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Cranks and dissections in Ramanujan's lost notebook

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Abstract

In his lost notebook, Ramanujan offers several results related to the crank, the existence of which was first conjectured by F. J. Dyson and later established by G.E. Andrews and F.G. Garvan. Using an obscure identity found on p. 59 of the lost notebook, we provide uniform proofs of several congruences in the ring of formal power series for the generating function $F(q)$ of cranks. All are found, sometimes in abbreviated form, in the lost notebook, and imply dissections of $F(q)$. Consequences of our work are interesting new q-series identities and congruences in the spirit of Atkin and Swinnerton-Dyer.

MSC: primary 11P82; secondary 11P83

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1. Introduction

In attempting to find a combinatorial interpretation for Ramanujan's famous congruences for the partition function $p(n)$, the number of ways of representing the positive integer n as a sum of positive integers, in 1944, Dyson [4] defined the *rank of a partition* to be the largest part minus the number of parts. Dyson offered several conjectures, including combinatorial interpretations of Ramanujan's famous congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. These conjectures, as well as further conjectures of Dyson, were first proved by Atkin and Swinnerton-Dyer [2] in 1954.

The corresponding analogue does not hold for $p(11n+6) \equiv 0 \pmod{11}$, and so Dyson conjectured the existence of a *crank*. In his doctoral dissertation [8], Garvan defined vector partitions which became the forerunners of the crank. The *true crank* was discovered by Andrews and Garvan [1].

Definition 1.1. For a partition π, let $\lambda(n)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be

$$c(\pi) = \begin{cases} \lambda(\pi) & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi) & \text{if } \mu(\pi) > 0. \end{cases} \tag{1.1}$$

The crank not only leads to a combinatorial interpretation of $p(11n+6) \equiv 0 \pmod{11}$, as predicted by Dyson, but also to similar interpretations for $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leqslant 1$, we set [10]

$$M(m,n) = \begin{cases} -1 & \text{if } (m,n) = (0,1), \\ 1 & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Andrews and Garvan [1] showed that the generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{1.2}$$

In fact, in his lost notebook [17], Ramanujan records several entries about cranks, mostly about the generating function (1.2). At the top of p. 179 in his lost notebook [17], Ramanujan defines a function $F(q)$ and coefficients λ_n, $n \geqslant 0$, by

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n. \tag{1.3}$$

Thus, by (1.2), for $n > 1$,

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n)a^m.$$

He then offers two congruences for $F(q)$. These congruences, like others in the sequel, are to be regarded as congruences in the ring of formal power series in the two variables a and q. First, however, we need to define Ramanujan's theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.4}$$

which satisfies the Jacobi triple product identity [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

and the elementary identity [3, p. 34, Entry 18(iv)]

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}) \tag{1.6}$$

for any integer n. Both of these identities will be used many times in the sequel, possibly without comment. The two congruences are then given by

$$F(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right)\sqrt{q}\,\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \left(\mathrm{mod}\, a^2 + \frac{1}{a^2}\right) \tag{1.7}$$

and

$$\begin{aligned} F(q^{1/3}) \equiv{}& \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ &+ \left(a - 1 + \frac{1}{a}\right) q^{1/3}\,\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ &+ \left(a^2 + \frac{1}{a^2}\right) q^{2/3}\,\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \left(\mathrm{mod}\, a^3 + 1 + \frac{1}{a^3}\right). \end{aligned} \tag{1.8}$$

Note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \,(\mathrm{mod}\, \lambda_2)$ and $a^8 \equiv 1 \,(\mathrm{mod}\, \lambda_2)$. Thus, in (1.7), a behaves like a primitive eighth root of unity modulo λ_2. On the other hand, $\lambda_3 = a^3 + 1 + a^{-3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1 \,(\mathrm{mod}\, \lambda_3)$. So in (1.8), a behaves like a primitive ninth root of unity modulo λ_3.

This leads us to the following definition.

Definition 1.2. Let $P(q)$ denote any power series in q. Then the t-dissection of P is given by

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t). \tag{1.9}$$

Thus, if we let $a = \exp(2\pi i/8)$ and replace q by q^2, (1.7) implies the 2-dissection of $F(q)$, while if we let $a = \exp(2\pi i/9)$ and replace q by q^3, (1.8) implies the 3-dissection of $F(q)$.

Ramanujan gives the 5-dissection of $F(q)$ on p. 20 of his lost notebook [17]. It is interesting that Ramanujan does not give the alternative form, analogous to those in (1.7) and (1.8), from which the 5-dissection would follow by setting a to be a primitive fifth root of unity. Proofs of the 5-dissection have been given by Garvan [10] and Ekin [6].

The first explicit statement and proof of the 7-dissection of $F(q)$ was given by Garvan [10, Theorem 5.1]. Although Ramanujan did not state the 7-dissection of $F(q)$, he clearly knew it, because the six quotients of theta functions that appear in the 7-dissection are found on the bottom of p. 71 (written upside down) in his lost notebook. The first appearance of the 11-dissection of $F(q)$ in the literature can also be found in Garvan's paper [10, Theorem 6.7]. Further proofs have been given by Hirschhorn [12] and Ekin [5,6], who also gave a different proof of the 7-dissection. However, again, it is very likely that Ramanujan knew the 11-dissection, since he offers the quotients of theta functions which appear in the 11-dissection on p. 70 of his lost notebook [17].

On p. 59 in his lost notebook [17], Ramanujan records a quotient of two power series, with the highest power of the numerator being q^{21} and the highest power of the denominator being q^{22}. Underneath he records another power series with the highest power being q^5. Although not claimed by Ramanujan, the two expressions are equal. We state Ramanujan's "claim" in the following theorem.

Theorem 1.3. *If*

$$A_n := a^n + a^{-n}, \tag{1.10}$$

then

$$\frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} = \frac{1 - \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(A_{n+1} - A_n)}{(q;q)_\infty}. \tag{1.11}$$

The primary purpose of this paper is to employ an alternative version of Theorem 1.3 to give uniform proofs of the 2, 3, 5, 7, and 11-dissections of $F(q)$ in Sections 3–7. However, we emphasize that our results will be formulated in terms of congruences. An interesting byproduct of our work is that several interesting q-series identities naturally arise in our proofs. Some of these identities appear to be new (see (4.8)–(4.10)), while others (see Theorems 4.2, 5.2, and 7.2) can also be proved using identities discovered by Ekin [6]. We emphasize here that our approach to these q-series identities is much simpler than that of Ekin. For example, Ekin's proof of Theorem 7.2 requires the verifications of 55 identities [6, p. 2154], while in our proof, only Winquist's identity and Theorem 2.1 are needed.

We also employ a method of "rationalization" to provide alternative proofs of the congruences for $F(q)$ corresponding to the 2, 3, 5, 7, and 11-dissections. These proofs of the congruences for 2, 5, 7, and 11 are similar to those of Garvan [10,11] for the identities associated with $F_a(q)$ when the variable a is replaced by the corresponding primitive root of unity, but on the other hand are more detailed and more systematic, because of the use

of Ramanujan's addition formula for theta functions in Lemma 2.2. Special cases of our theorems yield congruences of Atkin and Swinnerton-Dyer [2].

In Section 8, we, in fact, show that the formulations in terms of congruences are equivalent to those in terms of roots of unity. This was claimed without proof by one of us [15, pp. 85–86], who unfortunately was unable to convince the other three present authors. Garvan convinced all of us by providing a proof, a modification of which is given in Section 8. An advantage, however, of the formulations in terms of congruences is that they yield congruences like those of Atkin and Swinnerton-Dyer as corollaries.

2. Preliminary results

It is easily seen that Ramanujan's Theorem 1.3 is equivalent to a theorem discovered independently by Kač and Wakimoto [14] and by Evans [7, Eq. (3.1)], which we now give. The notation a_k below will be used throughout the sequel.

Theorem 2.1. *Let* $a_k = (-1)^k q^{k(k+1)/2}$. *Then*

$$\frac{(q;q)_\infty^2}{(q/x;q)_\infty (qx;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{a_k(1-x)}{1-xq^k}. \tag{2.1}$$

Several times in the sequel, we shall use an addition theorem for theta functions found in Chapter 16 of Ramanujan's second notebook [16,3, p. 48, Entry 31].

Lemma 2.2. *If* $U_n = \alpha^{n(n+1)/2}\beta^{n(n-1)/2}$ *and* $V_n = \alpha^{n(n-1)/2}\beta^{n(n+1)/2}$ *for each integer* n, *then*

$$f(U_1, V_1) = \sum_{k=0}^{N-1} U_k f\left(\frac{U_{N+k}}{U_k}, \frac{V_{N-k}}{U_k}\right). \tag{2.2}$$

Also useful for us is the quintuple product identity [3, p. 80, Eq. (38.2)].

Lemma 2.3 (*Quintuple product identity*). *Let* $f(a,b)$ *be defined as in* (1.4), *and let*

$$f(-q) := f(-q, -q^2) = (q;q)_\infty, \tag{2.3}$$

by (1.5). *Then*

$$f(P^3Q, Q^5/P^3) - P^2 f(Q/P^3, P^3Q^5) = f(-Q^2)\frac{f(-P^2, -Q^2/P^2)}{f(PQ, Q/P)}. \tag{2.4}$$

Lastly, we need Winquist's identity [18]. From [10, Eq. (6.15)], Winquist's identity can be put in the following form.

Lemma 2.4 (*Winquist's identity*). *If*

$$(b_1, b_2, \dots, b_n; q)_\infty := \prod_{j=1}^{n} (b_j; q)_\infty,$$

then

$$\begin{aligned}(\alpha, q/\alpha, \beta, q/\beta, \alpha\beta, q/(\alpha\beta), \alpha/\beta, \beta q/\alpha, q, q; q)_\infty \\ = f(-\alpha^3, -q^3/\alpha^3)\{f(-\beta^3 q, -q^2/\beta^3) - \beta f(-\beta^3 q^2, -q/\beta^3)\} \\ -\alpha\beta^{-1} f(-\beta^3, -q^3/\beta^3)\{f(-\alpha^3 q, -q^2/\alpha^3) - \alpha f(-\alpha^3 q^2, -q/\alpha^3)\}. \end{aligned} \tag{2.5}$$

3. The 2-dissection for $F(q)$

Theorem 3.1. *Recall that $F(q) = F_a(q)$ is defined by (1.3) and that $f(a, b)$ is defined by (1.4). Then*

$$F_a(q) \equiv \frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty} + \left(a - 1 + \frac{1}{a}\right) q \, \frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty} \pmod{A_2}, \tag{3.1}$$

where A_2 is defined in (1.10).

Note that (3.1) is equivalent to (1.7), with $\sqrt{q}$ in (1.7) replaced by q.

The first proof of Theorem 3.1 that we give uses the method of "rationalization" and is an elaboration and an extension of Garvan's proof [10]. This method does not work in general, but only for those n-dissections when n is "small." The method used in our second proof is longer, but it is more general. Furthermore, we obtain very interesting identities, (3.12) and (3.13), along the way.

First proof of Theorem 3.1. Throughout the proof, we assume that $|q| < |a| < 1/|q|$. We also shall frequently use the facts that $a^4 \equiv -1 \pmod{A_2}$ and that $a^8 \equiv 1 \pmod{A_2}$.

Write

$$\frac{(q; q)_\infty}{(aq; q)_\infty (q/a; q)_\infty} = (q; q)_\infty \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} (aq^n)^k\right) \left(\sum_{k=0}^{\infty} (q^n/a)^k\right). \tag{3.2}$$

We now subdivide the series under the product sign into residue classes modulo 8 and then sum the series. Using repeatedly congruences modulo 8 for the powers of a, we readily find from (3.2) that

$$\begin{aligned}&\frac{(q; q)_\infty}{(aq; q)_\infty (q/a; q)_\infty} \\ &\quad \equiv (q; q)_\infty \prod_{n=1}^{\infty} \frac{\left(1 + aq^n + a^2 q^{2n} + a^3 q^{3n}\right)\left(1 + a^{-1} q^n + a^{-2} q^{2n} + a^{-3} q^{3n}\right)}{(1 + q^{4n})^2}\end{aligned}$$

$$\equiv \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} \prod_{n=1}^{\infty} (1+aq^n)(1+a^{-1}q^n) \pmod{A_2}, \tag{3.3}$$

upon multiplying out the polynomials in the product on the previous line and using congruences for powers of a modulo A_2.

Next, using Lemma 2.2 with $\alpha = a$ and $\beta = q/a$, (1.6), and congruences for powers of a modulo A_2, we find that

$$\begin{aligned}
&(q;q)_\infty(-aq;q)_\infty(-q/a;q)_\infty \\
&= (q;q)_\infty \frac{(-a;q)_\infty}{1+a}(-q/a;q)_\infty \\
&= \frac{1}{1+a}\{f(a^4q^6, q^{10}/a^4) + af(q^6/a^4, a^4q^{10}) \\
&\quad + a^2qf(q^2/a^4, a^4q^{14}) + (q/a)f(a^4q^2, q^{14}/a^4)\} \\
&\equiv \frac{1}{1+a}\{(1+a)f(-q^6,-q^{10}) + (a^2+1/a)qf(-q^2,-q^{14})\} \\
&\equiv f(-q^6,-q^{10}) + (A_1-1)qf(-q^2,-q^{14}) \pmod{A_2}.
\end{aligned} \tag{3.4}$$

Using (3.4) in (3.3), we complete the proof of Theorem 3.1. □

Second proof of Theorem 3.1. From (2.1), we deduce that

$$\begin{aligned}
\frac{(q;q)_\infty^2}{(q/x;q)_\infty(qx;q)_\infty} &= 1 + \sum_{k=1}^{\infty} a_k \frac{1-x}{1-xq^k} + \sum_{k=1}^{\infty} a_k \frac{1-x^{-1}}{1-q^k/x} \\
&= 1 + (1-x)\sum_{k=1,m=0}^{\infty} a_k q^{km} x^m \\
&\quad + (1-x^{-1})\sum_{k=1,m=0}^{\infty} a_k q^{km} x^{-m}.
\end{aligned}$$

Hence, we deduce that

$$(q;q)_\infty F_a(q) = \frac{(q;q)_\infty^2}{(q/a;q)_\infty(qa;q)_\infty} = 1 + \sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m - A_{m+1}), \tag{3.5}$$

where A_m is defined in (1.10). Observe that

$$A_m - A_{m+1} \equiv A_j - A_{j+1} \pmod{A_2},$$

whenever $m \equiv j \pmod 8$. Therefore, if

$$S_{i,j} := \sum_{\substack{k=1,m=0 \\ m\equiv i,j \pmod 8}}^{\infty} a_k q^{km},$$

we conclude that

$$\begin{aligned}
&1+\sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m - A_{m+1})\\
&\equiv 1+(2-A_1)\{S_{0,3}-S_{4,7}\}+A_1\{S_{1,2}-S_{5,6}\}\\
&\equiv 1+(A_1-1)\{S_{1,2}-S_{5,6}-S_{0,3}+S_{4,7}\}\\
&\quad +\{S_{1,2}-S_{5,6}+S_{0,3}-S_{4,7}\} \pmod{A_2}.
\end{aligned}$$

Summing the series on m, and then converting the sums into bilateral series, we conclude that

$$\begin{aligned}
&1+\sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m - A_{m+1})\\
&\equiv (A_1-1)\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^{4k}} + \sum_{k=-\infty}^{\infty} a_k \frac{q^k+1}{1+q^{4k}} \pmod{A_2}. \qquad (3.6)
\end{aligned}$$

We are now ready to complete the proof of (3.1). Let $\omega = e^{\pi i/4}$. By calculating the partial fraction decomposition, we find that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^{4k}}\\
&= -\frac{1}{4}\sum_{k=-\infty}^{\infty} a_k \left(\frac{1+\omega^3}{1-\omega q^k}+\frac{1+\omega}{1-\omega^3 q^k}+\frac{1+\omega^7}{1-\omega^5 q^k}+\frac{1+\omega^5}{1-\omega^7 q^k}\right). \qquad (3.7)
\end{aligned}$$

By (2.1), we may rewrite (3.7) as

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^{4k}} &= -\frac{1}{4}(q;q)_\infty^2 \left\{\frac{1+\omega^3}{(\omega;q)_\infty(q/\omega;q)_\infty}+\frac{1+\omega}{(\omega^3;q)_\infty(q/\omega^3;q)_\infty}\right.\\
&\left. +\frac{1+\omega^7}{(\omega^5;q)_\infty(q/\omega^5;q)_\infty}+\frac{1+\omega^5}{(\omega^7;q)_\infty(q/\omega^7;q)_\infty}\right\}. \qquad (3.8)
\end{aligned}$$

Since $1-\omega^j q^k = 1-q^k/\omega^{8-j}$, we may simplify (3.8) and obtain

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^{4k}}\\
&= -\frac{\sqrt{2}}{4}(q;q)_\infty^2\left(\frac{1}{(\omega^3 q;q)_\infty(q/\omega^3;q)_\infty}-\frac{1}{(\omega q;q)_\infty(q/\omega;q)_\infty}\right)\\
&= -\frac{\sqrt{2}}{4}\frac{(q;q)_\infty}{(-q^4;q^4)_\infty}((q;q)_\infty(\omega q;q)_\infty(q/\omega;q)_\infty\\
&\quad -(q;q)_\infty(\omega^3 q;q)_\infty(q/\omega^3;q)_\infty). \qquad (3.9)
\end{aligned}$$

From the second equality of (3.4), with a replaced by $-\omega$ and $-\omega^3$, respectively, we find that

$$(q;q)_\infty(q\omega;q)_\infty(q/\omega;q)_\infty = f(-q^6,-q^{10}) + (-\omega-\omega^7-1)qf(-q^2,-q^{14}), \tag{3.10}$$

$$(q;q)_\infty(q\omega^3;q)_\infty(q/\omega^3;q)_\infty = f(-q^6,-q^{10}) + (-\omega^3-\omega^5-1)qf(-q^2,-q^{14}). \tag{3.11}$$

Employing (3.10) and (3.11) in (3.9) and simplifying yields

$$\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^{4k}} = q\frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^2,-q^{14}). \tag{3.12}$$

Using exactly the same method, we can show that

$$\sum_{k=-\infty}^{\infty} a_k \frac{q^k+1}{1+q^{4k}} = \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^6,-q^{10}). \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.6), we obtain (3.1) by eliminating the factor $(q;q)_\infty$ in (3.5). □

4. The 3-dissection for $F(q)$

As in the case of (3.1), we will prove instead the congruence given below. Surprisingly, the 3-dissection is considerably more difficult to prove than the 2 and 5-dissections, for example. We give two proofs. The first uses the method of "rationalization" and is shorter than our second proof, which depends on Ramanujan's key theorem, Theorem 2.1. However, we were only able to find the first proof because of insights gained from the second proof. When $a = e^{2\pi i/9}$, Theorem 4.1 yields the 3-dissection of $F_a(q)$, which was first proved by Garvan [11] using the Macdonald identity for the root system A_2. Garvan's proof can be modified to give another proof of Theorem 4.1.

Theorem 4.1. *If A_n is given by* (1.10), *then*

$$F_a(q) \equiv \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty} + (A_1-1)q\frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty} + A_2q^2\frac{f(-q^3,-q^{24})f(-q^6,-q^{21})}{(q^{27};q^{27})_\infty} \pmod{A_3+1}. \tag{4.1}$$

First proof of Theorem 4.1. We first record the identities that we will need in our proof.

Substituting $(P, Q) = (-q^{3/2}, q^{27/2}), (-q^{15/2}, q^{27/2})$ and $(-q^{21/2}, q^{27/2})$ into the quintuple product identity (2.4), we find that

$$B - q^3C = \frac{f(-q^3, -q^{24})(q^{27}; q^{27})_\infty}{f(-q^{12}, -q^{15})}, \tag{4.2}$$

$$A + q^6C = A - q^{15} f(-q^{-9}, -q^{90}) = \frac{f(-q^{12}, -q^{15})(q^{27}; q^{27})_\infty}{f(-q^6, -q^{21})} \tag{4.3}$$

and

$$A + q^3B = A - q^{21} f(-q^{-18}, -q^{99}) = \frac{f(-q^6, -q^{21})(q^{27}; q^{27})_\infty}{f(-q^3, -q^{24})}, \tag{4.4}$$

respectively, where $A = f(-q^{45}, -q^{36})$, $B = f(-q^{63}, -q^{18})$, and $C = f(-q^{72}, -q^9)$.

Substituting $\alpha = -a^2$, $\beta = -q/a^2$, and $N = 9$ into (2.2) and simplifying, we deduce that

$$\begin{aligned}&(q; q)_\infty (a^2q; q)_\infty (q/a^2; q)_\infty \\ &\quad \equiv -(1 + A_2)q(q^{27}; q^{27})_\infty + A - (A_1 - 1)q^3B + A_1q^6C \pmod{A_3 + 1}.\end{aligned} \tag{4.5}$$

After these preliminary steps, we now complete our proof of the 3-dissection.

From the generating function (1.3),

$$\begin{aligned}F_a(q) &= \frac{(q; q)_\infty}{(aq; q)_\infty (a^8q; q)_\infty} \\ &\equiv \frac{(q; q)_\infty (q^3; q^3)_\infty (a^2q; q)_\infty (a^4q; q)_\infty (a^5q; q)_\infty (a^7q; q)_\infty}{(q^9; q^9)_\infty} \\ &\equiv \frac{(q^3; q^3)_\infty}{(q; q)_\infty (q^9; q^9)_\infty} \\ &\quad \times\{-(1 + a^2 + a^7)q(q^{27}; q^{27})_\infty + A - (a + a^8 - 1)q^3B + (a + a^8)q^6C\} \\ &\quad \times\{-(1 + a^4 + a^5)q(q^{27}; q^{27})_\infty + A - (a^2 + a^7 - 1)q^3B \\ &\quad +(a^2 + a^7)q^6C\} \pmod{A_3 + 1},\end{aligned}$$

where we have applied (4.5) in the last equality.

Arranging the terms in the "right" order, with knowledge from our second proof being helpful, we find that

$$\begin{aligned}F_a(q) \equiv \frac{(q^3; q^3)_\infty}{(q; q)_\infty (q^9; q^9)_\infty} &\{(-q^2(q^{27}; q^{27})_\infty^2 - q(q^{27}; q^{27})_\infty (A + q^3B) \\ &+(A + q^3B)(A + q^6C)) + [a - 1 + a^8](-q^2(q^{27}; q^{27})_\infty^2 \\ &+q(q^{27}; q^{27})_\infty (A + q^6C) - q^3(B - q^3C)(A + q^6C))\end{aligned}$$

$$+[a^2+a^7](q^2(q^{27};q^{27})_\infty^2 - q^4(q^{27};q^{27})_\infty(B-q^3C)$$
$$-q^3(A+q^3B)(B-q^3C))\} \pmod{A_3+1}.$$

Substituting (4.2)–(4.4) into the terms on the right-hand side and simplifying, we find that

$$\begin{aligned}
F_a(q) &\equiv \frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty} \\
&\quad\times\left\{\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^3,-q^{24})}\right. \\
&\quad+[a-1+a^8]q\,\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^6,-q^{21})} \\
&\quad\left.+[a^2+a^7]q^2\,\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q^{12},-q^{15})}\right\} \\
&\equiv \frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty}\left\{\frac{(q;q)_\infty}{f(-q^3,-q^{24})}+[a-1+a^8]q\frac{(q;q)_\infty}{f(-q^6,-q^{21})}\right. \\
&\quad\left.+[a^2+a^7]q^2\frac{(q;q)_\infty}{f(-q^{12},-q^{15})}\right\} \pmod{A_3+1},
\end{aligned}$$

where we have applied [3, p. 349, Entry 2(v)] in the last equality, namely,

$$(q;q)_\infty = f(-q^{12},-q^{15})-qf(-q^6,-q^{21})-q^2f(-q^3,-q^{24}). \tag{4.6}$$

Finally, note that [3, p. 349, Entry 2(vi)]

$$\begin{aligned}
\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^3,-q^{24})} &= \frac{f(-q^6,-q^{21})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}, \\
\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^6,-q^{21})} &= \frac{f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty}, \\
\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^{12},-q^{15})} &= \frac{f(-q^3,-q^{24})f(-q^6,-q^{21})}{(q^{27};q^{27})_\infty},
\end{aligned}$$

and this completes the first proof of the 3-dissection given in Theorem 4.1. □

Second proof of Theorem 4.1. First, we observe that

$$A_m - A_{m+1} \equiv A_j - A_{j+1} \pmod{A_3+1},$$

whenever $m \equiv j \pmod 9$, where A_m is defined in (1.10). Proceeding as before, if we set

$$T_{i,j,l} := \sum_{\substack{k=1,m=0 \\ m\equiv i,j,l \pmod 9}}^{\infty} a_k q^{km},$$

we find that

$$\begin{aligned} &1 + \sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m - A_{m+1}) \\ &\quad \equiv 1 + \{T_{0,1,2} - T_{6,7,8}\} + (A_1 - 1)\{T_{1,3,8} - T_{0,5,7}\} \\ &\qquad + A_2\{T_{2,3,7} - T_{1,5,6}\} \pmod{A_3 + 1}. \end{aligned}$$

Simplifying, we find that

$$\begin{aligned} &1 + \sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m - A_{m+1}) \\ &\quad \equiv \sum_{k=-\infty}^{\infty} a_k \frac{1 + q^k + q^{2k}}{1 + q^{3k} + q^{6k}} + (A_1 - 1) \sum_{k=-\infty}^{\infty} a_k \frac{q^k - 1}{1 + q^{3k} + q^{6k}} \\ &\qquad + A_2 \sum_{k=-\infty}^{\infty} a_k \frac{q^{2k} - q^k}{1 + q^{3k} + q^{6k}} \pmod{A_3 + 1}. \end{aligned} \tag{4.7}$$

The proof of (4.1) now follows from Theorem 1.3 and the following identities, which are analogues of (3.12) and (3.13). □

Theorem 4.2. *We have*

$$\sum_{k=-\infty}^{\infty} a_k \frac{q^k - 1}{1 + q^{3k} + q^{6k}} = q(q;q)_\infty \frac{f(-q^3, -q^{24}) f(-q^{12}, -q^{15})}{(q^{27}; q^{27})_\infty}, \tag{4.8}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{q^{2k} - q^k}{1 + q^{3k} + q^{6k}} = q^2(q;q)_\infty \frac{f(-q^3, -q^{24}) f(-q^6, -q^{21})}{(q^{27}; q^{27})_\infty}, \tag{4.9}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{1 + q^k + q^{2k}}{1 + q^{3k} + q^{6k}} = (q;q)_\infty \frac{f(-q^6, -q^{21}) f(-q^{12}, -q^{15})}{(q^{27}; q^{27})_\infty}. \tag{4.10}$$

We give only the proof of (4.8). The other two identities can be established using the same method.

Proof of (4.8). Let $\zeta = e^{2\pi i/9}$. Proceeding as in the second proof of the 2-dissection, we calculate the partial fraction decomposition

$$\frac{9(1-q^k)}{1+q^{3k}+q^{6k}} = (1-\zeta^6)\left(\frac{1-\zeta^8}{1-\zeta q^k}+\frac{1-\zeta^5}{1-\zeta^4 q^k}+\frac{1-\zeta^2}{1-\zeta^7 q^k}\right)$$
$$+(1-\zeta^3)\left(\frac{1-\zeta}{1-\zeta^8 q^k}+\frac{1-\zeta^4}{1-\zeta^5 q^k}+\frac{1-\zeta^7}{1-\zeta^2 q^k}\right).$$

Hence, we find that

$$\begin{aligned}
&9\sum_{k=-\infty}^{\infty} a_k \frac{1-q^k}{1+q^{3k}+q^{6k}}\\
&\quad=(1-\zeta^6)\left[\frac{1-\zeta^8}{1-\zeta}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta}{1-\zeta q^k}\right.\\
&\qquad\left.+\frac{1-\zeta^5}{1-\zeta^4}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^4}{1-\zeta^4 q^k}+\frac{1-\zeta^2}{1-\zeta^7}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^7}{1-\zeta^7 q^k}\right]\\
&\qquad+(1-\zeta^3)\left[\frac{1-\zeta}{1-\zeta^8}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^8}{1-\zeta^8 q^k}\right.\\
&\qquad\left.+\frac{1-\zeta^4}{1-\zeta^5}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^5}{1-\zeta^5 q^k}+\frac{1-\zeta^7}{1-\zeta^2}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^2}{1-\zeta^2 q^k}\right]. \qquad (4.11)
\end{aligned}$$

Using (2.1) and the identity $1-\zeta^6 = -\zeta^6(1-\zeta^3)$, we rewrite (4.11) as

$$\begin{aligned}
&9\sum_{k=-\infty}^{\infty} a_k \frac{1-q^k}{1+q^{3k}+q^{6k}}\\
&\quad=-(1-\zeta^3)(q;q)_\infty^2\\
&\qquad\times\left(\frac{\zeta-\zeta^5}{(\zeta q;q)_\infty(\zeta^8 q;q)_\infty}+\frac{\zeta^4-\zeta^2}{(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty}+\frac{\zeta^7-\zeta^8}{(\zeta^2 q;q)_\infty(\zeta^7 q;q)_\infty}\right)\\
&\quad=-(1-\zeta^3)\frac{(q;q)_\infty^2(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\\
&\qquad\times[(\zeta-\zeta^5)(\zeta^2 q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty(\zeta^7 q;q)_\infty\\
&\qquad+(\zeta^4-\zeta^2)(\zeta q;q)_\infty(\zeta^2 q;q)_\infty(\zeta^7 q;q)_\infty(\zeta^8 q;q)_\infty\\
&\qquad+(\zeta^7-\zeta^8)(\zeta q;q)_\infty(\zeta^4 q;q)_\infty(\zeta^5 q;q)_\infty(\zeta^8 q;q)_\infty]. \qquad (4.12)
\end{aligned}$$

Note that when $a = \zeta^j$ and $\gcd(j, 9) = 1$, we can deduce from (4.5) that

$$\begin{aligned}
&(q;q)_\infty(\zeta^{2j}q;q)_\infty(q/\zeta^{2j};q)_\infty \\
&\quad = -\left(1+\frac{1}{\zeta^{2j}}+\zeta^{2j}\right)q(q^{27};q^{27})_\infty \\
&\qquad +A-\left(\zeta^j+\frac{1}{\zeta^j}-1\right)q^3B+\left(\zeta^j+\frac{1}{\zeta^j}\right)q^6C. \qquad (4.13)
\end{aligned}$$

Using (4.13) six times, we rewrite (4.12) as

$$\begin{aligned}
&9\sum_{k=-\infty}^{\infty} a_k\,\frac{1-q^k}{1+q^{3k}+q^{6k}} \\
&\quad = -9q\,\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}[-q(q^{27};q^{27})_\infty^2+q^6(q^{27};q^{27})_\infty C \\
&\qquad +(q^{27};q^{27})_\infty A-q^8BC-q^2AB+q^{11}C^2+q^5AC] \\
&\quad = -9q\,\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}[-q(q^{27};q^{27})_\infty^2+(q^{27};q^{27})_\infty(A+q^6C) \\
&\qquad -q^2(A+q^6C)(B-q^3C)]. \qquad (4.14)
\end{aligned}$$

Substituting (4.2) and (4.3) into (4.14), we deduce that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k\,\frac{1-q^k}{1+q^{3k}+q^{6k}} \\
&\quad = -q\,\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\left(-q(q^{27};q^{27})_\infty^2+\frac{f(-q^{12},-q^{15})(q^{27};q^{27})_\infty^2}{f(-q^6,-q^{21})}\right. \\
&\qquad \left.-q^2\,\frac{f(-q^3,-q^{24})(q^{27};q^{27})_\infty}{f(-q^{12},-q^{15})}\,\frac{f(-q^{12},-q^{15})(q^{27};q^{27})_\infty}{f(-q^6,-q^{21})}\right) \\
&\quad = -q\,\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q^9;q^9)_\infty f(-q^6,-q^{21})}(-qf(-q^6,-q^{21}) \\
&\qquad +f(-q^{12},-q^{15})-q^2f(-q^3,-q^{24})) \\
&\quad = -q\,\frac{(q^3;q^3)_\infty}{(q^9;q^9)_\infty}\,\frac{(q^{27};q^{27})_\infty^2(q;q)_\infty}{f(-q^6,-q^{21})} \\
&\quad = -q\,\frac{(q;q)_\infty f(-q^3,-q^{24})f(-q^{12},-q^{15})}{(q^{27};q^{27})_\infty},
\end{aligned}$$

where we have applied (4.6) in the penultimate equality. This completes the proof of (4.8). □

5. The 5-dissection for $F(q)$

For this and the remaining sections, it will be convenient to define

$$S_n(a) := \sum_{k=-n}^{n} a^k. \tag{5.1}$$

Note that when p is an odd prime,

$$S_{(p-1)/2}(a) = a^{(1-p)/2}\Phi_p(a),$$

where $\Phi_n(a)$ is the minimal, monic polynomial for a primitive nth root of unity.

In this section, $\zeta = e^{2\pi i/5}$.We provide two proofs of the congruence corresponding to the 5-dissection. The first proof is similar to Garvan's proof [10] of the 5-dissection of $F_\zeta(q)$. Note that if we set $a = 1$ in Theorem 5.1, we recover Atkin and Swinnerton-Dyer's result [2, Theorem 1].

Theorem 5.1. *With $f(-q)$ and S_2 as defined above and A_n defined by* (1.10),

$$\begin{aligned} F_a(q) \equiv{}& \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})} f^2(-q^{25}) + (A_1-1)q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} \\ &+A_2q^2\frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - A_1q^3\frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})} f^2(-q^{25}) \pmod{S_2}. \end{aligned} \tag{5.2}$$

In his lost notebook [17, pp. 58, 59, 182], Ramanujan factored the coefficients of $F_a(q)$ as functions of a. In particular, he sought factors of S_2 in the coefficients.

First proof of Theorem 5.1. It is easy to see that

$$F_a(q) \equiv \frac{(q;q)_\infty^2(a^2q;q)_\infty(a^3q;q)_\infty}{(q^5;q^5)_\infty} \pmod{S_2}. \tag{5.3}$$

We shall use later a famous formula for the Rogers–Ramanujan continued fraction $R(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1}\frac{q}{+1}\frac{q^2}{+1}\frac{q^3}{+1}+\cdots, \quad |q|<1,$$

namely [3, p. 265, Entry 11(iii)],

$$\frac{1}{R(q)} - R(q) - 1 = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}. \tag{5.4}$$

Using the well-known fact [3, p. 266, Entry 11(iii)],

$$R(q) = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)},$$

we can rewrite (5.4) in the form

$$\frac{f(-q)}{qf(-q^{25})} = \frac{f(-q^{10},-q^{15})}{qf(-q^5,-q^{20})} - \frac{qf(-q^5,-q^{20})}{f(-q^{10},-q^{15})} - 1. \tag{5.5}$$

By (1.5) and Lemma 2.2 with $(\alpha,\beta,n) = (-a^2,-q/a^2,5)$, we find that

$$\begin{aligned}&(q;q)_\infty (a^2q;q)_\infty (a^3q;q)_\infty\\ &\quad\equiv \frac{f(-a^2,-q/a^2)}{(1-a^2)}\\ &\quad\equiv f(-q^{10},-q^{15}) + qA_1 f(-q^{20},-q^5) \pmod{S_2}.\end{aligned} \tag{5.6}$$

Substituting (5.6) and (5.5) into (5.3) yields (5.2). □

Second proof of Theorem 5.1. We apply Theorem 2.1. Define

$$T_i := \sum_{\substack{k=1,m=0\\ m\equiv i\,(\mathrm{mod}\,5)}}^{\infty} a_k q^{km}.$$

Then

$$\begin{aligned}\frac{(q;q)_\infty^2}{(aq;q)_\infty (q/a;q)_\infty} &= 1 + \sum_{k=1,m=0}^{\infty} a_k q^{km}\,(A_m - A_{m+1})\\ &\equiv 1 + (2-A_1)\{T_0 - T_4\} + (A_1 - A_2)\{T_1 - T_3\}\\ &\equiv 1 + \{2T_0 - 2T_4 + T_1 - T_3\}\\ &\quad + A_1\{T_4 - T_0 + 2T_1 - 2T_3\} \pmod{S_2},\end{aligned} \tag{5.7}$$

where A_m is defined in (1.10). Note that

$$\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} a_k \frac{q^{2k}}{1-q^{5k}} = 0.$$

This enables us to simplify (5.7) to conclude that

$$\begin{aligned}\frac{(q;q)_\infty^2}{(aq;q)_\infty (q/a;q)_\infty} &\equiv \sum_{k=-\infty}^{\infty} a_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}}\\ &\quad + A_1 \sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^k+q^{2k}+q^{3k}+q^{4k}} \pmod{S_2}.\end{aligned}$$

The proof of the 5-dissection now follows from the following identities. □

Theorem 5.2. *We have*

$$\sum_{k=-\infty}^{\infty} a_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}}$$

$$= \frac{(q;q)_\infty (q^{25};q^{25})_\infty^2 f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}$$

$$-q\,\frac{(q;q)_\infty (q^{25};q^{25})_\infty^2}{f(-q^5,-q^{20})} - q^2 \frac{(q;q)_\infty (q^{25};q^{25})_\infty^2}{f(-q^{10},-q^{15})}, \tag{5.8}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{q^k-1}{1+q^k+q^{2k}+q^{3k}+q^{4k}}$$

$$= q\,\frac{(q;q)_\infty (q^{25};q^{25})_\infty^2}{f(-q^5,-q^{20})} - q^2\,\frac{(q;q)_\infty (q^{25};q^{25})_\infty^2}{f(-q^{10},-q^{15})}$$

$$-q^3\,\frac{(q;q)_\infty (q^{25};q^{25})_\infty^2 f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}. \tag{5.9}$$

Proof. We prove only (5.8), since the proof of (5.9) is similar.

We begin with the partial fraction decomposition

$$\frac{5(2+3q^k)}{1+q^k+q^{2k}+q^{3k}+q^{4k}} = \frac{(1-\zeta^4)(2+3\zeta^4)}{1-\zeta q^k} + \frac{(1-\zeta^3)(2+3\zeta^3)}{1-\zeta^2 q^k}$$

$$+\frac{(1-\zeta^2)(2+3\zeta^2)}{1-\zeta^3 q^k} + \frac{(1-\zeta)(2+3\zeta)}{1-\zeta^4 q^k}.$$

Therefore,

$$G(q) := 5\sum_{k=-\infty}^{\infty} a_k \frac{2+3q^k}{1+q^k+q^{2k}+q^{3k}+q^{4k}}$$

$$= \frac{(1-\zeta^4)(2+3\zeta^4)}{1-\zeta} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta}{1-\zeta q^k}$$

$$+\frac{(1-\zeta^3)(2+3\zeta^3)}{1-\zeta^2} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^2}{1-\zeta^2 q^k}$$

$$+\frac{(1-\zeta^2)(2+3\zeta^2)}{1-\zeta^3} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^3}{1-\zeta^3 q^k}$$

$$+\frac{(1-\zeta)(2+3\zeta)}{1-\zeta^4} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^4}{1-\zeta^4 q^k}.$$

Applying Theorem 2.1 on the right-hand side and simplifying, we find that

$$\begin{aligned} G(q) &= -(2\zeta^4+3\zeta^3+2\zeta+3\zeta^2)\frac{(q;q)_\infty^2}{(\zeta q;q)_\infty(\zeta^4 q;q)_\infty} \\ &\quad -(2\zeta^3+3\zeta+2\zeta^2+3\zeta^4)\frac{(q;q)_\infty^2}{(\zeta^2 q;q)_\infty(\zeta^3 q;q)_\infty} \\ &= (2-\zeta^2-\zeta^3)\frac{(q;q)_\infty^3(\zeta^2 q;q)_\infty(\zeta^3 q;q)_\infty}{(q^5;q^5)_\infty} \\ &\quad +(2-\zeta-\zeta^4)\frac{(q;q)_\infty^3(\zeta q;q)_\infty(\zeta^4 q;q)_\infty}{(q^5;q^5)_\infty}. \end{aligned} \tag{5.10}$$

Applying (5.6) two times on the right-hand side of (5.10) with $a=\zeta$ and ζ^3, respectively, we find that

$$\begin{aligned} G(q) &= \frac{(q;q)_\infty^2}{(q^5;q^5)_\infty}\left\{5f(-q^{10},-q^{15})\right. \\ &\quad \left.+\left[(2-\zeta^2-\zeta^3)\frac{\zeta^4-\zeta^3}{1-\zeta^2}+(2-\zeta-\zeta^4)\frac{\zeta^2-\zeta^4}{1-\zeta}\right]f(-q^5,-q^{20})\right\} \\ &= 5\,\frac{(q;q)_\infty^2 f(-q^{10},-q^{15})}{(q^5;q^5)_\infty}. \end{aligned} \tag{5.11}$$

From (5.5), we find that

$$\begin{aligned} (q;q)_\infty &= -q(q^{25};q^{25})_\infty \\ &\quad +(q^{25};q^{25})_\infty\left\{\frac{f(-q^{10},-q^{15})}{f(-q^5,-q^{20})}-q^2\frac{f(-q^5,-q^{20})}{f(-q^{10},-q^{15})}\right\}. \end{aligned} \tag{5.12}$$

Substituting (5.12) into (5.11) and dividing by five, we find that $G(q)/5$ equals the right-hand side of (5.8). □

Theorem 5.2 can also be proved using identities established by Ekin [6, bottom of p. 2149].

6. The 7-dissection for $F(q)$

We offer two proofs of the 7-dissection of $F_a(q)$. The first is an extension and elaboration of that of Garvan [10], while the second uses the theorem of Ramanujan, Kač and Wakimoto [14] and Evans [7], Theorem 2.1. Note that if we substitute $a=1$ in Theorem 6.1, we immediately obtain [2, Theorem 2]. In this section, $\zeta=e^{2\pi i/7}$.

Theorem 6.1. *With $f(a,b)$ defined by (1.4), $f(-q)$ defined by (2.3), A_n defined by (1.10), and S_m defined by (5.1),*

$$\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} \equiv \frac{1}{f(-q^7)}(A^2+(A_1-1)qAB+A_2q^2B^2+(A_3+1)q^3AC - A_1q^4BC-(A_2+1)q^6C^2) \pmod{S_3}, \tag{6.1}$$

where $A=f(-q^{21},-q^{28})$, $B=f(-q^{35},-q^{14})$, and $C=f(-q^{42},-q^7)$.

First proof of Theorem 6.1. Rationalizing and using Jacobi's triple product identity (1.5), we find that

$$\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty^2(qa^2;q)_\infty(qa^{-2};q)_\infty(qa^3;q)_\infty(qa^{-3};q)_\infty}{(q^7;q^7)_\infty} \equiv \frac{1}{f(-q^7)}\frac{f(-a^2,-q/a^2)}{(1-a^2)}\frac{f(-a^3,-q/a^3)}{(1-a^3)} \pmod{S_3}. \tag{6.2}$$

Using Lemma 2.2, with $(\alpha,\beta,N)=(-a^2,-q/a^2,7)$ and $(-a^3,-q/a^3,7)$, respectively, we find that

$$\frac{f(-a^2,-q/a^2)}{(1-a^2)} \equiv A - q\frac{(a^5-a^4)}{(1-a^2)}B + q^3\frac{(a^3-a^6)}{(1-a^2)}C \pmod{S_3} \tag{6.3}$$

and

$$\frac{f(-a^3,-q/a^3)}{(1-a^3)} \equiv A - q\frac{(a^4-a^6)}{(1-a^3)}B + q^3\frac{(a-a^2)}{(1-a^3)}C \pmod{S_3}. \tag{6.4}$$

Substituting (6.3) and (6.4) into (6.2) and simplifying, we complete the proof of Theorem 6.1. □

Second proof of Theorem 6.1. Set

$$T_i := \sum_{\substack{k=1,m=0\\ m\equiv i \pmod 7}}^{\infty} a_k q^{km}.$$

As in our proofs of the 2, 3 and 5-dissections, we begin by using Theorem 2.1 to deduce that

$$\begin{aligned}\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} &= 1+\sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m-A_{m+1})\\ &\equiv 1+(2-A_1)\{T_0-T_6\}+(A_1-A_2)\{T_1-T_5\}\\ &\quad +(A_2-A_3)\{T_2-T_4\}\\ &\equiv (2-A_1)\sum_{k=-\infty}^{\infty} a_k\frac{1+q^k+q^{2k}}{1+q^k+\cdots+q^{6k}}\end{aligned}$$

$$+(A_1 - A_2) \sum_{k=-\infty}^{\infty} a_k \frac{q^k + q^{2k}}{1 + q^k + \cdots + q^{6k}}$$

$$+(A_2 - A_3) \sum_{k=-\infty}^{\infty} a_k \frac{q^{2k}}{1 + q^k + \cdots + q^{6k}}$$

$$\equiv \sum_{k=-\infty}^{\infty} a_k \frac{2 + 2q^k + 3q^{2k}}{1 + q^k + \cdots + q^{6k}}$$

$$+A_1 \sum_{k=-\infty}^{\infty} a_k \frac{q^{2k} - 1}{1 + q^k + \cdots + q^{6k}}$$

$$+A_2 \sum_{k=-\infty}^{\infty} a_k \frac{q^{2k} - q^k}{1 + q^k + \cdots + q^{6k}} \pmod{S_3}, \tag{6.5}$$

where A_m is defined in (1.10).

The proof of Theorem 6.1 now follows from the following identities. Indeed, if we substitute the identities of Theorem 6.2 into (6.5) and collect terms, we complete the second proof of Theorem 6.1. □

Theorem 6.2. *We have*

$$\sum_{k=-\infty}^{\infty} a_k \frac{7}{1 + q^k + \cdots + q^{6k}}$$

$$= \frac{(q;q)_\infty}{(q^7;q^7)_\infty}\{A^2 - 6qAB + 2q^2B^2 + 3q^3AC + 5q^4BC - 3q^6C^2\}, \tag{6.6}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{7q^k}{1 + q^k + \cdots + q^{6k}}$$

$$= \frac{(q;q)_\infty}{(q^7;q^7)_\infty}\{A^2 + qAB - 5q^2B^2 + 3q^3AC - 2q^4BC + 4q^6C^2\}, \tag{6.7}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{7q^{2k}}{1 + q^k + \cdots + q^{6k}}$$

$$= \frac{(q;q)_\infty}{(q^7;q^7)_\infty}\{A^2 + qAB + 2q^2B^2 - 4q^3AC - 2q^4BC - 3q^6C^2\}, \tag{6.8}$$

where A, B and C are given in Theorem 6.1.

Proof. We prove only (6.6), since the proofs of the remaining two identities are similar. We first calculate the partial fraction decomposition

$$\frac{7}{1 + q^k + \cdots + q^{6k}} = \frac{1 - \zeta^6}{1 - \zeta q^k} + \frac{1 - \zeta^5}{1 - \zeta^2 q^k} + \frac{1 - \zeta^4}{1 - \zeta^3 q^k}$$

$$+\frac{1-\zeta^3}{1-\zeta^4q^k}+\frac{1-\zeta^2}{1-\zeta^5q^k}+\frac{1-\zeta}{1-\zeta^6q^k}.$$

Therefore we deduce that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k \frac{7}{1+q^k+\cdots+q^{6k}}\\
&=\frac{1-\zeta^6}{1-\zeta}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta}{1-\zeta q^k}+\frac{1-\zeta^5}{1-\zeta^2}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^2}{1-\zeta^2q^k}\\
&\quad+\frac{1-\zeta^4}{1-\zeta^3}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^3}{1-\zeta^3q^k}+\frac{1-\zeta^3}{1-\zeta^4}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^4}{1-\zeta^4q^k}\\
&\quad+\frac{1-\zeta^2}{1-\zeta^5}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^5}{1-\zeta^5q^k}+\frac{1-\zeta}{1-\zeta^6}\sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^6}{1-\zeta^6q^k}. \qquad (6.9)
\end{aligned}$$

From the identity (6.9) and Theorem 2.1, we find that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} a_k \frac{7}{1+q^k+\cdots+q^{6k}} &= \left\{\frac{1-\zeta^6}{1-\zeta}+\frac{1-\zeta}{1-\zeta^6}\right\}\frac{(q;q)_\infty^2}{(\zeta q;q)_\infty(\zeta^6q;q)_\infty}\\
&\quad+\left\{\frac{1-\zeta^5}{1-\zeta^2}+\frac{1-\zeta^2}{1-\zeta^5}\right\}\frac{(q;q)_\infty^2}{(\zeta^2q;q)_\infty(\zeta^5q;q)_\infty}\\
&\quad+\left\{\frac{1-\zeta^4}{1-\zeta^3}+\frac{1-\zeta^3}{1-\zeta^4}\right\}\frac{(q;q)_\infty^2}{(\zeta^3q;q)_\infty(\zeta^4q;q)_\infty}.
\end{aligned}$$

Rationalizing the denominators of the infinite products on the right-hand side and applying the elementary identity $1-\zeta^n=-\zeta^n(1-\zeta^{7-n})$, we deduce that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k \frac{7}{1+q^k+\cdots+q^{6k}}\\
&=\frac{(q;q)_\infty^3}{(q^7;q^7)_\infty}\{(-\zeta^6-\zeta)(\zeta^2q;q)_\infty(\zeta^3q;q)_\infty(\zeta^4q;q)_\infty(\zeta^5q;q)_\infty\\
&\quad+(-\zeta^5-\zeta^2)(\zeta q;q)_\infty(\zeta^3q;q)_\infty(\zeta^4q;q)_\infty(\zeta^6q;q)_\infty\\
&\quad+(-\zeta^4-\zeta^3)(\zeta q;q)_\infty(\zeta^2q;q)_\infty(\zeta^5q;q)_\infty(\zeta^6q;q)_\infty\}. \qquad (6.10)
\end{aligned}$$

Applying Lemma 2.2 with $\alpha=-a$, $\beta=-q/a$, and $N=7$, we deduce that

$$(aq;q)_\infty(q/a;q)_\infty(q;q)_\infty \equiv A+\frac{a^2-a^6}{1-a}\,qB+\frac{a^5-a^3}{1-a}\,q^3C \pmod{S_3}. \qquad (6.11)$$

Applying (6.11) six times with $a=\zeta^2,\zeta^3,\zeta,\zeta^3,\zeta,\zeta^2$ in (6.10) and simplifying, we complete the proof of the first identity in Theorem 6.2. □

Theorem 6.2 can also be found in [6, Eqs. (4.13)–(4.15)]. Our method of proof is different from that of Ekin.

7. The 11-dissection for $F(q)$

In this section, $\zeta = e^{2\pi i/11}$. If we set $a = 1$ in Theorem 7.1 below, we recover [2, Theorem 3]. An elementary proof of [2, Theorem 3] has been given by Hirschhorn [13].

Theorem 7.1. *With A_m defined by (1.10) and S_5 defined by (5.1), we have*

$$\begin{aligned} F_a(q) \equiv \frac{1}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2} &(ABCD + \{A_1 - 1\}qA^2BE \\ &+\{A_2\}q^2AC^2D + \{A_3 + 1\}q^3ABD^2 \\ &+\{A_2 + A_4 + 1\}q^4ABCE - \{A_2 + A_4\}q^5B^2CE \\ &+\{A_1 + A_4\}q^7ABDE - \{A_2 + A_5 + 1\}q^{19}CDE^2 \\ &-\{A_4 + 1\}q^9ACDE - \{A_3\}q^{10}BCDE) \pmod{S_5}, \end{aligned}$$

where $A = f(-q^{55}, -q^{66})$, $B = f(-q^{77}, -q^{44})$, $C = f(-q^{88}, -q^{33})$, $D = f(-q^{99}, -q^{22})$, and $E = f(-q^{110}, -q^{11})$.

Before we begin our proofs of Theorem 7.1, we first state some results that will be useful in our proofs. Specializing (2.5) with $\alpha = a^m$ and $\beta = a^n$, we find that

$$\begin{aligned} &(a^m q, a^{11-m}q, a^n q, a^{11-n}q, a^{m+n}q, a^{11-m-n}q, a^{m-n}q, a^{11-m+n}q, q, q; q)_\infty \\ &\quad\equiv \frac{1}{(1-a^m)(1-a^n)(1-a^{m+n})(1-a^{m-n})} \\ &\qquad\times\{G(a^{3m})H(a^n) - a^{m-n}G(a^{3n})H(a^m)\} \pmod{S_5}, \end{aligned} \tag{7.1}$$

where

$$G(x) := f(-x, -x^{10}q^3) \quad \text{and} \quad H(x) := f(-x^3q, -x^8q^2) - xf(-x^3q^2, -x^8q).$$

Using Lemma 2.2 with $N = 11$ and $(\alpha, \beta) = (-x, -x^{10}q^3)$, $(-x^3q, -x^8q^2)$ and $(-x^3q^2, -x^8q)$, taking congruences modulo S_5, and using the fact that $f(-1, b) = 0$ for every complex number b with $|b| < 1$ [3, p. 34, Entry 18(iii)], we find that, for every positive integer n,

$$\begin{aligned} G(a^n) \equiv (1-a^n)P(15) &+ (a^{2n} - a^{10n})q^3P(12) + (a^{9n} - a^{3n})q^9P(9) \\ &+(a^{4n} - a^{8n})q^{18}P(6) + (a^{7n} - a^{5n})q^{30}P(3) \pmod{S_5} \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} H(a^n) \equiv (1-a^n)[P(16) - q^{22}P(5)] &+ q(a^{9n} - a^{3n})[P(14) - q^{11}P(8)] \\ &+q^2(a^{4n} - a^{8n})[P(13) - q^{33}P(2)] + q^{15}(a^{10n} - a^{2n})[P(7) + q^{11}P(4)] \end{aligned}$$

$$+q^7(a^{5n}-a^{7n})[P(10)+q^{33}P(1)] \pmod{S_5}, \tag{7.3}$$

where

$$P(k) := f(-q^{11k}, -q^{363-11k}). \tag{7.4}$$

Furthermore, we obtain the following 10 identities (7.5)–(7.14) from Winquist's identity (2.5) by replacing (α, β, q) by $(q^{55}, q^{22}, q^{121})$, $(q^{55}, q^{11}, q^{121})$, $(q^{55}, q^{33}, q^{121})$, $(q^{44}, q^{22}, q^{121})$, $(q^{44}, q^{11}, q^{121})$, $(q^{44}, q^{33}, q^{121})$, $(q^{55}, q^{44}, q^{121})$, $(q^{22}, q^{11}, q^{121})$, $(q^{33}, q^{22}, q^{121})$, and $(q^{33}, q^{11}, q^{121})$:

$$P(15)[P(16)-q^{22}P(5)]-q^{33}P(6)[P(7)+q^{11}P(4)]=\frac{ABCD}{(q^{121};q^{121})_\infty^2}, \tag{7.5}$$

$$P(15)[P(14)-q^{11}P(8)]-q^{44}P(3)[P(7)+q^{11}P(4)]=\frac{A^2BE}{(q^{121};q^{121})_\infty^2}, \tag{7.6}$$

$$P(15)[P(13)-q^{33}P(2)]-q^{22}P(9)[P(7)+q^{11}P(4)]=\frac{AC^2D}{(q^{121};q^{121})_\infty^2}, \tag{7.7}$$

$$P(12)[P(16)-q^{22}P(5)]-q^{22}P(6)[P(10)+q^{33}P(1)]=\frac{ABD^2}{(q^{121};q^{121})_\infty^2}, \tag{7.8}$$

$$P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)]=\frac{ABCE}{(q^{121};q^{121})_\infty^2}, \tag{7.9}$$

$$P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)]=\frac{B^2CE}{(q^{121};q^{121})_\infty^2}, \tag{7.10}$$

$$P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)]=\frac{ABDE}{(q^{121};q^{121})_\infty^2}, \tag{7.11}$$

$$P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)]=\frac{CDE^2}{(q^{121};q^{121})_\infty^2}, \tag{7.12}$$

$$P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)]=\frac{ACDE}{(q^{121};q^{121})_\infty^2}, \tag{7.13}$$

$$P(9)[P(14)-q^{11}P(8)]-q^{22}P(3)[P(13)-q^{33}P(2)]=\frac{BCDE}{(q^{121};q^{121})_\infty^2}. \tag{7.14}$$

We now begin our first proof of the 11-dissection of the generating function $F_a(q)$ for cranks.

First proof of Theorem 7.1. Beginning, as usual, with the generating function for $F_a(q)$ and rationalizing, we find that

$$F_a(q) \equiv \frac{(q;q)_\infty}{(aq;q)_\infty(a^{10}q;q)_\infty}$$

$$\equiv \frac{(a^2q,a^3q,a^4q,a^5q,a^6q,a^7q,a^8q,a^9q,q,q;q)_\infty}{(q^{11};q^{11})_\infty}$$

$$\equiv \frac{1}{(q^{11};q^{11})_\infty}$$

$$\times\left(\frac{1}{(1-a^2)(1-a^5)(1-a^7)(1-a^8)}\{G(a^6)H(a^5)-a^8G(a^4)H(a^2)\}\right)$$

$$\equiv \frac{1}{(q^{11};q^{11})_\infty}(P(15)[P(16)-q^{22}P(5)]-q^{33}P(6)[P(7)+q^{11}P(4)]$$

$$+\{A_1-1\}q(P(15)[P(14)-q^{11}P(8)]-q^{44}P(3)[P(7)+q^{11}P(4)])$$

$$+\{A_2\}q^2(P(15)[P(13)-q^{33}P(2)]-q^{22}P(9)[P(7)+q^{11}P(4)])$$

$$+\{A_3+1\}q^3(P(12)[P(16)-q^{22}P(5)]-q^{22}P(6)[P(10)+q^{33}P(1)])$$

$$+\{A_2+A_4+1\}q^4(P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)])$$

$$-\{A_2+A_4\}q^5(P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)])$$

$$+\{A_1+A_4\}q^7(P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)])$$

$$-\{A_2+A_5+1\}q^{19}(P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)])$$

$$-\{A_4+1\}q^9(P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)])$$

$$-\{A_3\}q^{10}(P(9)[P(14)-q^{11}P(8)]$$

$$-q^{22}P(3)[P(13)-q^{33}P(2)])) \pmod{S_5},$$

where, in the last congruence, we applied (7.1) with $m=5$ and $n=2$, (7.2) with $n=4,6$, and (7.3) with $n=5,2$.

Applying (7.5)–(7.14) to each of the dissection factors, respectively, above, we complete the first proof of Theorem 7.1. □

Second proof of Theorem 7.1. As in our second proofs of the 2, 3, 5 and 7-dissections, we apply Theorem 2.1 and divide the series into residue classes modulo 11. If we set

$$T_i := \sum_{\substack{k=1,m=0\\ m\equiv i \pmod{11}}}^{\infty} a_k q^{km},$$

we deduce that

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty} = 1+\sum_{k=1,m=0}^{\infty} a_k q^{km}(A_m-A_{m+1})$$

$$\equiv 1+(2-A_1)\{T_0-T_{10}\}+(A_1-A_2)\{T_1-T_9\}$$

$$
\begin{aligned}
&+(A_2 - A_3)\{T_2 - T_8\} + (A_3 - A_4)\{T_3 - T_7\}\\
&+(A_4 - A_5)\{T_4 - T_6\}\\
&\equiv \sum_{k=-\infty}^{\infty} a_k \frac{2 + 2q^k + 2q^{2k} + 2q^{3k} + 3q^{4k}}{1 + q^k + \cdots + q^{10k}}\\
&+A_1 \sum_{k=-\infty}^{\infty} a_k \frac{q^{4k} - 1}{1 + q^k + \cdots + q^{10k}}\\
&+ A_2 \sum_{k=-\infty}^{\infty} a_k \frac{q^{4k} - q^k}{1 + q^k + \cdots + q^{10k}}\\
&+A_3 \sum_{k=-\infty}^{\infty} a_k \frac{q^{4k} - q^{2k}}{1 + q^k + \cdots + q^{10k}}\\
&+A_4 \sum_{k=-\infty}^{\infty} a_k \frac{q^{4k} - q^{3k}}{1 + q^k + \cdots + q^{10k}} \pmod{S_5}. \qquad (7.15)
\end{aligned}
$$

The second proof of Theorem 7.1 now follows from the following identities. Indeed, if we substitute the identities of Theorem 7.2 into (7.15) and collect terms, we complete the second proof of Theorem 7.1. □

Theorem 7.2. *We have*

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_k \frac{11}{1 + q^k + \cdots + q^{10k}}\\
&= \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}\\
&\times\{ABCD - 10qA^2BE + 2q^2AC^2D + 3q^3ABD^2 + 5q^4ABCE\\
&-4q^5B^2CE - 7q^7ABDE - 5q^{19}CDE^2 - 3q^9ACDE\\
&-2q^{10}BCDE\}, \qquad (7.16)\\
&\sum_{k=-\infty}^{\infty} a_k \frac{11q^k}{1 + q^k + \cdots + q^{10k}}\\
&= \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}\\
&\times\{ABCD + qA^2BE - 9q^2AC^2D + 3q^3ABD^2 - 6q^4ABCE\\
&+7q^5B^2CE + 4q^7ABDE + 6q^{19}CDE^2 - 3q^9ACDE
\end{aligned}
$$

$$-2q^{10}BCDE\},\tag{7.17}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{11q^{2k}}{1+q^k+\cdots+q^{10k}}$$

$$= \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}$$

$$\times\{ABCD + qA^2BE + 2q^2AC^2D - 8q^3ABD^2 + 5q^4ABCE$$

$$-4q^5B^2CE + 4q^7ABDE - 5q^{19}CDE^2$$

$$-3q^9ACDE + 9q^{10}BCDE\},\tag{7.18}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{11q^{3k}}{1+q^k+\cdots+q^{10k}}$$

$$= \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}$$

$$\times\{ABCD + qA^2BE + 2q^2AC^2D + 3q^3ABD^2 - 6q^4ABCE$$

$$+7q^5B^2CE - 7q^7ABDE - 5q^{19}CDE^2$$

$$+8q^9ACDE - 2q^{10}BCDE\},\tag{7.19}$$

$$\sum_{k=-\infty}^{\infty} a_k \frac{11q^{4k}}{1+q^k+\cdots+q^{10k}}$$

$$= \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}$$

$$\times\{ABCD + qA^2BE + 2q^2AC^2D + 3q^3ABD^2 + 5q^4ABCE$$

$$-4q^5B^2CE + 4q^7ABDE + 6q^{19}CDE^2$$

$$-3q^9ACDE - 2q^{10}BCDE\},\tag{7.20}$$

where $A = f(-q^{55}, -q^{66})$, $B = f(-q^{77}, -q^{44})$, $C = f(-q^{88}, -q^{33})$, $D = f(-q^{99}, -q^{22})$, *and* $E = f(-q^{110}, -q^{11})$.

We present only the proof of (7.16), since the proofs of the remaining four identities are similar.

Proof. We calculate the partial fraction decomposition

$$\frac{11}{1+q^k+\cdots+q^{10k}}$$

$$= \frac{1-\zeta^{10}}{1-\zeta q^k} + \frac{1-\zeta^9}{1-\zeta^2 q^k} + \frac{1-\zeta^8}{1-\zeta^3 q^k} + \frac{1-\zeta^7}{1-\zeta^4 q^k} + \frac{1-\zeta^6}{1-\zeta^5 q^k}$$
$$+ \frac{1-\zeta^5}{1-\zeta^6 q^k} + \frac{1-\zeta^4}{1-\zeta^7 q^k} + \frac{1-\zeta^3}{1-\zeta^8 q^k} + \frac{1-\zeta^2}{1-\zeta^9 q^k} + \frac{1-\zeta}{1-\zeta^{10} q^k}. \tag{7.21}$$

From the identity (7.21) and Theorem 2.1, we find that

$$I(q) =: \sum_{k=-\infty}^{\infty} a_k \frac{11}{1+q^k+\cdots+q^{10k}}$$
$$= \frac{1-\zeta^{10}}{1-\zeta} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta}{1-\zeta q^k} + \frac{1-\zeta^9}{1-\zeta^2} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^2}{1-\zeta^2 q^k}$$
$$+ \frac{1-\zeta^8}{1-\zeta^3} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^3}{1-\zeta^3 q^k} + \frac{1-\zeta^7}{1-\zeta^4} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^4}{1-\zeta^4 q^k}$$
$$+ \frac{1-\zeta^6}{1-\zeta^5} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^5}{1-\zeta^5 q^k} + \frac{1-\zeta^5}{1-\zeta^6} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^6}{1-\zeta^6 q^k}$$
$$+ \frac{1-\zeta^4}{1-\zeta^7} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^7}{1-\zeta^7 q^k} + \frac{1-\zeta^3}{1-\zeta^8} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^8}{1-\zeta^8 q^k}$$
$$+ \frac{1-\zeta^2}{1-\zeta^9} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^9}{1-\zeta^9 q^k} + \frac{1-\zeta}{1-\zeta^{10}} \sum_{k=-\infty}^{\infty} a_k \frac{1-\zeta^{10}}{1-\zeta^{10} q^k}$$
$$= \left\{ \frac{1-\zeta^{10}}{1-\zeta} + \frac{1-\zeta}{1-\zeta^{10}} \right\} \frac{(q;q)_\infty^2}{(\zeta q;q)_\infty (\zeta^{10} q;q)_\infty}$$
$$+ \left\{ \frac{1-\zeta^9}{1-\zeta^2} + \frac{1-\zeta^2}{1-\zeta^9} \right\} \frac{(q;q)_\infty^2}{(\zeta^2 q;q)_\infty (\zeta^9 q;q)_\infty}$$
$$+ \left\{ \frac{1-\zeta^8}{1-\zeta^3} + \frac{1-\zeta^3}{1-\zeta^8} \right\} \frac{(q;q)_\infty^2}{(\zeta^3 q;q)_\infty (\zeta^8 q;q)_\infty}$$
$$+ \left\{ \frac{1-\zeta^7}{1-\zeta^4} + \frac{1-\zeta^4}{1-\zeta^7} \right\} \frac{(q;q)_\infty^2}{(\zeta^4 q;q)_\infty (\zeta^7 q;q)_\infty}$$
$$+ \left\{ \frac{1-\zeta^6}{1-\zeta^5} + \frac{1-\zeta^5}{1-\zeta^6} \right\} \frac{(q;q)_\infty^2}{(\zeta^5 q;q)_\infty (\zeta^6 q;q)_\infty}. \tag{7.22}$$

Applying the elementary identity $1-\zeta^n=-\zeta^n(1-\zeta^{11-n})$, and rationalizing the denominator, we find that

$$\begin{aligned}I(q)=\frac{(q;q)_\infty^3}{(q^{11};q^{11})_\infty}&(\{-\zeta-\zeta^{10}\}(\zeta^2q,\zeta^3q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^8q,\zeta^9q;q)_\infty\\&+\{-\zeta^2-\zeta^9\}(\zeta q,\zeta^3q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^8q,\zeta^{10}q;q)_\infty\\&+\{-\zeta^3-\zeta^8\}(\zeta q,\zeta^2q,\zeta^4q,\zeta^5q,\zeta^6q,\zeta^7q,\zeta^9q,\zeta^{10}q;q)_\infty\\&+\{-\zeta^4-\zeta^7\}(\zeta q,\zeta^2q,\zeta^3q,\zeta^5q,\zeta^6q,\zeta^8q,\zeta^9q,\zeta^{10}q;q)_\infty\\&+\{-\zeta^5-\zeta^6\}(\zeta q,\zeta^2q,\zeta^3q,\zeta^4q,\zeta^7q,\zeta^8q,\zeta^9q,\zeta^{10}q;q)_\infty).\end{aligned}\tag{7.23}$$

Next, applying (7.1) with $(a,m,n)=(\zeta,5,2)$, $(\zeta,4,1)$, $(\zeta,5,4)$, $(\zeta,3,2)$, and $(\zeta,3,1)$, respectively, on each summand of (7.23) and simplifying, we find that

$$\begin{aligned}I(q)=\frac{(q;q)_\infty}{(q^{11};q^{11})_\infty}&(P(15)[P(16)-q^{22}P(5)]-q^{33}P(6)[P(7)+q^{11}P(4)]\\&-10q\{P(15)[P(14)-q^{11}P(8)]-q^{44}P(3)[P(7)+q^{11}P(4)]\}\\&+2q^2\{P(15)[P(13)-q^{33}P(2)]-q^{22}P(9)[P(7)+q^{11}P(4)]\}\\&+3q^3\{P(12)[P(16)-q^{22}P(5)]-q^{22}P(6)[P(10)+q^{33}P(1)]\}\\&+5q^4\{P(12)[P(14)-q^{11}P(8)]-q^{33}P(3)[P(10)+q^{33}P(1)]\}\\&-4q^5\{P(12)[P(13)-q^{33}P(2)]-q^{11}P(9)[P(10)+q^{33}P(1)]\}\\&-7q^7\{P(15)[P(10)+q^{33}P(1)]-q^{11}P(12)[P(7)+q^{11}P(4)]\}\\&-5q^{19}\{P(6)[P(14)-q^{11}P(8)]-q^{11}P(3)[P(16)-q^{22}P(5)]\}\\&-3q^9\{P(9)[P(16)-q^{22}P(5)]-q^{11}P(6)[P(13)-q^{33}P(2)]\}\\&-2q^{10}\{P(9)[P(14)-q^{11}P(8)]-q^{22}P(3)[P(13)-q^{33}P(2)]\}).\end{aligned}$$

Finally, applying (7.5)–(7.14) to each of the dissection factors, respectively, we obtain the right-hand side of (7.16), which completes the proof of Theorem 7.2. □

If we let a be a primitive 11th root of unity in Theorem 7.1, then we recover the identity discovered by Hirschhorn [12]. Hirschhorn's identity is a simplification of Garvan's identity given in [10, Theorem 6.7]. The proof of Hirschhorn's identity was first given by Ekin [5, pp. 286–287]. The idea illustrated in our first proof here is similar to that of Ekin.

Theorem 7.2 can also be proved using identities found in Ekin's paper [6, Eqs. (5.13)–(5.17), p. 2153]. Our approach of Theorem 7.2 is different from that of Ekin.

8. Conclusion

In the beginning of this article, we mention that by substituting a by the corresponding primitive root of unity, we obtain Garvan's identities proved in [10,11]. Garvan highlighted

to us that the identities in [10,11] imply the congruences established in this paper. We briefly explain his observation here.

Suppose, for some function $G_a(q)$, we want to show that

$$F_a(q) \equiv G_a(q) \pmod{S_{(p-1)/2}}.$$

Let $H_a(q) = F_a(q) - G_a(q)$. Then

$$H_a(q) = \sum_{n=0}^{\infty} h(a,n)q^n,$$

where $h(a,n) \in \mathbf{Z}[a,1/a]$. Let $h(a,n) = a^{-t(n)}\tilde{h}(a,n)$, where now, $\tilde{h}(a,n)$ is a polynomial in $\mathbf{Z}[a]$ and $t(n)$ is the largest integer k for which $1/a^k$ appears in $h(a,n)$. Garvan's identities show that $\tilde{h}(\zeta,n) = 0$ for all roots of the cyclotomic polynomial $\Phi_p(a)$. Since $\tilde{h}(a,n) \in \mathbf{Z}[a]$, this implies that $\Phi_p(a)$ divides $\tilde{h}(a,n)$. Therefore,

$$h(a,n) = a^{-t(n)}\Phi_p(a)Q(a,n) = a^{-t(n)+(p-1)/2}S_{(p-1)/2}(a)Q(a,n),$$

where $Q(a,n) \in \mathbf{Z}[a]$. This implies that

$$H_a(q) \equiv 0 \pmod{S_{(p-1)/2}}.$$

Garvan's observation allows us to deduce from [11, Eq. (2.16), 10, Theorem 8.16] respectively, the 5-dissection of $F_a(q) \pmod{a^{-4}\Phi_{10}(a)}$ and the congruence

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{q^{m^2}}{(aq;q)_m(q/a;q)_m} \\
&\quad\equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})}f^2(-q^{25}) + \left(a+\frac{1}{a}-2\right)\phi(q^5) + q\frac{f^2(-q^{25})}{f(-q^5,-q^{20})} \\
&\qquad + \left(a+\frac{1}{a}\right)q^2\frac{f^2(-q^{25})}{f(-q^{10},-q^{15})} - \left(a+\frac{1}{a}\right)q^3\frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})}f^2(-q^{25}) \\
&\qquad - \left(2a+\frac{2}{a}+1\right)\frac{\psi(q^5)}{q^2} \pmod{S_2},
\end{aligned}$$

where

$$\phi(q) = \frac{q}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{15n(n+1)/2}}{1-q^{5n+1}}$$

and

$$\psi(q) = \frac{q^2}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{15n(n+1)/2}}{1-q^{5n+2}}.$$

A direct proof of the congruence above in the spirit of our second method illustrated in this paper has not been found. Note that if we substitute $a = 1$ in the congruence above and use

Euler's identity [10, Eq. (7.1)]

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m^2} = \frac{1}{(q;q)_\infty},$$

we recover the Atkin–Swinnerton-Dyer congruences [2, Theorem 1]. This also provides an explanation to the "curious fact" raised by Garvan [10, second paragraph, p. 52].

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