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Constituent groups of Clifford semigroups arising from t-closure ☆

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ABSTRACT

The t-class semigroup of an integral domain R, denoted $S_t(R)$, is the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals with the operation induced by ideal t-multiplication. We recently proved that if R is a Krull-type domain [M. Griffin, Rings of Krull type, J. Reine Angew. Math. 229 (1968) 1–27], then $S_t(R)$ is a Clifford semigroup [S. Kabbaj, A. Mimouni, t-Class semigroups of integral domains, J. Reine Angew. Math. 612 (2007) 213–229]. In this paper, we aim to describe the idempotents of $S_t(R)$ and the structure of their associated groups. We extend and recover well-known results on class semigroups of valuation domains and Prüfer domains of finite character.

1. Introduction

The t-operation in integral domains is considered as one of the keystones of multiplicative ideal theory. It originated in Jaffard's 1960 book "Les Systèmes d'Idéaux" [23] and was investigated by many authors in the 1980's. From the t-operation stemmed the notion of class group of an arbitrary domain, extending both notions of divisor class group (in Krull domains) and Picard group (in Prüfer domains). Class groups were first introduced and developed by Bouvier [10] and Bouvier and Zafrullah [11], and have been, since then, extensively studied in the literature. In the 1990's, the attention of

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some authors moved from the class group to the class semigroup, first considering orders in number fields [32] and valuation domains [9], and then more general contexts [5–8,24,25]. The basic idea is to look at those domains that have Clifford class semigroup.

Let R be an integral domain with quotient field K. For a nonzero fractional ideal I of R, let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \bigcup J_v$ where J ranges over the set of finitely generated subideals of I. The ideal I is said to be a v-ideal if $I_v = I$, and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$, the set $F_t(R)$ of fractional t -ideals of R is a semigroup with unit R. An invertible element for this operation is called a t-invertible t-ideal of R. So the set $\mathrm{Inv}_t(R)$ of t-invertible fractional t-ideals of R is a group with unit R (cf. [15]). Let $F(R)$, $\mathrm{Inv}(R)$, and $P(R)$ denote the sets of nonzero, invertible, and nonzero principal fractional ideals of R, respectively. Under this notation, the Picard group [3,4,16], class group [10,11], t-class semigroup [26], and class semigroup [9,24,25,32] of R are defined as follows: $\mathrm{Pic}(R) := \mathrm{Inv}(R)/P(R)$, $\mathrm{Cl}(R) := \mathrm{Inv}_t(R)/P(R)$, $S_t(R) := F_t(R)/P(R)$, and $S(R) := F(R)/P(R)$. We have the set-theoretic inclusions

$$\mathrm{Pic}(R) \subseteq \mathrm{Cl}(R) \subseteq S_t(R) \subseteq S(R)$$

where the first and third inclusions turn into equality in the class of Prüfer domains and the second inclusion turns into equality in the class of Krull domains.

A commutative semigroup S is said to be Clifford if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x^2a = x$. The importance of a Clifford semigroup S resides in its ability to stand as a disjoint union of subgroups G_e, where e ranges over the set of idempotents of S, and G_e is the largest subgroup of S with identity equal to e; namely, $G_e = \{ae \mid abe = e \text{ for some } b \in S\}$ (cf. [21]). Often, the G_e's are called the constituent groups of S.

A domain R is called a PVMD (Prüfer v-multiplication domain) if R_M is a valuation domain for each t-maximal ideal M of R. Ideal t-multiplication converts ring notions such as PID, Dedekind, Bezout, and Prüfer, respectively to UFD, Krull, GCD, and PVMD. Recall at this point that the PVMDs of finite t-character (i.e., each proper t-ideal is contained in only finitely many t-maximal ideals) are exactly the Krull-type domains introduced by Griffin in 1967–1968 [17,18].

Divisibility properties of R are often reflected in semigroup-theoretic properties of $S(R)$ and $S_t(R)$. Obviously, Dedekind (resp., Krull) domains have Clifford class (resp., t-class) semigroup. In 1994, Zanardo and Zannier proved that all orders in quadratic fields have Clifford class semigroup [32]. They also showed that the ring of all entire functions in the complex plane (which is Bezout) fails to have this property. In 1996, Bazzoni and Salce proved that any arbitrary valuation domain has Clifford semigroup [9]. In [5–7], Bazzoni examined the case of Prüfer domains of finite character, showing that these, too, have Clifford class semigroup. In 2001, she completely resolved the problem for the class of integrally closed domains by stating that "*an integrally closed domain has Clifford class semigroup if and only if it is a Prüfer domain of finite character*" [8, Theorem 4.5]. Recently, we extended this result to the class of PVMDs of finite t-character; namely, "*a PVMD has Clifford t-class semigroup if and only if it is a Krull-type domain*" [26, Theorem 3.2].

This paper extends Bazzoni and Salce's study of groups in the class semigroup of a valuation domain [9] or a Prüfer domain of finite character [6,7] to a larger class of integral domains. Precisely, we describe the idempotents of $S_t(R)$ and the structure of their associated groups when R is a Krull-type domain. Indeed, we prove that there are two types of idempotents in $S_t(R)$: those represented by certain fractional overrings of R and those represented by finite intersections of t-maximal ideals of certain fractional overrings of R. Further, we show that the group associated with an idempotent of the first type equals the class group of the fractional overring, and characterize the elements of the group associated with an idempotent of the second type in terms of their localizations at t-prime ideals.

Our findings recover Bazzoni's results on the constituents groups of the class semigroup of a Prüfer domain of finite character. Recall for convenience, that in a Prüfer domain R, the notions of class group and t-class semigroup coincide, that is $S_t(R) = S(R)$.

All rings considered in this paper are integral domains. For the convenience of the reader, Fig. 1 displays a diagram of implications summarizing the relations between the main classes of integral domains involved in this work. It also places the Clifford property in a ring-theoretic perspective.

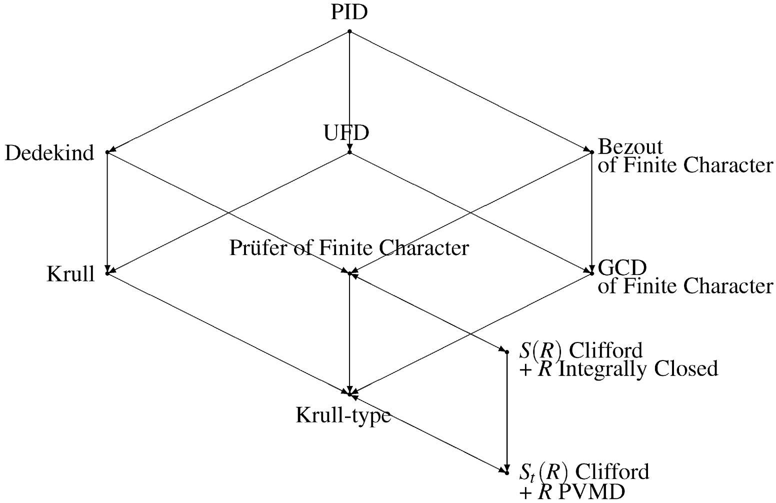

Fig. 1. A ring-theoretic perspective for Clifford property.

2. Main result

An overring T of a domain R is t-linked over R if $I^{-1} = R \Rightarrow (T : IT) = T$, for each finitely generated ideal I of R [2,30]. In Prüfer domains, the t-linked property collapses merely to the notion of overring (since every finitely generated proper ideal is invertible and then its inverse is a fortiori different from R). This concept plays however a crucial role in any attempt to extend classical results on Prüfer domains to PVMDs (via t-closure). Recall also that an overring T of R is fractional if T is a fractional ideal of R; in this case, any (fractional) ideal of T is a fractional ideal of R. Of significant importance too for the study of t-class semigroups is the notion of t-idempotence; namely, a t-ideal I of a domain R is t-idempotent if $(I^2)_t = I$.

The following discussion, connected with the t-ideal structure of a PVMD, will be of use in the sequel without explicit mention. Let R be a PVMD. Note that prime ideals of R contained in a t-maximal ideal are necessarily t-ideals and form a chain [27, Corollary 2.47] or [29, Theorem 1.1]. Also, recall that t-linked overrings of R are exactly the subintersections of R; precisely, T is a t-linked overring of R if and only if $T = \bigcap R_P$, where P ranges over some set of t-prime ideals of R [28, Theorem 3.8] or [12]. Further, every t-linked overring of R is a PVMD [28, Corollary 3.9]; in fact, this condition characterizes the notion of PVMD [14, Theorem 2.10]. Now let I be an arbitrary t-ideal of R. Then $(I : I)$ is a subintersection of R. Indeed, let $\mathrm{Max}_t(R, I) := \{M \in \mathrm{Max}_t(R) \mid I \subseteq M\}$ and $\overline{\mathrm{Max}_t}(R, I) := \{M \in \mathrm{Max}_t(R) \mid I \nsubseteq M\}$, where $\mathrm{Max}_t(R)$ denotes the set of t-maximal ideals of R. Then $(I : I) = (\bigcap_\alpha R_{M_\alpha}) \cap (\bigcap_\beta R_{N_\beta \cap R})$, where M_α ranges over $\overline{\mathrm{Max}_t}(R, I)$ and N_β denotes the set of zerodivisors (a fortiori a prime ideal) of R_{M_β} modulo IR_{M_β} where M_β ranges over $\mathrm{Max}_t(R, I)$. Consequently, $(I : I)$ is a fractional t-linked overring of R and hence a PVMD. Finally, given M_1 and M_2 two t-maximal ideals of R, we will denote by $M_1 \wedge M_2$ the largest prime ideal of R contained in $M_1 \cap M_2$.

Throughout, we shall use $\bar{I}$ to denote the isomorphy class of an ideal I of R in $S_t(R)$ and $\mathrm{qf}(R)$ to denote the quotient field of R. Recall that the class group of an integral domain R, denoted $\mathrm{Cl}(R)$, is the group of fractional t-invertible t-ideals modulo its subgroup of nonzero principal fractional ideals. Also we shall use v_1 and t_1 to denote the v- and t-operations with respect to an overring T of R. By [26, Theorem 3.2], if R is a Krull-type domain, then $S_t(R)$ is Clifford and hence a disjoint union of subgroups $G_{\overline{J}}$, where $\overline{J}$ ranges over the set of idempotents of $S_t(R)$ and $G_{\overline{J}}$ is the largest subgroup of $S_t(R)$ with unit $\overline{J}$. Notice for convenience that in valuation and Prüfer domains the t- and trivial operations (and hence the t-class and class semigroups) coincide. At this point, it is worthwhile recalling the Bazzoni–Salce result that valuation domains have Clifford class semigroup [9].

Next we announce the main result of this paper.

Theorem 2.1. *Let R be a Krull-type domain and I a t-ideal of R. Set $T := (I : I)$ and $\Gamma(I) :=$ {finite intersections of t-idempotent t-maximal ideals of T}. Then $\bar{I}$ is an idempotent of $S_t(R)$ if and only if there exists a unique $J \in \{T\} \cup \Gamma(I)$ such that $\bar{I} = \bar{J}$. Moreover,*

(1) *if $J = T$, then $G_{\bar{J}} \cong \mathrm{Cl}(T)$;*
(2) *if $J = \bigcap_{1\leqslant i\leqslant r} Q_i \in \Gamma(I)$, then the following sequence of natural group homomorphisms is exact*

$$0 \longrightarrow \mathrm{Cl}(T) \xrightarrow{\phi} G_{\bar{J}} \xrightarrow{\psi} \prod_{1\leqslant i\leqslant r} G_{\overline{Q_iT_{Q_i}}} \longrightarrow 0$$

where $G_{\overline{Q_iT_{Q_i}}}$ denotes the constituent group of the Clifford semigroup $S(T_{Q_i})$ associated with $\overline{Q_iT_{Q_i}}$.

The proof of the theorem involves several preliminary lemmas, some of which are of independent interest. We will often appeal to some of them without explicit mention.

Lemma 2.2. *(See [26, Lemma 2.1].) Let I be a t-ideal of a domain R. Then $\bar{I}$ is regular in $S_t(R)$ if and only if $I = (I^2(I : I^2))_t$.*

Lemma 2.3. *Let R be a PVMD, T a t-linked overring of R, and Q a t-prime ideal of T. Then $P = Q \cap R$ is a t-prime ideal of R with $R_P = T_Q$. If, in addition, Q is supposed to be t-idempotent in T, then so is P in R.*

Proof. Since R is a PVMD, by [30, Proposition 2.10], T is t-flat over R. Hence $R_P = T_Q$. Moreover, since T is t-linked over R, then $P_t \subsetneq R$ [14, Proposition 2.1]. Hence P is a t-prime ideal of R [27, Corollary 2.47] or [29, Theorem 1.1]. Next assume that $(Q^2)_{t_1} = Q$. Then $P^2R_P = Q^2T_Q = (Q^2)_{t_1}T_Q = QT_Q = PR_P$ by [26, Lemma 3.3]. Now, let M be an arbitrary t-maximal ideal of R. We claim that $P^2R_M = PR_M$. Indeed, without loss of generality we may assume $P \subseteq M$. So $R_M \subseteq R_P$ and hence $PR_M \subseteq PR_P$. If $PR_M \subsetneq PR_P$ and $x \in PR_P \setminus PR_M$, necessarily $PR_M \subset xR_M$ since R_M is a valuation domain. Hence, by [22, Theorem 3.8 and Corollary 3.6], $x^{-1} \in (R_M : PR_M) = (PR_M : PR_M) = (R_M)_{PR_M} = R_P$, absurd. Therefore $PR_M = PR_P$. It follows that $P^2R_M = P^2R_P = PR_P = PR_M$. By [27, Theorem 2.19] or [1, Theorem 6], $(P^2)_t = P$, as desired. □

Lemma 2.4. *Let R be a PVMD and T a t-linked overring of R. Let J be a common (fractional) ideal of R and T. Then the following assertions hold:*

(1) *$J_{t_1} = J_t$.*
(2) *J is a t-idempotent t-ideal of R if and only if J is a t-idempotent t-ideal of T.*

Proof. (1) Let $x \in J_{t_1}$. Then there exists a finitely generated ideal $B := \sum_{1\leqslant i\leqslant n} a_iT$ of T such that $B \subseteq J$ and $x(T : B) \subseteq T$. Clearly, $A := \sum_{1\leqslant i\leqslant n} a_iR$ is a finitely generated ideal of R with $AT = B$. Therefore $(R : A) \subseteq (T : B)$ and hence $xA(R : A) \subseteq xB(T : B) \subseteq B \subseteq J$. Moreover A is t-invertible in R since R is a PVMD. It follows that $xR = x(A(R : A))_t = (xA(R : A))_t \subseteq J_t$. Hence $J_{t_1} \subseteq J_t$. Conversely, let $x \in J_t$. Then there exists a finitely generated subideal A of J such that $x(R : A) \subseteq R$. Let $N \in \mathrm{Max}_t(T)$ with $M := N \cap R$. So $x(AT_N)^{-1} = x(AR_M)^{-1} = xA^{-1}R_M \subseteq R_M = T_N$. Therefore x lies in the v-closure of AT_N in the valuation domain T_N. Further AT_N is principal (since it is finitely generated), hence a v-ideal. So that $x \in AT_N \subseteq JT_N$. Consequently, $x \in \bigcap_{N\in \mathrm{Max}_t(T)} JT_N = J_{t_1}$, as desired.

(2) Straightforward via (1). □

Lemma 2.5. *Let R be a PVMD, I a t-ideal of R, and $T := (I : I)$. Let $J := \bigcap_{1\leqslant i\leqslant r} Q_i$, where each Q_i is a t-idempotent t-maximal ideal of T. Then J is a fractional t-idempotent t-ideal of R.*

Proof. Notice that $J=(\prod_{1\leqslant i\leqslant r} Q_i)_{t_1}$; this can be seen by localizing J and $\prod_{1\leqslant i\leqslant r} Q_i$ with respect to t-maximal ideals of T. Clearly, J is a t-ideal of T and hence a fractional t-ideal of R by Lemma 2.4, with $J=(\prod_{1\leqslant i\leqslant r} Q_i)_{t_1}=(\prod_{1\leqslant i\leqslant r} Q_i)_t$. Further, J^2 is a common ideal of R and T, whence $(J^2)_t=(J^2)_{t_1}=(\prod_{1\leqslant i\leqslant r} Q_i^2)_{t_1}=(\prod_{1\leqslant i\leqslant r}(Q_i^2)_{t_1})_{t_1}=(\prod_{1\leqslant i\leqslant r} Q_i)_{t_1}=J$, as desired. □

Lemma 2.6. *Let R be a PVMD, I a t-idempotent t-ideal of R, and M a t-maximal ideal of R containing I. Then IR_M is an idempotent (prime) ideal of R_M.*

Proof. By [26, Lemma 3.3], $I^2R_M=(I^2)_tR_M=IR_M$. Therefore IR_M is an idempotent ideal which is necessarily prime since R_M is a valuation domain. □

Lemma 2.7. *Let R be a Krull-type domain, L a t-ideal of R, and J a t-idempotent t-ideal of R. Then $\bar{L}\in G_{\bar{J}}$ if and only if $(L:L)=(J:J)$ and $(JL(L:L^2))_t=(L(L:L^2))_t=(L(J:L))_t=J$.*

Proof. Suppose $\bar{L}\in G_{\bar{J}}$. Then $J=(LK)_t$ and $xL=(AJ)_t$ for some fractional ideals K and A of R and some $0\neq x\in \mathrm{qf}(R)$. Moreover, $x(LJ)_t=(xLJ)_t=((AJ)_tJ)_t=(AJ^2)_t=(A(J^2)_t)_t=(AJ)_t=xL$. So $(LJ)_t=L$. We have $(L:L)\subseteq(LK:LK)\subseteq((LK)_t:(LK)_t)=(J:J)$ and $(J:J)\subseteq(LJ:LJ)\subseteq((LJ)_t:(LJ)_t)=(L:L)$, so that $(L:L)=(J:J)$. Further, $(JL(L:L^2))_t=(L(L:L^2))_t=(L((J:J):L))_t=(L(J:JL))_t=(L(J:(JL)_t))_t=(L(J:L))_t$. Clearly, $(L(J:L))_t\subseteq J$. Also $LK\subseteq(LK)_t=J$. Then $K\subseteq(J:L)$, hence $J=(LK)_t\subseteq(L(J:L))_t$. Conversely, take $K:=J(L:L^2)$ and notice that $(LK)_t=J$ and $(LJ)_t=(L^2(L:L^2))_t=L$ (since $S_t(R)$ is Clifford), completing the proof of the lemma. □

Lemma 2.8. *Let R be a PVMD and I a t-ideal of R. Then*

(1) *I is a t-ideal of $(I:I)$.*
(2) *If R is Clifford t-regular, then so is $(I:I)$.*

Proof. (1) $(I:I)$ is a t-linked overring of R and then apply Lemma 2.4(1).

(2) Let J be a t-ideal of $T:=(I:I)$. By Lemma 2.4(1), J is a t-ideal of R. Next, assume that R is Clifford t-regular. By [26, Lemma 3.3], $(J^2(J:J^2))_{t_1}T_N=(J^2(J:J^2))T_N=(J^2(J:J^2))R_M=(J^2(J:J^2))_tR_M=JR_M=JT_N$. Hence $(J^2(J:J^2))_{t_1}=J$ and therefore T is Clifford t-regular. □

Proof of Main Theorem. On account of Lemma 2.5, we need only prove the "only if" assertion.

Uniqueness: Suppose there exist $J,F\in\{T\}\cup\Gamma(I)$ such that $\overline{J}=\overline{F}$. Then there is $0\neq q\in\mathrm{qf}(R)$ such that $qJ=F=(F^2)_t=(q^2J^2)_t=q^2(J^2)_t=q^2J$. So $J=qJ=F$.

Existence: Let $J:=(I(T:I))_{t_1}=(I(T:I))_t$, a t-ideal of T and a fractional t-ideal of R (by Lemma 2.4). Since $(I^2)_t=qI$ for some $0\neq q\in\mathrm{qf}(R)$, then $(I:I^2)=(I:(I^2)_t)=(I:qI)=q^{-1}(I:I)=q^{-1}T$. Hence $J=(I(T:I))_t=(I(I:I^2))_t=(q^{-1}I)_t=q^{-1}I$. Therefore $\overline{J}=\bar{I}$. Now R is Clifford t-regular, then $I=(I^2(I:I^2))_t=(IJ)_t$. So $(I:I)\subseteq(J:J)\subseteq((IJ)_t:(IJ)_t)=(I:I)$, whence $(J:J)=T$. Moreover, $(T:J)=((I:I):J)=(I:IJ)=(I:(IJ)_t)=(I:I)=T$. It follows that $(J:J^2)=((J:J):J)=(T:J)=T$. Consequently, $J=(J^2(J:J^2))_t=(J^2)_t$ and thus J is a fractional t-idempotent t-ideal of R, and hence a t-idempotent t-ideal of T by Lemma 2.4.

Now assume that $J\neq T$. Then we shall prove that $J\in\Gamma(I)$. By [26, Theorem 3.2] and Lemma 2.8, T is a Krull-type domain. Then J is contained in a finite number of t-maximal ideals of T, say, $N_1,\dots,N_r$. By Lemma 2.6, for each $i\in\{1,\dots,r\}$, JT_{N_i} is an idempotent prime ideal of the valuation domain T_{N_i}. So $JT_{N_i}=Q_iT_{N_i}$ for some prime ideal $Q_i\subseteq N_i$ of T; moreover, Q_i is minimal over J. Then $J=\bigcap_{N\in\mathrm{Max}_t(T)}JT_N=(\bigcap_{1\leqslant i\leqslant r}Q_iT_{N_i})\cap(\bigcap_{N'}JT_{N'})$, where N' ranges over the t-maximal ideals of T which do not contain J. The contraction to T of both sides yields $J=\bigcap_{1\leqslant i\leqslant r}Q_i$. One may assume the Q_i's to be distinct. Since JT_{N_i} is idempotent, $Q_iT_{N_i}=Q_i^2T_{N_i}$. We claim that N_i is a unique t-maximal ideal of T containing Q_i. Otherwise, if $Q_i\subseteq N_j$ for some $j\neq i$, then Q_i and Q_j are t-prime ideals [27, Corollary 2.47] or [29, Theorem 1.1] contained in the same t-maximal ideal

N_j in the PVMD T. So Q_i and Q_j are comparable under inclusion. By minimality, we get $Q_i = Q_j$, absurd. It follows that

$$Q_i = \bigcap Q_i T_N = Q_i T_{N_i} \bigcap \Bigl(\bigcap_{N \neq N_i} T_N \Bigr) = Q_i^2 T_{N_i} \bigcap \Bigl(\bigcap_{N \neq N_i} T_N \Bigr) = \bigcap Q_i^2 T_N = \bigl(Q_i^2\bigr)_t$$

where N ranges over all t-maximal ideals of T. Finally, for each $i \in \{1, \dots r\}$, $T \subseteq (T : Q_i) \subseteq (T : J) = (J : J) = T$. Then $(T : Q_i) = (Q_i : Q_i) = T$. By [26, Lemma 3.6], Q_i is a t-maximal ideal of T, completing the proof of the first statement.

Next, we describe the constituent groups $G_{\overline{J}}$ of $S_t(R)$. We write G_J instead of $G_{\overline{J}}$, since we can always choose J to be a unique fractional t-idempotent t-ideal (of R) representing $\overline{J}$. We shall use $[L]$ to denote the elements of the class group $\mathrm{Cl}(T)$. Notice that for any two common t-ideals L, L' of R and T, we have $[L] = [L']$ if and only if $\overline{L} = \overline{L'}$ if and only if $L = xL'$ for some $0 \neq x \in \mathrm{qf}(R) = \mathrm{qf}(T)$.

(1) Assume that $J = T$. Let $[L] \in \mathrm{Cl}(T)$, where L is a t-invertible t-ideal of T. Then $T \subseteq (L : L) \subseteq ((L(T : L))_{t_1} : (L(T : L))_{t_1}) = (T : T) = T$. Hence $(L : L) = T = (J : J)$. We obtain, via Lemma 2.4, $(JL(L : L^2))_t = (JL(T : L))_t = (L(T : L))_t = (L(T : L))_{t_1} = T = J$. Whence $\overline{L} \in G_J$. Conversely, let $\overline{L} \in G_J$ for some t-ideal L of R. By Lemma 2.7, $(L : L) = (J : J)$ and $(JL(L : L^2))_t = J$. By Lemma 2.8, L is a t-ideal of $(L : L) = (J : J) = T$. Moreover, via Lemma 2.4, $(L(T : L))_{t_1} = (L(T : L))_t = (L(L : L^2))_t = (JL(L : L^2))_t = J = T$. Therefore L is a t-invertible t-ideal of T and thus $[L] \in \mathrm{Cl}(T)$. Consequently, $G_J \cong \mathrm{Cl}(T)$.

(2) Assume $J = \bigcap_{1 \leqslant i \leqslant r} Q_i$, where the Q_i's are distinct t-idempotent t-maximal ideals of T.

Claim 1. $(T : J) = (J : J) = T$.

Indeed, for each i, we have $(Q_i : Q_i) = T$ and hence $(T : Q_i) = ((Q_i : Q_i) : Q_i) = (Q_i : Q_i^2) = (Q_i : (Q_i^2)_t) = (Q_i : Q_i) = T$. Obviously, $\bigcap_{1 \leqslant i \leqslant r} Q_i$ is an irredundant intersection, so $(T : J)$ is a ring by [19, Proposition 3.13]. Further since $\{Q i\}_{1 \leqslant i \leqslant r}$ equals the set of minimal primes of J in the PVMD T, then, by [19, Theorem 4.5], $(T : J) = (\bigcap_{1 \leqslant i \leqslant r} T_{Q_i}) \cap (\bigcap_{N'} T_{N'})$, where N' ranges over the t-maximal ideals of T which do not contain J. Hence $(T : J) = T$, as claimed.

Claim 2. *ϕ is well-defined and injective.*

Let $[L] \in \mathrm{Cl}(T)$ for some t-invertible t-ideal L of T, that is, $(L(T : L))_t = (L(T : L))_{t_1} = T$. The homomorphism ϕ is given by $\phi([L]) = \overline{(LJ)_t}$. We have $(J : J) \subseteq (LJ : LJ) \subseteq ((LJ)_t : (LJ)_t)$. Conversely, let $x \in ((LJ)_t : (LJ)_t)$. Then $x(LJ)_t \subseteq (LJ)_t$, hence $x(LJ)_t(T : L) \subseteq (LJ)_t(T : L)$. So $xJ = xJT = x(JT)_t = x(J(L(T : L))_t)_t = x(JL(T : L))_t \subseteq (JL(T : L))_t = (J(L(T : L))_t)_t = J$. Therefore $x \in (J : J)$ and hence $T = (J : J) = ((LJ)_t : (LJ)_t)$. Moreover, $((LJ)_t(T : (LJ)_t))_t = (JL(T : JL))_t = (JL((T : J) : L))_t = (JL(T : L))_t = (J(L(T : L))_t)_t = JT = J$. By Lemma 2.7, $\overline{(LJ)_t} \in G_J$ and thus ϕ is well-defined.

Now, let $[A]$ and $[B]$ in $\mathrm{Cl}(T)$ with $\overline{(AJ)_t} = \overline{(BJ)_t}$. So there exists $x \neq 0 \in \mathrm{qf}(R) = \mathrm{qf}(T)$ such that $(AJ)_t = x(BJ)_t$. Since A and B are t-invertible t-ideals of T, then A and B are v-ideals of T. Further $(T : A) = ((T : J) : A) = (T : JA) = (T : (AJ)_t) = (T : x(BJ)_t) = x^{-1}(T : (BJ)_t) = x^{-1}(T : JB) = x^{-1}((T : J) : B) = x^{-1}(T : B)$. Hence $A = A_{v_1} = xB_{v_1} = xB$, whence $[A] = [B]$, proving that ϕ is injective.

Claim 3. *Let Q be a t-idempotent t-maximal ideal of T and L a t-ideal of T such that $\overline{LT_Q} \in G_{QT_Q}$. Then there exists a t-ideal A of T such that $\overline{LT_Q} = \overline{AT_Q}$, Q is a unique t-maximal ideal of T containing A, and $\overline{A} \in G_Q$ in $S_t(T)$.*

We may assume $LT_Q \neq T_Q$, i.e., $L \subseteq Q$. By [26, Theorem 3.2] and Lemma 2.8, T is a Krull-type domain. So let $\{Q, Q_1, \dots, Q_s\}$ be the set of all t-maximal ideals of T containing L. Since $\{Q \wedge Q_i\}_{1 \leqslant i \leqslant s}$ is linearly ordered, we may assume that $Q \wedge Q_i \subseteq P := Q \wedge Q_1$ for each i. Necessarily, $PT_Q \subsetneqq QT_Q$. On the other hand, Lemma 2.7 yields $(LT_Q : LT_Q) = (QT_Q : QT_Q)$ and $LT_Q(QT_Q : LT_Q) = QT_Q$;

notice that in the valuation domain T_Q, the t- and trivial operations coincide. Now, by [9, Lemma 2], $QT_Q = LT_Q^{\sharp} := \bigcap r^{-1}LT_Q$ where r describes the set $T_Q \setminus LT_Q$. Therefore there exists $r \in T_Q$ such that $PT_Q \subsetneqq r^{-1}LT_Q \subseteq T_Q$. Let $A := r^{-1}LT_Q \cap T$. As in the proof of [7, Proposition 3.3(1)], $AT_{Q_i} = T_{Q_i}$ for every $i = 1, \ldots, s$. Consequently, Q is a unique t-maximal ideal of T containing A. Finally, one can assume A to be a t-ideal since $A \subseteq A_{t_1} \subseteq Q$ and $A_{t_1}T_Q = AT_Q$ by [26, Lemma 3.3].

Next we show that $\overline{A} \in G_Q$ via Lemma 2.7. Since $\overline{AT_Q} \in G_{QT_Q}$, then $(AT_Q : AT_Q) = (QT_Q : QT_Q) = T_Q$ and $AT_Q(T_Q : AT_Q) = QT_Q$. Now, $(A : A) \supseteq T$ since A is an integral ideal of T. Conversely, we readily have $(A : A) \subseteq (AT_Q : AT_Q) = T_Q$ and $(A : A) \subseteq T_N$ for each t-maximal ideal $N \neq Q$ of T. Therefore $(A : A) \subseteq T$ and hence $(A : A) = T$. Let $x \in (T : A)$. Then $x \in (T_Q : AT_Q) = (QT_Q : AT_Q)$ since $AT_Q(T_Q : AT_Q) = QT_Q$. Therefore $xA \subseteq xAT_Q \subseteq QT_Q$ and hence $xA \subseteq QT_Q \cap T = Q$, i.e., $x \in (Q : A)$. It follows that $(T : A) = (Q : A)$ and thus $A(T : A) = A(Q : A)$. Consequently, $(A(T : A))_{t_1} \subseteq Q$. Now, by the first statement of the theorem applied to T, there exists a unique t-idempotent t-ideal E of T such that $\overline{A} \in G_E$ with either $E = S$ for some fractional t-linked overring S of T or $E = \bigcap_{1 \leqslant i \leqslant s} N_i$, where the N_i's are distinct t-idempotent t-maximal ideals of S. If $E = S$, then $\overline{A} \in G_S$ implies that $(A : A) = (S : S) = S$ and $(A(S : A))_{t_1} = S$. So $T = (A : A) = S = (A(S : A))_{t_1} = (A(T : A))_{t_1} \subseteq Q$, absurd. Hence, necessarily, $E = \bigcap_{1 \leqslant i \leqslant s} N_i$. It follows that $T = (A : A) = (E : E) = S$ (by the first claim) and $(A(T : A))_{t_1} = E$. Therefore $A \subseteq E \subseteq N_i$ for each i, hence $E = Q$, a unique t-maximal ideal of T containing A. Thus, $\overline{A} \in G_Q$, proving the claim.

Claim 4. *ψ is well-defined and surjective.*

Let $\bar{L} \in G_J$ for some t-ideal L of R. Notice that L is also a t-ideal of $(L : L) = (J : J) = T$. The homomorphism ψ is given by $\psi(\bar{L}) = (\overline{LT_{Q_i}})_{1 \leqslant i \leqslant r}$. We prove that ψ is well-defined via a combination of [26, Lemma 3.3], Lemmas 2.4, and 2.7. Indeed it suffices to show that $(LT_{Q_i} : LT_{Q_i}) = (Q_iT_{Q_i} : Q_iT_{Q_i}) = T_{Q_i}$ and $LT_{Q_i}(T_{Q_i} : LT_{Q_i}) = Q_iT_{Q_i}$. Let $i \in \{1, \ldots, r\}$ and $x \in (LT_{Q_i} : LT_{Q_i})$. Then $xLT_{Q_i} \subseteq LT_{Q_i}$ and hence $xL(T : L)T_{Q_i} \subseteq L(T : L)T_{Q_i}$. Whence $JT_{Q_i} = (L(T : L))_tT_{Q_i} = (L(T : L))_{t_1}T_{Q_i} = (L(T : L))T_{Q_i}$. Further $JT_{Q_i} = Q_iT_{Q_i}$. It follows that $xQ_iT_{Q_i} \subseteq Q_iT_{Q_i}$, as desired. On the other hand, we have $Q_iT_{Q_i} = JT_{Q_i} = (L(T : L))T_{Q_i} \subseteq LT_{Q_i}(T_{Q_i} : LT_{Q_i}) \subseteq T_{Q_i}$. The last containment is necessarily strict. Otherwise $LT_{Q_i} = aT_{Q_i}$ for some $0 \neq a \in L$. Therefore $L = (LJ)_t$ implies $aT_{Q_i} = LT_{Q_i} = (LJ)_tT_{Q_i} = (LJ)_{t_1}T_{Q_i} = LJT_{Q_i} = aQ_iT_{Q_i}$, absurd. Consequently $LT_{Q_i}(T_{Q_i} : LT_{Q_i}) = Q_iT_{Q_i}$. So ψ is well-defined.

Next we show that ψ is surjective. Let $(\overline{L_iT_{Q_i}})_{1 \leqslant i \leqslant r} \in \prod G_{Q_iT_{Q_i}}$. By the above claim, for each i, there exists a t-ideal A_i of T such that $\overline{L_iT_{Q_i}} = \overline{A_iT_{Q_i}}$, $\overline{A_i} \in G_{Q_i}$, and Q_i is a unique t-maximal ideal of T containing A_i. Set $A := (A_1A_2 \ldots A_rJ)_{t_1} = (A_1A_2 \ldots A_rJ)_t$. Let $j \in \{1, \ldots, r\}$. By [26, Lemma 3.3], $AT_{Q_j} = (A_1A_2 \ldots A_rJ)T_{Q_j} = A_jQ_jT_{Q_j}$ since $JT_{Q_j} = Q_jT_{Q_j}$ and $A_iT_{Q_j} = T_{Q_j}$ for each $i \neq j$. So $\overline{AT_{Q_j}} = \overline{A_jQ_jT_{Q_j}} = \overline{A_jT_{Q_j}}\ \overline{Q_jT_{Q_j}} = \overline{A_jT_{Q_j}} = \overline{L_jT_{Q_j}}$. Therefore $\psi(\overline{A}) = (\overline{L_iT_{Q_i}})_{1 \leqslant i \leqslant r}$.

Next we show that $\overline{A} \in G_J$. First notice that $Q_1, \ldots, Q_r$ are the only t-maximal ideals of T containing A. For, let Q be a t-maximal ideal of T such that $A \subseteq Q$. Then either $J \subseteq Q$ or $A_i \subseteq Q$ for some i. In both cases, $Q = Q_j$ for some j, as desired. Now A is an ideal of T, then $(J : J) = T \subseteq (A : A)$. Conversely, for each j, $A_jQ_jT_{Q_j} = a_jA_jT_{Q_j}$, for some nonzero $a_j \in \mathrm{qf}(T)$, since $\overline{A_jT_{Q_j}} \in G_{Q_jT_{Q_j}}$. So $(A : A) \subseteq (AT_{Q_j} : AT_{Q_j}) = (A_jQ_jT_{Q_j} : A_jQ_jT_{Q_j}) = (a_jA_jT_{Q_j} : a_jA_jT_{Q_j}) = (A_jT_{Q_j} : A_jT_{Q_j}) = T_{Q_j}$. Further, for each $N \in \overline{\mathrm{Max}_t}(T, A)$, we clearly have $(A : A) \subseteq T_N$. It follows that $(A : A) \subseteq T = (J : J)$ and hence $(A : A) = T = (J : J)$. Next we prove that $(A(T : A))_t = J$. We have $(A_i : A_i) = T$ and $(A_i(T : A_i))_t = (A_i(T : A_i))_{t_1} = Q_i$, for each i, since $\overline{A_i} \in G_{Q_i}$. Let $j \in \{1, \ldots, r\}$ and set $F_j := \prod_{i \neq j} A_i$. Clearly $A = (JF_jA_j)_t$. Now, since $A \subseteq A_j$, then $(A(T : A_j))_t \subseteq (A(T : A))_t$. However $(A(T : A_j))_t = ((JF_jA_j)_t(T : A_j))_t = (JF_jA_j(T : A_j))_t = (JF_j(A_j(T : A_j))_t)_t = (JF_jQ_j)_t$. Hence $(JF_jQ_j)_t \subseteq (A(T : A))_t$. So $JQ_jT_{Q_j} = JF_jQ_jT_{Q_j} = (JF_jQ_j)_tT_{Q_j} \subseteq (A(T : A))_tT_{Q_j}$ since $F_jT_{Q_j} = T_{Q_j}$. Then $JT_{Q_j} = Q_jT_{Q_j} = Q_j^2T_{Q_j} = JQ_jT_{Q_j} \subseteq (A(T : A))_tT_{Q_j}$. Since $\mathrm{Max}_t(T, A) = \mathrm{Max}_t(T, J)$ and $\overline{\mathrm{Max}_t}(T, A) = \overline{\mathrm{Max}_t}(T, J)$, it follows that $J \subseteq (A(T : A))_t$. Conversely, let $j \in \{1, \ldots, r\}$. By the proof of the third claim, $(T : A_j) = (Q_j : A_j)$. Then $(T : A) = (T : (JF_jA_j)_t) = (T : JF_jA_j) = ((T : A_j) : JF_j) =$

$((Q_j : A_j) : JF_j) = (Q_j : JF_jA_j) = (Q_j : (JF_jA_j)_t) = (Q_j : A)$. Hence $(T : A) = \bigcap_{1\leqslant j\leqslant r}(Q_j : A) = ((\bigcap_{1\leqslant j\leqslant r} Q_j) : A) = (J : A) \subseteq (T : A)$. So $(T : A) = (J : A)$. Therefore $(A(T : A))_t = (A(J : A))_t \subseteq J$. Consequently, $J = (A(T : A))_t$ and thus $\overline{A} \in G_J$, as desired.

Claim 5. $\mathrm{Im}(\phi) = \mathrm{Ker}(\psi)$.

Indeed, let $[A] \in \mathrm{Cl}(T)$ for some t-invertible t-ideal A of T. Then there exists a finitely generated ideal B of T such that $A = B_{v_1} = B_{t_1}$. Hence $\psi(\phi([A])) = \psi(\overline{(AJ)_t}) = (\overline{(AJ)_tT_{Q_i}})_{1\leqslant i\leqslant r}$. For each i, we have $(AJ)_tT_{Q_i} = (AJ)_{t_1}T_{Q_i} = (BJ)_{t_1}T_{Q_i} = BJT_{Q_i} = BQ_iT_{Q_i} = b_iQ_iT_{Q_i}$ for some nonzero $b_i \in B$. Then $\overline{(AJ)_tT_{Q_i}} = \overline{Q_iT_{Q_i}}$ in $G_{Q_iT_{Q_i}}$. It follows that $\mathrm{Im}(\phi) \subseteq \mathrm{Ker}(\psi)$.

Conversely, let $\bar{L} \in G_J$ such that $\overline{LT_{Q_i}} = \overline{Q_iT_{Q_i}}$ for each $i \in \{1,\dots r\}$, that is, there exists $a_i \neq 0 \in \mathrm{qf}(T)$ such that $a_iQ_iT_{Q_i} = LT_{Q_i} \subseteq T_{Q_i}$. Then $a_i \in (T_{Q_i} : Q_iT_{Q_i}) = (Q_iT_{Q_i} : Q_i^2T_{Q_i}) = (Q_iT_{Q_i} : Q_iT_{Q_i}) = T_{Q_i}$ for each i. Let $B := \sum_{1\leqslant k\leqslant r} Ta_i$ and $A := B_{t_1}$. Clearly, A is a fractional t-invertible t-ideal of T, i.e., $[A] \in \mathrm{Cl}(T)$. Further, for each i, $(AJ)_tT_{Q_i} = (AJ)_{t_1}T_{Q_i} = (BJ)_{t_1}T_{Q_i} = BJT_{Q_i} = BQ_iT_{Q_i} = a_kQ_iT_{Q_i}$ for some a_k $(\neq 0)$, hence $\overline{(AJ)_tT_{Q_i}} = \overline{LT_{Q_i}}$. Therefore $\phi([A]) = \bar{L}$. Hence $\mathrm{Ker}(\psi) \subseteq \mathrm{Im}(\phi)$, as desired.

Consequently, the sequence is exact, completing the proof of the theorem. □

A domain R is said to be strongly t-discrete if it has no t-idempotent t-prime ideals, i.e., for every t-prime ideal P of R, $(P^2)_t \subsetneq P$.

Corollary 2.9. *Let R be a Krull-type domain which is strongly t-discrete. Then $S_t(R)$ is a disjoint union of subgroups $\mathrm{Cl}(T)$, where T ranges over the set of fractional t-linked overrings of R.*

Proof. Recall first the fact that every fractional t-linked overring T of R has the form $T = (I : I)$ for some t-ideal I of R such that $\bar{I}$ is an idempotent of $S_t(R)$; precisely, $I := aT$, for some $0 \neq a \in (R : T)$, with $(I^2)_t = (I^2)_{t_1} = a^2T = aI$. Now, Lemma 2.3 forces each T to be strongly t-discrete, that is, T has no t-idempotent t-maximal ideals. So Theorem 2.1 leads to the conclusion (via the identification $G_{\overline{T}} \cong \mathrm{Cl}(T)$). □

Since in a Prüfer domain the t-operation coincides with the trivial operation, we recover Bazzoni's results on Prüfer domains of finite character.

Corollary 2.10. *(See [6, Theorem 3.1] and [7, Theorem 3.5].) Let R be a Prüfer domain of finite character. Then $\overline{J}$ is an idempotent of $S(R)$ if and only if there exists a unique nonzero idempotent fractional ideal L such that $\overline{J} = \bar{L}$ and L satisfies one of the following two conditions*:

(1) *$L := D$ where D is a fractional overring of R, or*
(2) *$L := P_1.P_2\dots P_nD$, where each P_i is a nonzero idempotent prime ideal of R, and D is a fractional overring of R. Moreover, the following sequence is exact*

$$0 \longrightarrow \mathrm{Cl}(D) \longrightarrow G_L \longrightarrow \prod_{1\leqslant i\leqslant r} G_{P_iR_{P_i}} \longrightarrow 0.$$

Proof. The result follows readily from Theorem 2.1 since T is flat over R and every prime ideal Q of T is of the form $Q = PT$ for some prime P of R, and Q is idempotent if and only if so is P. □

3. Examples

One can develop numerous illustrative examples via Theorem 2.1 and Corollary 2.9. We will provide two families of such examples by means of polynomial rings over valuation domains. For this purpose, we first state the following lemma.

Lemma 3.1. *Let V be a nontrivial valuation domain, X an indeterminate over V, and $R := V[X]$. Then the following statements hold*:

(1) *R is a Krull-type domain which is not Prüfer.*
(2) *Every fractional t-linked overring of R has the form $V_p[X]$ for some nonzero $p \in \operatorname{Spec}(V)$.*
(3) *Every t-idempotent t-prime ideal of R has the form $p[X]$ for some idempotent $p \in \operatorname{Spec}(V)$.*

Proof. (1) The notion of PVMD is stable under adjunction of indeterminates [20]. So R is a PVMD and has finite t-character by [26, Proposition 4.2], as desired. Further, a polynomial ring over a domain is a Prüfer domain only if the coefficient ring is a field. Hence, in the current setting, R is not a Prüfer domain.

(2) Let $p \neq 0 \in \operatorname{Spec}(V)$, then $V_p[X]$ is a fractional t-linked overring of R. Indeed, let $S := V \setminus p$ and let J be a finitely generated ideal of R such that $J^{-1} = R$. We have $(V_p[X] : JV_p[X]) = (S^{-1}(V[X]) : S^{-1}J) = S^{-1}(R : J) = S^{-1}R = V_p[X]$. Hence $V_p[X]$ is t-linked over R. Now suppose p is not maximal. Since V is a conducive domain (since valuation), then $(V : V_p) = p$ [13]. Hence $(V[X] : V_p[X]) = (V : V_p)[X] = p[X]$. It follows that $V_p[X]$ is a fractional overring of R. If p is maximal, then $V_p[X] = V[X] = R$ is trivially a fractional overring of R. Next let T be a fractional t-linked overring of R, $0 \neq a \in (R : T)$, and $I := aT$. By Lemma 2.4, I is a common t-ideal of both R and T. Set $A := I \cap V$. If $A \neq 0$, then A is a t-ideal of V and hence $I = A[X]$; if $A = (0)$, then $I = fB[X]$ where $f \neq 0 \in \operatorname{qf}(V)[X]$ and B is a t-ideal of V [31]. If $I = A[X]$, then $T = (I : I) = (A[X] : A[X]) = (A : A)[X]$; and if $I = fB[X]$, then $T = (I : I) = (fB[X] : fB[X]) = (B[X] : B[X]) = (B : B)[X]$. Moreover, $(A : A)$ and $(B : B)$ are overrings (and hence localizations) of V. Therefore, in both cases, $T = V_p[X]$ for some nonzero prime ideal p of V, as desired.

(3) Let p be an idempotent prime ideal of V. Then $((p[X])^2)_t = (p^2[X])_t = (p[X])_t = p[X]$, recall here that the t-operation with respect to V is trivial. Next let P be a t-idempotent t-prime ideal of R and $p := P \cap V$. Assume that $p = (0)$ and set $S := V \setminus \{0\}$. Then, by [26, Lemma 2.6], $S^{-1}P$ is an idempotent (nonzero) ideal of the PID $S^{-1}R = \operatorname{qf}(V)[X]$, absurd. It follows that $p \neq (0)$. Since V is integrally closed, then $P = p[X]$ [31]. Moreover $p[X] = (p^2[X])_t = p^2[X]$, hence $p = p^2$, as desired. $\square$

Example 3.2. Let n be an integer $\geqslant 1$. Let V be an n-dimensional strongly discrete valuation domain and let $(0) \subset p_1 \subset p_2 \subset \cdots \subset p_n$ denote the chain of its prime ideals. Let $R := V[X]$, a Krull-type domain. A combination of Lemma 3.1 and Corollary 2.9 yields $S_t(R) \cong \bigcup_{1 \leqslant i \leqslant n} \operatorname{Cl}(V_{p_i}[X])$. Moreover $\operatorname{Cl}(V_{p_i}[X]) = \operatorname{Cl}(V_{p_i}) = 0$, so that $S_t(R)$ is a disjoint union of n groups all of them are trivial. Precisely, $S_t(R) = \{V_{p_1}[X], V_{p_2}[X], \ldots, V_{p_n}[X]\}$ where, for each i, $\overline{V_{p_i}[X]}$ is identified with $V_{p_i}[X]$ (due to the uniqueness stated by Theorem 2.1).

Example 3.3. Let V be a one-dimensional valuation domain with idempotent maximal ideal M and X an indeterminate over V. Let $R := V[X]$, a Krull-type domain. By Theorem 2.1 and Lemma 3.1, $S_t(R)$ is a disjoint union of $\operatorname{Cl}(R)$ and $G_{M[X]}$. Now, $\operatorname{Cl}(R) = \operatorname{Cl}(V[X]) = \operatorname{Cl}(V) = 0$. So $S_t(R) = \{\overline{R}\} \cup \{\overline{I} \mid I \ t\text{-ideal of } R \text{ with } (II^{-1})_t = M[X]\}$.

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