = \pm 225$ for $y^2 + 28 = x^3$. Putting $u = u_1 - v_1$ and $v = v_1$, Eq. (4) becomes

$$\pm 2 = u_1^3 - 6u_1v_1^2 + 2v_1^3. \tag{5}$$

Thus, if (a, b) is a solution of Eq. (3), then $(a, -b)$ is a solution of Eq. (5), and conversely. Hence it will suffice to find all solutions to the Eq. (3). Putting $u = 2Y$, $Y \in Z$ and $v = -X - 2Y$, $X \in Z$, Eq. (3) becomes

$$X^3 - 12XY^2 - 12Y^3 = \pm 1. \tag{6}$$

We will find all integral solutions of Eq. (6).

Other ways of reducing an equation $y^2 + k = x^3$ to a finite set of Thue equations can be found in [7].

3. Let $f(X, Y) = X^3 - 12XY^2 - 12Y^3$; we will be working in the field $\mathbf{Q}(\theta)$ where $f(\theta, 1) = 0$. Routine calculation shows that an integral basis is $\{1, \theta, \theta^2/2\}$ and that a pair of fundamental units are

$$\eta_1 = -7 - 4\theta + (3\theta^2/2) \qquad \text{and} \qquad \eta_2 = 11 + \theta - \theta^2.$$

Later we will have occasion to work with rational approximations to θ, η_1, η_2 and their conjugates. We relegate this numerical information to an appendix, to be quoted when required.

If $x, y \in \mathbf{Z}$ are such that $f(x, y) = \pm 1$, we have

$$(x - \theta^{(1)}y)(x - \theta^{(2)}y)(x - \theta^{(3)}y) = \pm 1.$$

Putting $\beta = x - \theta y$ we see that β is a unit in $\mathbf{Q}(\theta)$ and so $\beta = \pm\eta_1^{b_1}\eta_2^{b_2}$ for some $b_1, b_2 \in \mathbf{Z}$. It follows from this that

$$\log|\beta^{(j)}| = b_1 \log|\eta_1^{(j)}| + b_2 \log|\eta_2^{(j)}| \qquad \text{for} \quad 1 \leqslant j \leqslant 3$$

and so

$$b_r = \{\log|\beta^{(j)}| \cdot \log|\eta_s^{(i)}| - \log|\beta^{(i)}| \cdot \log|\eta_s^{(j)}|\}/\Delta,$$

where $r, s = \{1, 2\}$, $r \neq s$ and

$$\Delta = \log|\eta_1^{(j)}| \cdot \log|\eta_2^{(i)}| - \log|\eta_1^{(i)}| - \log|\eta_2^{(j)}|.$$

If $H = \max\{|b_1|, |b_2|\}$ and $M = \max\{\log|\eta_1^{(i)}|, \log|\eta_2^{(i)}|\}$ we deduce that $H \leqslant \{|\log|\beta^{(j)}|| + |\log|\beta^{(i)}||\} \cdot M/|\Delta|$ and hence $\max\{\log|\beta^{(j)}|\} \geqslant |\Delta| \cdot H/M = \delta H$.

From the appendix we see that $M = 2.745588\ldots$, $|\Delta| = 2.15632779\ldots$ and so $\delta \geqslant 2.6730415\ldots$. Since $\log|\beta^{(1)}| + \log|\beta^{(2)}| + \log|\beta^{(3)}| = 0$ it follows that for at least one superscript (l) we must have $\log|\beta^{(l)}| \leqslant -(\delta/2) H$. Unfortunately we do not know for which values of l this holds. When we come to our numerical calculations we will have to do three calculations, one for each of the possible values of l. Now,

$$|\beta^{(1)}| \cdot |\beta^{(2)}| \cdot |\beta^{(3)}| = 1 \to |\beta^{(k)}| \cdot |\beta^{(j)}| = |\beta^{(l)}|^{-1} \geqslant \exp\left(\frac{\delta H}{2}\right)$$

and it follows that at least one of $|\beta^{(k)}|$ and $|\beta^{(j)}|$, say $|\beta^{(k)}|$, must be larger than $\exp(\delta H/4)$. Consequently we infer that $|\beta^{(l)}/\beta^{(k)}| \leqslant \exp(-\delta H/4)$. Again we do not know the numerical value of k (other than $l \neq k$) so there are six possible choices for the pair (k, l) and all must be tried when we do our computations.

Recalling that $\beta^{(j)} = x - \theta^{(j)}y$; $\beta^{(k)} = x - \theta^{(k)}y$; $\beta^{(l)} = x - \theta^{(l)}y$ and eliminating x and y we obtain

$$(\theta^{(k)} - \theta^{(l)})\,\beta^{(j)} + (\theta^{(j)} - \theta^{(k)})\,\beta^{(l)} + (\theta^{(l)} - \theta^{(j)})\,\beta^{(k)} = 0$$

$$\frac{\beta^{(j)}}{\beta^{(k)}} + \frac{\theta^{(l)} - \theta^{(j)}}{\theta^{(k)} - \theta^{(l)}} = \frac{\theta^{(j)} - \theta^{(k)}}{\theta^{(l)} - \theta^{(k)}} \cdot \frac{\beta^{(l)}}{\beta^{(k)}} = \omega.$$

Now $\beta = \eta_1^{b_1}\eta_2^{b_2}$ and putting

$$\alpha_1 = \frac{\eta_1^{(j)}}{\eta_1^{(k)}}, \qquad \alpha_2 = \frac{\eta_2^{(j)}}{\eta_2^{(k)}}, \qquad \alpha_3 = \frac{\theta^{(l)} - \theta^{(j)}}{\theta^{(k)} - \theta^{(l)}}$$

we obtain $\alpha_1^{b_1}\alpha_2^{b_2} + \alpha_3 = \omega$. From the appendix we see that

$$\max \left| \frac{\theta^{(j)} - \theta^{(k)}}{\theta^{(l)} - \theta^{(k)}} \right| \leqslant e^{\alpha},$$

where $\alpha = 1.392517959378504...$, thus $|\omega| \leqslant \exp(\alpha - (\delta H/4))$. Hence, we conclude that

$$|\alpha_1^{b_1}\alpha_2^{b_2} + \alpha_3| \leqslant \exp\left(\alpha - \frac{\delta H}{4}\right).$$

It will be convenient to rewrite this inequality in a different form: $|\alpha_1^{b_1}\alpha_2^{b_2}| = |-\alpha_3 + \omega|$ and upon taking logarithms

$$b_1 \log|\alpha_1| + b_2 \log|\alpha_2| = \log|\alpha_3 - \omega| = \log|\alpha_3| + \log\left|1 - \frac{\omega}{\alpha_3}\right|.$$

Thus,

$$\bigl|b_1 \log|\alpha_1| + b_2 \log|\alpha_2| - \log|\alpha_3|\bigr| = \left|\log\left|1 - \frac{\omega}{\alpha_3}\right|\right|$$

and

$$\left|\log\left|1 - \frac{\omega}{\alpha_3}\right|\right| = \left|\frac{\omega}{\alpha_3} + \frac{1}{2}\frac{\omega^2}{\alpha_3^2} + \cdots\right| \leqslant \left|\frac{\omega}{\alpha_3}\right| \cdot \frac{1}{1 - \left|\dfrac{\omega}{\alpha_3}\right|}.$$

We see that $|\omega/\alpha_3| \leqslant \exp(\alpha - (H\delta/4))$, so if $H \geqslant 6$. We certainly have $|\omega/\alpha_3| \leqslant 0.2$ and hence

$$\left|\log\left|1 - \frac{\omega}{\alpha_3}\right|\right| \leqslant 1.25\left|\frac{\omega}{\alpha_3}\right| \leqslant 6\exp\left(-\frac{\delta H}{4}\right).$$

Thus we certainly have

$$\bigl|b_1 \log|\alpha_1| + b_2 \log|\alpha_3| - \log|\alpha_3|\bigr| \leqslant \exp(-H \cdot 0.404) \qquad (7)$$

if $H \geqslant 20$.

In order to apply the result [2] of Baker we must find the heights of $\alpha_1, \alpha_2, \alpha_3$. This is easily done. The equation satisfied by α_1 is

$$X^6 - 132X^5 - 4773X^4 - 27236X^3 - 4773X^2 - 132X + 1 = 0.$$

The equation satisfied by α_2 is

$$X^6 + 30X^5 - 783X^4 - 2408X^3 - 783X^2 + 30X + 1 = 0.$$

The equation satisfied by α_3 is

$$21X^6 + 63X^5 - 198X^4 - 484X^3 - 198X^2 + 63X + 21 = 0.$$

Using Baker's theorem [2] we deduce that all the integral solutions of the inequality (7) satisfy

$$\max\{|b_1|, |b_2|\} \leqslant \{4^9(0.404...)^{-1} 6^6 \log 27236\}^{49} \leqslant 10^{563}.$$

In order to reduce this rather large upper bound we employ a simple lemma from diophantine approximation theory, due to Davenport.

LEMMA. *Suppose θ, β are given real numbers and M, $B > 6$ are given integers. Let p, q be integers such that $1 \leqslant q \leqslant BM$, $|\theta q - p| \leqslant 2(BM)^{-1}$. Then if $\|q\beta\| \geqslant 3B^{-1}$ there is no solution of the inequality $|b_1\theta + b_2 - \beta| \leqslant K^{-|b_1|}$ in integers b_1, b_2 with $\log(B^2M)/\log K \leqslant |b_1| \leqslant M$.*

Proof. $|b_1q\theta + b_2q - \beta q| \leqslant qK^{-|b_1|} \leqslant BMK^{-|b_1|}$ and if $q\theta = p + \omega$, where $|\omega| \leqslant 2(MB)^{-1}$, we have

$$|b_1(p + \omega) + b_2q - \beta q| \leqslant BMK^{-|b_1|}.$$

And since $\|\beta q\| \geqslant 3/B$ and $|b_1\omega| \leqslant 2/B$ it follows that $\|b_1\omega - \beta q\| \geqslant 1/B$. Thus we have

$$\frac{1}{B} \leqslant BMK^{-|b_1|}.$$

Hence

$$|b_1| \leqslant \frac{\log(B^2M)}{\log K}.$$

We will apply this lemma to the inequality

$$\big|b_1 \log|\alpha_1| + b_2 \log|\alpha_2| - \log|\alpha_3|\big| < \exp(-H \cdot 0404...)$$

by letting

$$\theta = \frac{\log|\alpha_1|}{\log|\alpha_2|}; \quad \beta = \frac{\log|\alpha_3|}{\log|\alpha_2|}; \quad K = e^{0.404}; \quad M = 10^{563}; \quad B = 10^{33}.$$

In order to use the lemma in a numerical example one must compute rational approximations to θ and β. Thus if a/b is a rational approximation to θ which satisfies $|\theta - a/b| < 1/(MB)^2$ and p/q is a rational approximation to a/b which sasitfies $1 \leqslant q \leqslant MB$, $|(a/b) - (p/q)| < 1/MB$, then p/q is a rational approximation to θ which satisfies

$$1 \leqslant q \leqslant MB; \qquad \left| \theta - \frac{p}{q} \right| < \frac{2}{MBq}.$$

We computed a/b such that

$$\left| \theta - \frac{a}{b} \right| < 10^{-1236}$$

and c/d such that $| \beta - (c/d)| < 10^{-650}$. The rational approximation p/q to a/b was found from the continued fraction expansion of a/b, where q is the largest convergent $\leqslant 10^{596}$. The computer calculated the quantities θ, β, p/q to the required degree of accuracy and found that

$$\left\| q \frac{c}{d} \right\| \geqslant 3 \times 10^{-33} \text{ in all cases.}$$

Thus all solutions of the inequality (7) satisfy

$$| b_1 | \leqslant \frac{\log 10^{629}}{0.404} \leqslant 3{,}585.$$

We applied the lemma a second time with $M = 4{,}500$, $B = 10^2$. Again in all cases the condition $\| \beta q \| \geqslant 3B^{-1}$ was satisfied, from which we concluded that all solutions of (7) satisfy

$$| b_1 | \leqslant \frac{\log(4.5 \times 10)}{0.404} \leqslant 44.$$

It was appropriate to use a desk calculator to find all integral solutions of the inequality

$$| \alpha_1^{b_1} \alpha_2^{b_2} + \alpha_3 | < \exp\left(\alpha - \frac{\delta H}{4}\right)$$

for $H \leqslant 6$ and the inequality

$$\left| b_1 \log | \alpha_1 | + b_2 \log | \alpha_2 | - \log | \alpha_3 | \right| < 6 \exp(-H \cdot 0.5438)$$

for H in the range $4 < H < 50$, say.

As was to be expected, there are many solutions of the inequality in this range. They are listed with the other numerical data, in the appendix. If (b_1, b_2) are pairs of integers which satisfied the inequality, we checked to see whether the units $\pm\eta_1^{b_1}\eta_2^{b_2}$ were of the form $x - \theta y$. The only integers which satisfy the inequality and are such that the corresponding unit is of the required form are $b_1 = 0$, $b_2 = 0$ and $b_1 = 0$, $b_2 = -1$. These give the solutions $x = 1$, $y = 0$ and $x = 1$, $y = -1$.

Thus, the only integral solutions of the equations

$$X^3 - 12XY^2 - 12Y^3 = \pm 1 \quad \text{are} \quad (X, Y) = (\pm 1, 0) \quad \text{and} \quad (\pm 1, \mp 1).$$

Hence the only solutions to Eqs. (3) are $(u, v) = (0, \pm 1)$ and $(\pm 2, \mp 1)$. These give as the solutions of $y^2 + 28 = x^3$, the following

$$(x, y) = (4, \pm 6) \qquad \text{and} \qquad (8, \pm 22).$$

Then the complete set of integral solutions of the equation $y^2 + 28 = x^3$ is

$$(x, y) = (4, \pm 6); \qquad (8, \pm 22); \qquad (37, \pm 225).$$

Appendix

Here we give some of the numerical data used in the investigation. The roots of the equation $f(\theta, 1) = 0$ are

$$\theta^{(1)} = -2.7687343052762\,82...$$
$$\theta^{(2)} = -1.115749396663048...$$
$$\theta^{(3)} = +3.88448370193933...$$

The equations satisfied by the fundamental units η_1 and η_2 are $X^3 - 15X^2 - 9X + 1 = 0$ and $X^3 - 9X^2 + 3X + 1 = 0$, respectively. The approximate roots of these equations are as follows:

$$\eta_1^{(1)} = 15.5737717009257510...; \qquad \eta_2^{(1)} = 0.565376041509972...$$
$$\eta_1^{(2)} = -0.6696573391168678...; \qquad \eta_2^{(2)} = 8.639353887182992...$$
$$\eta_1^{(3)} = 0.0958856381911168...; \qquad \eta_2^{(3)} = -0.2047299286929642...$$

$$\log | \eta_1^{(1)} | = 2.745588198059661...;$$
$$\log | \eta_2^{(1)} | = -0.5702642090280092...$$
$$\log | \eta_1^{(2)} | = -0.4009891315781089...;$$
$$\log | \eta_2^{(2)} | = 2.156327798443639...$$
$$\log | \eta_1^{(3)} | = -2.344599066481552...;$$
$$\log | \eta_2^{(3)} | = -1.586063589415630...$$

$$\max_i \left| | \log | \eta_1^{(i)} | - | \log | \eta_2^{(i)} || \right| = 2.175323989031652... .$$

The quantities $\log|\alpha_1|$ and $\log|\alpha_2|$ are computed from

$$\log\left|\frac{\eta_1^{(1)}}{\eta_1^{(3)}}\right| = 5.090187264541213...$$

$$\log\left|\frac{\eta_1^{(2)}}{\eta_1^{(3)}}\right| = 1.943609934903443...$$

$$\log\left|\frac{\eta_1^{(1)}}{\eta_1^{(2)}}\right| = 3.146577329637770...$$

$$\log\left|\frac{\eta_2^{(2)}}{\eta_2^{(1)}}\right| = 2.762592007471648...$$

$$\log\left|\frac{\eta_2^{(2)}}{\eta_2^{(3)}}\right| = 3.742391387859269...$$

$$\log\left|\frac{\eta_2^{(1)}}{\eta_2^{(3)}}\right| = 1.015799380387621...$$

The values of $\log|\alpha_3|$ are computed from the following quantities:

$$\begin{aligned}
|\theta^{(1)} - \theta^{(2)}| &= 1.652984908613233...\\
|\theta^{(1)} - \theta^{(3)}| &= 6.653218007215614...\\
|\theta^{(3)} - \theta^{(2)}| &= 5.00023309860238...\\
\log|\theta^{(1)} - \theta^{(2)}| &= 0.5025826891016480...\\
\log|\theta^{(1)} - \theta^{(3)}| &= 1.895100648480152...\\
\log|\theta^{(3)} - \theta^{(2)}| &= 1.60948453106791...
\end{aligned}$$

$$\max_{k\neq l}\left|\frac{\theta^{(j)} - \theta^{(k)}}{\theta^{(l)} - \theta^{(k)}}\right| \leqslant e^{\alpha}, \qquad \alpha = 1.392517959378504...$$

There are essentially three distinct linear inequalities

$$\left| b_1 \frac{\log|\alpha_1|}{\log|\alpha_2|} + b_2 - \frac{\log|\alpha_3|}{\log|\alpha_2|}\right| < 6\exp(-0.543H)$$

corresponding to the different choices of l and k mentioned earlier. They are as follows:

$$|b_1\theta_A + b_2 - \beta_A| < 6\exp(-0.543H) \qquad (A)$$

where,

$$\theta_A = -1.154033064...; \qquad \beta_A = -0.104752055...$$

and corresponds to the choices $l = 3$, $k = 1$ and $l = 3$, $k = 2$.

$$| b_1\theta_B + b_2 - \beta_B | < 6 \exp(-0.543H) \tag{B}$$

where,

$$\theta_B = 5.011016311...; \qquad \beta_B = -1.089685486...$$

and corresponds to the choices $l = 2$, $k = 1$ and $l = 2$, $k = 3$.

$$| b_1\theta_C + b_2 - \beta_C | < 6 \exp(-0.543H) \tag{C}$$

where,

$$\theta_C = 0.519349724...; \qquad \beta_C = -0.372093085...$$

and corresponds to the choices $l = 1$, $k = 2$ and $l = 1$, $k = 3$. For $M = 4500$, $B = 10^2$ the rational numbers p/q which occur in the Davenport lemma are as follows:

(A)	$(p, q) = (20989; 24{,}222)$;	$\| \beta_A q \| \geqslant 0.359104$;
(B)	$(p, q) = (153292; 30{,}591)$;	$\| \beta_B q \| \geqslant 0.431283$;
(C)	$(p, q) = (146091; 281296)$;	$\| \beta_C q \| \geqslant 0.296633$.

For $M = 10^{563}$, $B = 10^{33}$ the cost of letting the computer print out the decimal representation of the corresponding integers (p, q) was prohibitive. Only the first few decimals of the fractional parts of $\| \beta q \|$ were printed. They are as follows:

$$\| \beta_A q \| \geqslant 0.107819,$$
$$\| \beta_B q \| \geqslant 0.139308,$$
$$\| \beta_C q \| \geqslant 0.0452155.$$

The solutions to the linear inequalities A, B, C for H in the range $4 \leqslant H \leqslant 50$ are

Case A. $(b_1, b_2) = (-3, -4), (-4, -5); (-5, -6), (-6, -7), (3, 4), (4, 4), (4, 5)$.

Case B. $(b_1, b_2) = (1, -6), (-1, 4), (-2, 9)$.

Case C. $(b_1, b_2) = (4, -3), (5, -3), (7, -4), (-4, 2), (-5, 2), (-6, 3)$.

For each of the above choices of (b_1, b_2) we checked to see if $\eta_1^{b_1}\eta_2^{b_2}$ was of the form $A - B\theta$. No pair (b_1, b_2) has this property. The remaining possibility is that $| b_1 | \leqslant 5$ and $| b_2 | \leqslant 5$. In this range the only pairs (b_1, b_2) with $\eta_1^{b_1}\eta_2^{b_2}$ of the form $A - B\theta$ are $(0, 0)$ and $(0, -1)$.

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