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Siegel modular forms of degree two attached to Hilbert modular forms

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A R T I C L E   I N F O

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A B S T R A C T

Let $E/\mathbb{Q}$ be a real quadratic field and π_0 a cuspidal, irreducible, automorphic representation of $\mathrm{GL}(2,\mathbb{A}_E)$ with trivial central character and infinity type $(2,2n+2)$ for some non-negative integer n. We show that there exists a non-zero Siegel paramodular newform $F:\mathfrak{H}_2\to\mathbb{C}$ with weight, level, Hecke eigenvalues, epsilon factor and L-function determined explicitly by π_0. We tabulate these invariants in terms of those of π_0 for every prime p of $\mathbb{Q}$.

1. Introduction

In his 1980 paper [Y], Yoshida studied certain explicit theta liftings of Hilbert modular forms of weight $(2,2n+2)$ for real quadratic extensions of $\mathbb{Q}$ to Siegel modular forms of degree 2 and weight $n+2$ for the Siegel congruence subgroup $\Gamma_0(N)$ and an appropriate Dirichlet character χ. Yoshida calculated the action of the Hecke operators $T(1,1,p,p)$ and $T(1,p,p,p^2)$, defined below, on these lifts for $p\nmid N$, though Yoshida did not determine when these lifts are non-zero.

In this paper, we study an analogous problem. Given a Hilbert modular form of weight $(2,2n+2)$ we prove the existence of a non-zero Siegel modular form of degree 2 and weight $n+2$ for the paramodular congruence subgroup. Our main theorem completely characterizes the resulting Siegel modular form, including the Hecke eigenvalues at *every* rational prime p.

Main Theorem. *Let E be a real quadratic extension of $\mathbb{Q}$ with real archimedean places ∞_1 and ∞_2. Let π_0 be a cuspidal irreducible automorphic representation of* $\mathrm{GL}(2,\mathbb{A}_E)$ *with trivial central character. Let $\mathfrak{N}_0$ be*

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the level of π_0. Assume that π_0 is not Galois-invariant and $\pi_{0,\infty_1} = D_2$ and $\pi_{0,\infty_2} = D_{2n+2}$ with $n \geqslant 0$ a non-negative integer and D_k the holomorphic discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ *of lowest weight k. Let $N = d_E^2 \mathrm{N}_{\mathbb{Q}}^E(\mathfrak{N}_0)$, where d_E is the discriminant of $E/\mathbb{Q}$. Then there exists a non-zero Siegel paramodular newform $F : \mathfrak{H}_2 \to \mathbb{C}$ of weight $k = n+2$ and paramodular level N such that*:

i) *For every prime p,*

$$T(1,1,p,p)F = p^{k-3}\lambda_p F \quad \text{and} \quad T\bigl(1,p,p,p^2\bigr)F = p^{2(k-3)}\mu_p F \tag{1}$$

where the Hecke eigenvalues λ_p and μ_p are determined by the Hecke eigenvalues of π_0 as follows. If p splits, let w_1 and w_2 be the places above p. If p is non-split, let w be the place above p.

a) *If* $\mathrm{val}_p(N) = 0$,

$$\lambda_p = \begin{cases} p(\lambda_{w_1} + \lambda_{w_2}) & \text{if } p \text{ is split,} \\ 0 & \text{if } p \text{ is not split,} \end{cases} \qquad \mu_p = \begin{cases} p^2 + p\lambda_{w_1}\lambda_{w_2} - 1 & \text{if } p \text{ is split,} \\ -(p^2 + p\lambda_w + 1) & \text{if } p \text{ is not split.} \end{cases}$$

b) *If* $\mathrm{val}_p(N) = 1$, *then p splits and* $\mathrm{val}_{w_1}(\mathfrak{N}_0) = 1$, $\mathrm{val}_{w_2}(\mathfrak{N}_0) = 0$, *and*

$$\lambda_p = p\lambda_{w_1} + (p+1)\lambda_{w_2}, \qquad \mu_p = p\lambda_{w_1}\lambda_{w_2}.$$

c) *If* $\mathrm{val}_p(N) \geqslant 2$, *then*:

p inert:

$$\lambda_p = 0, \qquad \mu_p = -p^2 - p\lambda_w;$$

p ramified:

$$\lambda_p = p\lambda_w, \qquad \mu_p = \begin{cases} 0 & \text{if } \mathrm{val}_w(\mathfrak{N}_0) = 0, \\ -p^2 & \text{if } \mathrm{val}_w(\mathfrak{N}_0) \geqslant 1; \end{cases}$$

p split and $\mathrm{val}_{w_1}(\mathfrak{N}_0) \leqslant \mathrm{val}_{w_2}(\mathfrak{N}_0)$:

$$\lambda_p = p(\lambda_{w_1} + \lambda_{w_2}), \qquad \mu_p = \begin{cases} 0 & \text{if } \mathrm{val}_{w_1}(\mathfrak{N}_0) = 0, \\ -p^2 & \text{if } \mathrm{val}_{w_1}(\mathfrak{N}_0) \geqslant 1. \end{cases}$$

For particular π_0, λ_p and μ_p are given in Proposition 4.2.

ii) *For every prime $p|N$, let U_p be the Atkin–Lehner operator, defined below. Then,*

$$F|_k U_p = \varepsilon_p F \tag{2}$$

with

$$\varepsilon_p = \begin{cases} \varepsilon(1/2, \pi_{0,w_1}, \psi_p, dx_{\psi_p})\varepsilon(1/2, \pi_{0,w_2}, \psi_p, dx_{\psi_p}) & \text{if } p \text{ is split,} \\ \varepsilon(1/2, \pi_{0,w}, \psi_w, dx_{\psi_w})\omega_{E_w/\mathbb{Q}_p}(-1) & \text{if } p \text{ is not split,} \end{cases}$$

where ψ_w is an additive character of E_w with conductor $\mathfrak{o}_w$. For particular π_0, ε_p is given in Proposition 4.2.

iii) *For every prime p, we have an equality of Euler factors*

$$L_p\left(s+k-\frac{3}{2}, F\right) = L_p(s, \pi_0), \tag{3}$$

where $k = n+2$ and $L_p(s, F)$ is defined below for every finite place p of $\mathbb{Q}$. Moreover, we have the functional equation

$$\Lambda(2k-2-s, F) = (-1)^k\left(\prod_{p|N}\varepsilon_p\right)N^{s-k+1}\Lambda(s, F) \tag{4}$$

where the completed L-function is defined as the product

$$\Lambda(s, F) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\prod_{p<\infty}L_p(s, F).$$

Our main theorem has potential applications to arithmetic geometry. For example, Consani and Scholten [CS] studied the four dimensional Galois representation ρ of $G_{\mathbb{Q}}$ on the étale cohomology $H^3(\tilde{X}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(\sqrt{5}))$ of the desingularization $\tilde{X}$ of a quintic three-fold X. Consani and Scholten showed that ρ is induced from a representation σ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{5}))$. By work of Yi [Yi], σ corresponds to an automorphic representation π_0, as in our main theorem, with $n = 1$ and $\mathfrak{N}_0 = \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5^2 = (30)$. As a consequence of our main theorem, there exists a non-zero Siegel modular form F of degree 2 and weight 3 for the paramodular group of level $N = 2^2 3^2 5^4$ such that $\Lambda(s, F) = \Lambda(s, \rho)$.

The paper is organized as follows. In Section 2, we introduce notation and definitions which will be used throughout the paper. Section 3 supplies the proof of the Main Theorem. This proof depends on certain local results which are proved in Sections 4 and 5. In Section 4, we explicitly tabulate the local L-packets, Hecke eigenvalues, and epsilon factors used in the proof of the Main Theorem. The technical heart of the paper is in Section 5, where we calculate the gamma factors of the Novodvorsky zeta integrals of a generic supercuspidal representation of $\mathrm{GSp}(4, F)$ for a nonarchimedean local field F.

2. Notation and definitions

Let

$$J = \begin{bmatrix} & \mathbf{1}_2 \\ -\mathbf{1}_2 & \end{bmatrix}.$$

We define the algebraic $\mathbb{Q}$-group $\mathrm{GSp}(4)$ as the set of all $g \in \mathrm{GL}(4)$ such that ${}^t gJg = xJ$ for some $x \in \mathrm{GL}(1)$; we call x the multiplier of g and denote it by $\lambda(g)$. The kernel of $\lambda : \mathrm{GSp}(4) \to \mathrm{GL}(1)$ is the symplectic group $\mathrm{Sp}(4)$. For N a positive integer we define

$$\Gamma^{\mathrm{para}}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}).$$

Let p be a prime of $\mathbb{Q}$. For $n \geqslant 0$ a non-negative integer define the local paramodular group $\mathrm{K}(p^n)$ as the group of all $k \in \mathrm{GSp}(4, \mathbb{Q}_p)$ such that

$$k \in \begin{bmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p \\ \mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix}$$

and $\det(k) \in \mathbb{Z}_p^\times$. Note that $\mathrm{K}(p^0) = \mathrm{GSp}(4, \mathbb{Z}_p)$. We have

$$\Gamma^{\mathrm{para}}(N) = \mathrm{GSp}(4, \mathbb{Q}) \cap \mathrm{GSp}(4, \mathbb{R})^+ \prod_{p<\infty} \mathrm{K}\big(p^{\mathrm{val}_p(N)}\big).$$

Here, $p^{\mathrm{val}_p(N)}$ is the exact power of p dividing N, and $\mathrm{GSp}(4, \mathbb{R})^+$ is the subgroup of $g \in \mathrm{GSp}(4, \mathbb{R})$ such that $\lambda(g) > 0$. Let $\mathfrak{H}_2$ be the Siegel upper half-space of degree two. Then $\mathrm{GSp}(4, \mathbb{R})^+$ acts on $\mathfrak{H}_2$ by

$$g\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Z \in \mathfrak{H}_2$$

and we define $j(g, Z) = \det(CZ + D)$. If k is a positive integer, $g \in \mathrm{GSp}(4, \mathbb{R})^+$, and $F : \mathfrak{H}_2 \to \mathbb{C}$ is a function we define

$$(F|_k g)(Z) = \lambda(g)^k j(g, Z)^{-k} F\big(g\langle Z\rangle\big), \quad Z \in \mathfrak{H}_2.$$

A *Siegel modular form* of degree 2 and weight k with respect to $\Gamma^{\mathrm{para}}(N)$ is a holomorphic function $F : \mathfrak{H}_2 \to \mathbb{C}$ such that $F|_k\gamma = F$ for $\gamma \in \Gamma^{\mathrm{para}}(N)$. Let $M_k(\Gamma^{\mathrm{para}}(N))$ and $S_k(\Gamma^{\mathrm{para}}(N))$ be the spaces of all Siegel modular forms or cuspforms of degree 2 and weight k with respect to $\Gamma^{\mathrm{para}}(N)$, respectively. For each prime p, we define two Hecke operators $T(1, 1, p, p)$ and $T(1, p, p, p^2)$ on $M_k(\Gamma^{\mathrm{para}}(N))$ as follows. Let

$$\Gamma^{\mathrm{para}}(N) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \Gamma^{\mathrm{para}}(N) = \bigsqcup \Gamma^{\mathrm{para}}(N) h_i$$

and

$$\Gamma^{\mathrm{para}}(N) \begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{bmatrix} \Gamma^{\mathrm{para}}(N) = \bigsqcup \Gamma^{\mathrm{para}}(N) h'_j$$

be disjoint decompositions. Note that if $p \nmid N$ then

$$\Gamma^{\mathrm{para}}(N)\,\mathrm{diag}\big(p, 1, p, p^2\big)\Gamma^{\mathrm{para}}(N) = \Gamma^{\mathrm{para}}(N)\,\mathrm{diag}\big(1, p, p^2, p\big)\Gamma^{\mathrm{para}}(N).$$

For $F \in M_k(N)$ set

$$T(1, 1, p, p)F = p^{k-3} \sum_i F|_k h_i, \qquad T\big(1, p, p, p^2\big)F = p^{2(k-3)} \sum_j F|_k h'_j.$$

If $N=1$, this definition is the same as in, for example, (1.3.3) of [A1]. For each prime p dividing N, choose a matrix $\gamma_p \in \mathrm{Sp}(4,\mathbb{Z})$ such that

$$\gamma_p \equiv \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix} \mod p^{\mathrm{val}_p(N)} \quad \text{and} \quad \gamma_p \equiv \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix} \mod Np^{-\mathrm{val}_p(N)},$$

and define

$$U_p = \gamma_p \begin{bmatrix} p^{\mathrm{val}_p(N)}&&&\\ &p^{\mathrm{val}_p(N)}&&\\ &&1&\\ &&&1 \end{bmatrix}.$$

It can be verified that U_p normalizes $\Gamma^{\mathrm{para}}(N)$ and that U_p^2 is contained in $p^{\mathrm{val}_p(N)}\Gamma^{\mathrm{para}}(N)$, so that $F \mapsto F|_k U_p$ defines an involution of $S_k(\Gamma^{\mathrm{para}}(N))$ and $M_k(\Gamma^{\mathrm{para}}(N))$. Let $S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}(N))$ be defined as in [RS2]. Let $F \in S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}(N))$ and assume that

$$T(1,1,p,p)F = \lambda_{F,p}F, \qquad T\big(1,p,p,p^2\big)F = \mu_{F,p}F, \qquad F|_k U_p = \varepsilon_{F,p}F.$$

Then we define $L_p(s,F)$ as follows:

i) If $\mathrm{val}_p(N)=0$, then

$$L_p(s,F)^{-1} = 1 - \lambda_{F,p}p^{-s} + \big(p\mu_{F,p} + p^{2k-3} + p^{2k-5}\big)p^{-2s} - p^{2k-3}\lambda_{F,p}p^{-3s} + p^{4k-6}p^{-4s}.$$

ii) If $\mathrm{val}_p(N)=1$, then

$$L_p(s,F)^{-1} = 1 - \big(\lambda_{F,p} + p^{k-3}\varepsilon_{F,p}\big)p^{-s} + \big(p\mu_{F,p} + p^{2k-3}\big)p^{-2s} + \varepsilon_{F,p}p^{3k-5}p^{-3s}.$$

iii) If $\mathrm{val}_p(N) \geqslant 2$, then

$$L_p(s,F)^{-1} = 1 - \lambda_{F,p}p^{-s} + \big(p\mu_{F,p} + p^{2k-3}\big)p^{-2s}.$$

Note that the case $\mathrm{val}_p(N)=0$ agrees with the classical Euler factor at p as given in [A1], Theorem 3.1.1. The work [A1] uses $T(p^2)$ instead of $T(1,p,p,p^2)$. However, one has the relation $pT(1,p,p,p^2) + p(p^2+1)T(p,p,p,p) = T(p)^2 - T(p^2) - p^2T(p,p,p,p)$ by 3.3.38 of [A2]. Compare also Theorem 2 of [Sh]. The definitions in the cases $\mathrm{val}_p(N) \geqslant 1$ are motivated by the results of [RS1].

Additional notation. For k a positive integer, we let D_k denote the holomorphic discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of lowest weight k. Suppose that L is a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p} = \varpi\mathfrak{o} \subset \mathfrak{o}$. We define the character $\nu : L^\times \to \mathbb{C}^\times$ by $\nu(x) = |x|$ where $|\cdot|$ is the absolute value such that $|\varpi| = |\mathfrak{o}/\mathfrak{p}|^{-1}$. If $\chi : L^\times \to \mathbb{C}^\times$ is a character, then $a(\chi)$ is the smallest non-negative integer n such that $\chi(1+\mathfrak{p}^n) = 1$, where we take $1+\mathfrak{p}^0 = \mathfrak{o}^\times$. Let (τ, V) be a generic, irreducible, admissible representation of $\mathrm{GL}(2,L)$ with trivial central character. For n a non-negative integer, let $\Gamma_0(\mathfrak{p}^n)$ be the subgroup of $\left[\begin{smallmatrix} a&b\\ c&d \end{smallmatrix}\right]$ in $\mathrm{GL}(2,\mathfrak{o})$ such that $c \equiv 0\ (\mathfrak{p}^n)$. The group $\Gamma_0(\mathfrak{p}^n)$ is normalized by the Atkin–Lehner element $\left[\begin{smallmatrix} &1\\ -\varpi^n& \end{smallmatrix}\right]$. We define $a(\tau)$ to be the smallest non-negative integer n such that $V^{\Gamma_0(\mathfrak{p}^n)} \neq 0$; we call $\mathfrak{p}^{a(\tau)}$ the level of τ. The space $V^{\Gamma_0(\mathfrak{p}^{a(\tau)})}$ is spanned by a non-zero vector v. We have $\tau\big(\left[\begin{smallmatrix} &1\\ -\varpi^n& \end{smallmatrix}\right]\big)v = \varepsilon_\tau v$ for

$\varepsilon_\tau = \varepsilon(1/2, \tau, \psi) \in \{\pm 1\}$, where $\psi : L \to \mathbb{C}$ is a character with conductor $\mathfrak{o}$. We call ε_τ the Atkin–Lehner eigenvalue of τ. We also have $T(\mathfrak{p})v = \lambda_\tau v$ for some $\lambda_\tau \in \mathbb{C}$. Here, $T(\mathfrak{p})v = \sum_i \tau(h_i)v$, where $\Gamma_0(\mathfrak{p}^{a(\tau)}) \left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right] \Gamma_0(\mathfrak{p}^{a(\tau)}) = \bigsqcup h_i \Gamma_0(\mathfrak{p}^{a(\tau)})$ is a disjoint decomposition. We call λ_τ the Hecke eigenvalue of τ. The group $\mathrm{GSp}(2n)$ is defined with respect to $\left[\begin{smallmatrix} & 1_n \\ -1_n & \end{smallmatrix}\right]$, and $1_{\mathrm{GSp}(2n)}$ and $\mathrm{St}_{\mathrm{GSp}(2n)}$ are the trivial and Steinberg representations of $\mathrm{GSp}(2n, L)$, respectively.

3. Proof of the Main Theorem

Proof of the Main Theorem. We begin by constructing a certain cuspidal, irreducible, admissible representation π of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character; the desired Siegel modular form will correspond to a particular vector in π. For every place v of $\mathbb{Q}$, define $\pi_{0,v} = \bigotimes_{w|v} \pi_{0,w}$. Let $\varphi(\pi_{0,v}) : W'_{F_v} \to \mathrm{GSp}(4, \mathbb{C})$ be the L-parameter associated to $\pi_{0,v}$ as in (5) and (6); if v is non-split in E, we take $\eta = 1$. By [B] the representation $\pi_{0,w}$ is tempered for all finite places w of E. Let $\Pi(\varphi(\pi_{0,v}))$ be the L-packet of tempered, irreducible, admissible representations of $\mathrm{GSp}(4, \mathbb{Q}_v)$ with trivial central character associated to $\varphi(\pi_{0,v})$ as in [R]. For finite $v = p$, the packet $\Pi(\varphi(\pi_{0,p}))$ coincides with the packet associated to $\varphi(\pi_{0,p})$ in [GT], and contains a unique generic representation π_p of $\mathrm{GSp}(4, \mathbb{Q}_p)$. It is known that $\Pi(\varphi(\pi_{0,\infty}))$ contains the lowest weight representation π_k with $k = n + 2$; for the precise definition of π_k, see [AS], p. 184. We set $\pi_\infty = \pi_k$. Define π to be the restricted tensor product

$$\pi = \bigotimes_v \pi_v.$$

By Theorem 8.6 of [R], π is a cuspidal, irreducible, admissible, automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ with trivial central character.

To define the appropriate vector in π, for each finite prime p of $\mathbb{Q}$, let Φ_p be the local paramodular newform in π_p, and let $\Phi_\infty \in \pi_\infty$ be the non-zero smooth vector as in Lemma 3.4.2 of [AS]. Note that Φ_p exists and is unique up to scalars by [RS1]; we may assume that for almost all p, Φ_p is the unramified vector used to define the restricted tensor product. Also, $\pi_\infty(u)\Phi_\infty = j(u, I)^{-k}\Phi_\infty$ for $u \in U(2)$, where

$$I = \begin{bmatrix} i & \\ & i \end{bmatrix} \in \mathfrak{H}_2$$

and $U(2)$ is the subgroup of $u \in \mathrm{Sp}(4, \mathbb{R})$ such that $u\langle I \rangle = I$. We set

$$\Phi = \bigotimes_v \Phi_v.$$

For each finite prime p of $\mathbb{Q}$, because Φ_p is a local paramodular newform in π_p, we have $T_{1,0}(p)\Phi_p = \lambda_p \Phi_p$ and $T_{0,1}(p)\Phi_p = \mu_p \Phi_p$ for some complex numbers λ_p and μ_p and $\pi_p(u_p)\Phi_p = \varepsilon_p \Phi_p$ for some $\varepsilon_p \in \{\pm 1\}$; here, $T_{0,1}(p)$ and $T_{1,0}(p)$ are the Hecke operators from Chapter 6 of [RS1] and u_p is the Atkin–Lehner element defined in (2.2) of [RS1]. In fact, λ_p, μ_p and ε_p are as in i) and ii) by Proposition 4.2.

Next, define $F : \mathfrak{H}_2 \to \mathbb{C}$ by $F(Z) = \lambda(h)^{-k} j(h, I)^k \Phi(h_\infty)$ where $h \in \mathrm{GSp}(4, \mathbb{R})^+$ is such that $h\langle I \rangle = Z$. Then F is holomorphic by Lemma 3.2.1 of [AS], and an argument shows that $F \in S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}(N))$. A computation shows that

$$T(1, 1, p, p)F = \lambda_p p^{k-3} F, \qquad T\big(1, p, p, p^2\big)F = \mu_p p^{2(k-3)} F.$$

A similar computation shows that $F|_k U_p = \varepsilon_p F$ because $\pi_p(u_p)\Phi_p = \varepsilon_p \Phi_p$. This proves i) and ii). To prove iii), we note that the equality (3) follows by comparing the Euler factors at each finite prime

p of $\mathbb{Q}$ and using (1), (2) and (9). To deduce (4), we recall the functional equation for the completed L-function of π_0 (e.g., Theorem 6.2 of [J]),

$$L(1-s,\pi_0)=\varepsilon(1-s,\pi_0)L(s,\pi_0).$$

For every rational prime p, we have a canonical additive character $\psi_p(x)=e^{-2\pi i\lambda(x)}$ where λ is the composition

$$\lambda:\mathbb{Q}_p\to\mathbb{Q}_p/\mathbb{Z}_p\hookrightarrow\mathbb{Q}/\mathbb{Z}\hookrightarrow\mathbb{R}/\mathbb{Z},$$

and $\psi_\infty(x)=e^{2\pi ix}$. Then the function $\psi(x)=\prod_v\psi_v(x_v)$ defines a character of $\mathbb{A}/\mathbb{Q}$. For the quadratic extension E, we have the character $\tilde\psi=\psi\circ\mathrm{Tr}_{E/\mathbb{Q}}$ of $\mathbb{A}_E/E$. For each finite prime, w of E, we define $n(\tilde\psi_w)$ to be the least non-negative integer n such that $\tilde\psi_w(\mathfrak{p}_w^{-n})=1$. The epsilon factor is computed via

$$\begin{aligned}
\varepsilon(s,\pi_0)&=(-1)^n\prod_{w<\infty}\varepsilon(s,\pi_{0,w},\tilde\psi_w)\quad\big(\text{[Ge], Theorem 6.16}\big)\\
&=(-1)^n\prod_{p|N}\prod_{w|p}\varepsilon(\varphi_{0,w},\tilde\psi_w,dx_{\tilde\psi_w})q_w^{-s(2n(\tilde\psi_w)+a(\varphi_{0,w}))}\quad\big(\text{[Rohr], 11 Prop.}\big)\\
&=(-1)^n\prod_{\substack{p|N\\ \text{split}}}\varepsilon(\varphi_p,\psi_p,dx_{\psi_p})p^{-sa(\varphi_p)}\prod_{\substack{p|N\\ \text{nonsplit}}}\varepsilon(\varphi_{0,w},\tilde\psi_w,dx_{\tilde\psi_w})q_w^{-s(2n(\tilde\psi_w)+a(\varphi_{0,w}))}\\
&=(-1)^n\prod_{\substack{p|N\\ \text{split}}}\varepsilon(\varphi_p,\psi_p,dx_{\psi_p})p^{-sa(\varphi_p)}\\
&\quad\times\prod_{\substack{p|N\\ \text{nonsplit}}}\varepsilon(\varphi_{0,w},\tilde\psi_w,dx_{\tilde\psi_w})q_w^{-2sn(\tilde\psi_w)}p^{-s(a(\varphi_p)-2d(E_w/\mathbb{Q}_p))}\quad\big(\text{by (7)}\big)\\
&=(-1)^n\prod_{p|N}\varepsilon(\varphi_p,\psi_p,dx_{\psi_p})p^{-sa(\varphi_p)}\quad\big(\text{[Rohr], 11 Prop., }q_w^{n(\tilde\psi_w)}=p^{d(E_w/\mathbb{Q}_p)}\big)\\
&=(-1)^n\prod_{p|N}\varepsilon_p p^{-a(\varphi_p)(s-1/2)}\quad\big(\text{by (8)}\big)\\
&=(-1)^nN^{1/2-s}\prod_{p|N}\varepsilon_p.
\end{aligned}$$

Finally, the archimedean Euler factors of π_0 are given by

$$L(s,\pi_{0,\infty_1})L(s,\pi_{0,\infty_2})=(2\pi)^{-2s-n-1}\Gamma(s+1/2)\Gamma\big(s+(2n+1)/2\big).$$

Substituting into the functional equation for π_0 yields (4). □

4. Local results

Some definitions. Throughout this section let F be a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ with $\mathfrak{p}=\varpi\mathfrak{o}$, and let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Also, let E be a quadratic extension of F with $\mathrm{Gal}(E/F)=\{1,\sigma\}$ and associated quadratic character $\omega_{E/F}$. The residue class degree of E/F is denoted by $f = f(E/F)$ and the valuation of the discriminant by $d = d(E/F)$. Let

$$J' = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}, \qquad K = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

We note that in the work [RS1] the group GSp(4) is defined with respect to J', while throughout this work we define GSp(4) with respect to J, as in Section 2. However, it is easy to see that conjugation by K defines an isomorphism between the two realizations, in either direction.

Two families of *L*-parameters for GSp(4). We now consider two families of L-parameters for GSp(4) over F. The first family, which we will call the split case, is parameterized by pairs (π_1, π_2) where π_1 and π_2 are irreducible, admissible representations of $\mathrm{GL}(2, F)$ having the same central character, while the second family, which we refer to as the non-split case, is parameterized by triples (E, π_0, η) where E is a quadratic extension of F, π_0 is an irreducible, admissible representation of $\mathrm{GL}(2, E)$ with Galois invariant central character ω_{π_0}, and η is a character of $F^\times$ such that $\omega_{\pi_0} = \eta \circ \mathrm{N}^E_F$.

To define the parameter $\varphi(\pi_1, \pi_2) : W'_F \to \mathrm{GSp}(4, \mathbb{C})$ associated to a pair (π_1, π_2), let $\varphi_1 : W'_F \to \mathrm{GL}(2, \mathbb{C})$ and $\varphi_2 : W'_F \to \mathrm{GL}(2, \mathbb{C})$ be the L-parameters of π_1 and π_2, respectively. We define

$$\varphi(\pi_1, \pi_2)(x) = \begin{bmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{bmatrix}$$
$$\text{for } \varphi_1(x) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \ \varphi_2(x) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \text{ and } x \in W'_F. \tag{5}$$

Thus, $\varphi(\pi_1, \pi_2)$ is the symplectic direct sum of φ_1 and φ_2.

To define the L-parameter $\varphi(E, \pi_0, \eta) : W'_F \to \mathrm{GSp}(4, \mathbb{C})$ associated to a triple (E, π_0, η), let $\varphi_0 : W'_E \to \mathrm{GL}(2, \mathbb{C})$ be the L-parameter of π_0, and let V_0 be the space of φ_0. Let g_0 be a representative for the nontrivial coset of $W'_E \backslash W'_F$. We consider the representation of W'_F induced from φ_0 which we can realize as $V_0 \oplus V_0$ via the isomorphism

$$\mathrm{Ind}_{W'_E}^{W'_F} \varphi_0 \xrightarrow{\sim} V := V_0 \oplus V_0$$

that sends f to $f(1) \oplus f(g_0)$.

Notice that η can also be considered as a character of W'_F with the property that $\eta|_{W'_E} = \det(\varphi_0)$.

We define a non-degenerate symplectic bilinear form on V by

$$\langle v_1 \oplus v_2, v'_1 \oplus v'_2 \rangle = \eta(g_0) \langle v_1, v'_1 \rangle + \langle v_2, v'_2 \rangle.$$

A computation shows that $\langle x \cdot v, x \cdot v' \rangle = \eta(x) \langle v, v' \rangle$ for $v, v' \in V$ and $x \in W'_F$. The L-parameter $\varphi(E, \pi_0, \eta)$ is defined to be the representation V with this symplectic structure. We can choose a symplectic basis for V so that for $y \in W'_E$,

$$\varphi(E, \pi_0, \eta)(y) = \begin{bmatrix} a & & \eta(g_0)^{-1} b & \\ & a' & & b' \\ \eta(g_0) c & & d & \\ & c' & & d' \end{bmatrix}$$

$$\text{for } \varphi_0(y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \varphi_0\big(g_0 y g_0^{-1}\big) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \quad \text{and}$$

$$\varphi(E, \pi_0, \eta)(g_0) = \begin{bmatrix} & 1 & & \\ a_0 & & \eta(g_0)^{-1} b_0 & \\ & & & \eta(g_0) \\ c_0 & & \eta(g_0)^{-1} d_0 & \end{bmatrix} \quad \text{for } \varphi_0\big(g_0^2\big) = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}. \tag{6}$$

Associated L-packets. The local Langlands conjecture is proven for GSp(4) over F in [GT], and in the following proposition we tabulate the L-packets $\Pi(\varphi)$ associated to the L-parameters φ constructed above for all choices of π_1, π_2 and π_0. To list these L-packets we proceed as follows. First, the work [RS1] gives an explicit map, determined by the desiderata of the local Langlands conjecture for GSp(4, F), from the set of non-supercuspidal, irreducible, admissible representations of GSp(4, F) to the set of L-parameters; moreover, this map is the same as that in [GT]. Using this map, it is straightforward to determine all the non-supercuspidal elements in the L-packets $\Pi(\varphi)$ for φ as above. We remark that in some cases, to use [RS1], it is necessary to consider an L-parameter equivalent to φ. In [RS1] the non-supercuspidal representations are divided into eleven groups based on inducing data. Note that in particular each group contains a generic representation, which is designated by the letter "a" if the group contains more than one type of irreducible representation. We also use the notation of [ST] and [RS1] for the representations in Table 1.

Next, the L-packets $\Pi(\varphi)$ containing supercuspidal elements for φ as above can be described as follows. Assume that π_1 and π_2 are discrete series representations such that $\pi_1 \ncong \pi_2$. Then the L-packet $\Pi(\varphi(\pi_1, \pi_2))$ consists of two elements. These two representations of GSp(4, F) are theta lifts of the form $\theta_X(\sigma)$ and $\theta_{X'}(\sigma')$ where X and X' are the four dimensional hyperbolic and anisotropic quadratic spaces, respectively, and σ and σ' are irreducible, admissible representations of GO(X, F) and GO(X', F), respectively, arising from the pair π_1 and π_2. It is known that $\theta_X(\sigma)$ and $\theta_{X'}(\sigma')$ have central character $\omega_{\pi_1} = \omega_{\pi_2}$, $\theta_X(\sigma)$ is generic and tempered, and $\theta_{X'}(\sigma)$ is non-generic. When both π_1 and π_2 are not supercuspidal, i.e., $\pi_1 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ and $\pi_2 = \beta\,\mathrm{St}_{\mathrm{GL}(2)}$ for some characters α and β of $F^\times$, then $\theta_X(\sigma) = \delta([\alpha^{-1}\beta, \nu\alpha^{-1}\beta], \nu^{-1/2}\alpha)$; this representation belongs to group Va. On the other hand, $\theta_{X'}(\sigma')$ is supercuspidal, and to supplement the partition from [RS1], we will say that it belongs to group Vb*. If exactly one of π_1 and π_2 is supercuspidal, say $\pi_1 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ with α a character of $F^\times$, then $\theta_X(\sigma) = \delta(\nu^{1/2}\alpha^{-1}\pi_2, \nu^{-1/2}\alpha)$; this representation is of type XIa. Again, $\theta_{X'}(\sigma')$ is supercuspidal, and we say that it is of type XIb*. If both π_1 and π_2 are supercuspidal, then both $\theta_X(\sigma)$ and $\theta_{X'}(\sigma')$ are supercuspidal, and we say that they are of type XIIa* and XIIb*, respectively. Finally, assume that (E, π_0, η) is as above with π_0 supercuspidal and not Galois invariant, i.e., $\pi_0^\sigma \ncong \pi_0$. Then the L-packet $\Pi(\varphi(E, \pi_0, \eta))$ consists of a single representation. This representation is a theta lift of the form $\theta_{X_0}(\sigma_0)$ where X_0 is the four dimensional quadratic space having anisotropic component (E, N_F^E) and σ_0 is the supercuspidal, irreducible, admissible representation of GO(X_0, F) associated to π_0 and η. The representation $\theta_{X_0}(\sigma_0)$ is generic, supercuspidal and has central character η; we say that this representation is in group XIII*. Note that the superscript * indicates that representations of a given type are supercuspidal.

Table 1 of the following proposition summarizes the data obtained by this method.

Proposition 4.1. *Let $\varphi = \varphi(\pi_1, \pi_2)$ or $\varphi(E, \pi_0, \eta)$ be as defined in the previous subsection. Then the L-packet $\Pi(\varphi)$ associated to φ by the local Langlands correspondence is given in Table* 1.

Table 1 notation. If α is a non-Galois invariant character of $E^\times$, then $\pi(\alpha)$ is the associated supercuspidal representation of GL(2, F). If α is a Galois-invariant character of $E^\times$, then $\hat{\alpha}$ denotes a character of $F^\times$ such that $\hat{\alpha} \circ \mathrm{N}_F^E = \alpha$; if π_0 is Galois-invariant, then $\hat{\pi}_0$ denotes an irreducible, admissible representation of GL(2, F) such that the base change BC($\hat{\pi}_0$) of $\hat{\pi}_0$ is π_0. The central character of a representation π is denoted by ω_π.

Table 1
L-packets.

condition	$\Pi(\varphi)$	group
• $\pi_1 = \alpha_1 \times \alpha_2 \quad \pi_2 = \alpha_1' \times \alpha_2'$		
$\alpha_1'\alpha_2^{-1} \neq \nu^{\pm 1}$, $\alpha_2'\alpha_2^{-1} \neq \nu^{\pm 1}$	$\alpha_1'\alpha_2^{-1} \times \alpha_2'\alpha_2^{-1} \rtimes \alpha_2$	I
$\alpha_1'\alpha_2^{-1} \neq \nu^{\pm 1}$, $\alpha_2'\alpha_2^{-1} = \nu$	$\alpha_1'\alpha_2^{-1} \rtimes \nu^{1/2}\alpha_2 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} \neq \nu^{\pm 1}$, $\alpha_2'\alpha_2^{-1} = \nu^{-1}$	$\alpha_2'\alpha_1^{-1} \rtimes \nu^{1/2}\alpha_1 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu$, $\alpha_2'\alpha_2^{-1} \neq \nu^{\pm 1}$	$\alpha_2'\alpha_2^{-1} \rtimes \nu^{1/2}\alpha_2 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu$, $\alpha_2'\alpha_2^{-1} = \nu$	$\nu \rtimes \nu^{1/2}\alpha_2 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu$, $\alpha_2'\alpha_2^{-1} = \nu^{-1}$	$\nu \rtimes \nu^{-1/2}\alpha_1 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu^{-1}$, $\alpha_2'\alpha_2^{-1} \neq \nu^{\pm 1}$	$\alpha_1'\alpha_1^{-1} \rtimes \nu^{1/2}\alpha_1 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu^{-1}$, $\alpha_2'\alpha_2^{-1} = \nu$	$\nu \rtimes \nu^{-1/2}\alpha_1 1_{\mathrm{GSp}(2)}$	IIIb
$\alpha_1'\alpha_2^{-1} = \nu^{-1}$, $\alpha_2'\alpha_2^{-1} = \nu^{-1}$	$\nu \rtimes \nu^{1/2}\alpha_1 1_{\mathrm{GSp}(2)}$	IIIb
• $\pi_1 = \alpha_1 \times \alpha_2 \quad \pi_2 = \alpha 1_{\mathrm{GL}(2)}$		
$\alpha\alpha_2^{-1} \neq \nu^{\pm 3/2}$	$\alpha\alpha_2^{-1} 1_{\mathrm{GL}(2)} \rtimes \alpha_2$	IIb
$\alpha\alpha_2^{-1} = \nu^{\pm 3/2}$	$\alpha 1_{\mathrm{GSp}(4)}$	IVd
• $\pi_1 = \alpha_1 \times \alpha_2 \quad \pi_2 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$		
$\alpha\alpha_2^{-1} \neq \nu^{\pm 3/2}$	$\alpha\alpha_2^{-1}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \alpha_2$	IIa
$\alpha\alpha_2^{-1} = \nu^{\pm 3/2}$	$L(\nu^{3/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\alpha)$	IVc
• $\pi_1 = \alpha_1 \times \alpha_2 \quad \pi_2$ supercuspidal		
none	$\alpha_2^{-1}\pi_2 \rtimes \alpha_2$	X
• $\pi_1 = \alpha 1_{\mathrm{GL}(2)} \quad \pi_2 = \beta 1_{\mathrm{GL}(2)}$		
$\alpha \neq \beta$	$L(\nu\alpha^{-1}\beta, \alpha^{-1}\beta \rtimes \nu^{-1/2}\alpha)$	Vd
$\alpha = \beta$	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\alpha)$	VId
• $\pi_1 = \alpha 1_{\mathrm{GL}(2)} \quad \pi_2 = \beta\,\mathrm{St}_{\mathrm{GL}(2)}$		
$\alpha \neq \beta$	$L(\nu^{1/2}\alpha^{-1}\beta\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\alpha)$	Vb
$\alpha = \beta$	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\alpha)$	VIc
• $\pi_1 = \alpha 1_{\mathrm{GL}(2)} \quad \pi_2$ supercuspidal		
none	$L(\nu^{1/2}\alpha^{-1}\pi_2, \nu^{-1/2}\alpha)$	XIb
• $\pi_1 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)} \quad \pi_2 = \beta\,\mathrm{St}_{\mathrm{GL}(2)}$		
$\alpha \neq \beta$	$\delta([\alpha^{-1}\beta, \nu\alpha^{-1}\beta], \nu^{-1/2}\alpha)$, supercuspidal	Va, Vb*
$\alpha = \beta$	$\tau(S, \nu^{-1/2}\alpha)$, $\tau(T, \nu^{-1/2}\alpha)$	VIa, VIb
• $\pi_1 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)} \quad \pi_2$ supercuspidal		
none	$\delta(\nu^{1/2}\alpha^{-1}\pi_2, \nu^{-1/2}\alpha)$, supercuspidal	XIa, XIb*
• π_1 supercuspidal $\quad \pi_2$ supercuspidal		
$\pi_1 \ncong \pi_2$	two element supercuspidal L-packet	XIIa*, XIIb*
$\pi_1 \cong \pi_2$	$\tau(S, \pi_1)$, $\tau(T, \pi_1)$	VIIIa, VIIIb
• $\pi_0 = \alpha_1 \times \alpha_2$. If $\alpha_1^\sigma \neq \alpha_1$, then $\xi \neq 1$ is a character with $\xi^2 = 1$ and $\xi\pi(\alpha_1) \simeq \pi(\alpha_1)$		
$\alpha_1^\sigma \neq \alpha_1$, $\omega_{E/F}\eta^{-1}\alpha_2\vert_{F^\times} \neq \begin{cases} \xi\nu^{\pm 1}, \\ 1 \end{cases}$	$\omega_{E/F}\eta^{-1}\alpha_2\vert_{F^\times} \rtimes \pi(\alpha_1)$	VII
$\alpha_1^\sigma \neq \alpha_1$, $\omega_{E/F}\eta^{-1}\alpha_2\vert_{F^\times} = \xi\nu$	$L(\xi\nu, \pi(\alpha_1))$	IXb
$\alpha_1^\sigma \neq \alpha_1$, $\omega_{E/F}\eta^{-1}\alpha_2\vert_{F^\times} = \xi\nu^{-1}$	$L(\xi\nu, \pi(\alpha_2))$	IXb
$\alpha_1^\sigma \neq \alpha_1$, $\omega_{E/F}\eta^{-1}\alpha_2\vert_{F^\times} = 1$	$\tau(S, \pi(\alpha_2))$, $\tau(T, \pi(\alpha_2))$	VIIIa, VIIIb
$\alpha_1^\sigma = \alpha_1$, $\hat\alpha_1 \circ \mathrm{N}_F^E = \alpha_1$	$\omega_{E/F}\eta^{-1}\hat\alpha_1^2 \times \omega_{E/F} \rtimes \eta\hat\alpha_1^{-1}$	I

Table 1 (*continued*)

condition	$\Pi(\varphi)$	group
• $\pi_0 = \alpha 1_{\mathrm{GL}(2)}$		
$\alpha^\sigma \neq \alpha$	$L(\nu\omega_{E/F}\eta^{-1}\alpha\vert_{F^\times}, \nu^{-1/2}\pi(\alpha))$	IXb
$\alpha^\sigma = \alpha$, $\hat\alpha \circ \mathrm{N}_F^E = \alpha$, $\hat\alpha^2 = \eta$	$L(\nu\omega_{E/F}, \omega_{E/F} \rtimes \nu^{-1/2}\hat\alpha)$	Vd
$\alpha^\sigma = \alpha$, $\hat\alpha \circ \mathrm{N}_F^E = \alpha$, $\hat\alpha^2 = \eta\omega_{E/F}$	$\omega_{E/F} \rtimes \hat\alpha 1_{\mathrm{GSp}(2)}$	IIIb
• $\pi_0 = \alpha\, \mathrm{St}_{\mathrm{GL}(2)}$		
$\alpha^\sigma \neq \alpha$	$\delta(\nu\omega_{E/F}\eta^{-1}\alpha\vert_{F^\times}, \nu^{-1/2}\pi(\alpha))$	IXa
$\alpha^\sigma = \alpha$, $\hat\alpha \circ \mathrm{N}_F^E = \alpha$, $\hat\alpha^2 = \eta$	$\delta([\omega_{E/F}, \nu\omega_{E/F}], \nu^{-1/2}\hat\alpha)$, supercuspidal	Va, Vb*
$\alpha^\sigma = \alpha$, $\hat\alpha \circ \mathrm{N}_F^E = \alpha$, $\hat\alpha^2 = \eta\omega_{E/F}$	$\omega_{E/F} \rtimes \hat\alpha\, \mathrm{St}_{\mathrm{GSp}(2)}$	IIIa
• π_0 supercuspidal		
$\pi_0^\sigma \cong \pi_0$, $\mathrm{BC}(\hat\pi_0) = \pi_0$, $\omega_{\hat\pi_0} = \eta\omega_{E/F}$	$\omega_{E/F} \rtimes \hat\pi_0$	VII
$\pi_0^\sigma \ncong \pi_0$	one element supercuspidal L-packet	XIII*
$\pi_0^\sigma \cong \pi_0$, $\mathrm{BC}(\hat\pi_0) = \pi_0$, $\omega_{\hat\pi_0} = \eta$	two element supercuspidal L-packet	XIIa*, XIIb*

Degree four invariants. We now specialize to the case in which π_1, π_2, and π_0 are tempered representations with trivial central character. In this case, the L-packet $\Pi(\varphi)$ associated to φ, as described in the previous section, contains a unique generic representation π, which is also tempered and has trivial central character. The following proposition tabulates the level N_π, the Atkin–Lehner eigenvalue ε_π, and the Hecke eigenvalues μ_π and λ_π of the paramodular newform in π. We also fix two additive characters, ψ of F with conductor $\mathfrak{o}$ and ψ_E of E with conductor $\mathfrak{o}_E$. For the purposes of stating the next proposition we define certain Euler-type factors. Let N be a non-negative integer, let $\varepsilon = \pm 1$, and λ and μ be complex numbers. We define the factor $L(s, N, \varepsilon, \lambda, \mu)$ as follows. If $N = 0$, then we define:

$$L(s, 0, \varepsilon, \lambda, \mu)^{-1} = 1 - q^{-3/2}\lambda q^{-s} + \left(q^{-2}\mu + 1 + q^{-2}\right)q^{-2s} - q^{-3/2}\lambda q^{-3s} + q^{-4s}.$$

If $N = 1$, then we define

$$L(s, 1, \varepsilon, \lambda, \mu)^{-1} = 1 - q^{-3/2}(\lambda + \varepsilon)q^{-s} + \left(q^{-2}\mu + 1\right)q^{-2s} + \varepsilon q^{-1/2}q^{-3s}.$$

If $N \geqslant 2$, then we define

$$L(s, N, \varepsilon, \lambda, \mu)^{-1} = 1 - q^{-3/2}\lambda q^{-s} + \left(q^{-2}\mu + 1\right)q^{-2s}.$$

Proposition 4.2. *Let the notation be as in Proposition* 4.1. *Assume additionally that π_1, π_2 and π_0 are tempered, and that $\eta = 1$. Then $\Pi(\varphi)$ contains a unique generic element π, which is tempered and has trivial central character. Let N_π be the paramodular level of π, let ε_π be the Atkin–Lehner eigenvalue of the newform in π, and let λ_π and μ_π be the Hecke eigenvalues of the newform in π for the Hecke operators $T_{0,1}$ and $T_{1,0}$ from* 6.1 *of* [RS1]. *We have*

$$N_\pi = a(\varphi) = \begin{cases} a(\pi_1) + a(\pi_2) & \textit{in the split case,} \\ 2d(E/F) + f(E/F)a(\pi_0) & \textit{in the non-split case,} \end{cases} \tag{7}$$

$$\varepsilon_\pi = \varepsilon(1/2, \varphi, \psi, dx_\psi) = \begin{cases} \varepsilon(1/2, \pi_1, \psi, dx_\psi)\varepsilon(1/2, \pi_2, \psi, dx_\psi) & \textit{in the split case,} \\ \varepsilon(1/2, \pi_0, \psi_E, dx_{\psi_E})\omega_{E/F}(-1) & \textit{in the non-split case,} \end{cases} \tag{8}$$

and

$$L(s, N_\pi, \varepsilon_\pi, \lambda_\pi, \mu_\pi) = L(s, \varphi) = \begin{cases} L(s,\pi_1)L(s,\pi_2) & \text{in the split case,} \\ L(s,\pi_0) & \text{in the non-split case.} \end{cases} \tag{9}$$

Moreover, N_π, ε_π, λ_π and μ_π are given by Table 2. Note that in the table, the subscript ϖ indicates evaluation of the character at the uniformizer ϖ of F.

Proof. Let $\varphi = \varphi(\pi_1, \pi_2)$ or $\varphi(E, \pi_0, \eta)$ with π_1, π_2 and π_0 tempered, $\omega_{\pi_1} = \omega_{\pi_2} = 1$, and $\eta = 1$. Then using Proposition 4.1, [RS1] and the discussion preceding Proposition 4.1, one can verify that all the elements of $\Pi(\varphi)$ are tempered with trivial central character, and that exactly one element π is generic. Next, the second equalities in (7) and (9) follow from $(a'1)$, $(a'2)$, $(L1)$ and $(L2)$ in [Rohr], along with the local Langlands correspondence for GL(2) (see, for example, [Ku]). Similarly, the second equality in (8) follows from $(\epsilon'1)$ of [Rohr] in the split case. Finally, assume that we are in the non-split case and let $\tilde{\psi} = \psi \circ \mathrm{Tr}_F^E$. Then we have

$$\begin{aligned} \varepsilon(\varphi, \psi, dx_\psi) &= \varepsilon(\varphi_0, \tilde{\psi}, dx_{\tilde{\psi}}) \frac{\varepsilon(\mathrm{Ind}_{W'_E}^{W'_F} 1_E, \psi, dx_\psi)^2}{\varepsilon(1_E, \tilde{\psi}, dx_{\tilde{\psi}})^2} \quad \big([\mathrm{Rohr}], (\epsilon'2)\big) \\ &= \varepsilon(\varphi_0, \tilde{\psi}, dx_{\tilde{\psi}}) \frac{\varepsilon(1_F, \psi, dx_\psi)^2 \varepsilon(\omega_{E/F}, \psi, dx_\psi)^2}{\varepsilon(1_E, \tilde{\psi}, dx_{\tilde{\psi}})^2} \\ &= \varepsilon(\varphi_0, \psi_E, dx_{\psi_E}) \frac{\varepsilon(\omega_{E/F}, \psi, dx_\psi)^2}{\varepsilon(1_E, \psi_E, dx_E)^2} \quad \big([\mathrm{Rohr}], \text{11 Prop.}\big) \\ &= \varepsilon(\varphi_0, \psi_E, dx_{\psi_E}) \varepsilon(\omega_{E/F}, \psi, dx_\psi)^2 \\ &= \varepsilon(\varphi_0, \psi_E, dx_{\psi_E}) \omega_{E/F}(-1) q^d \quad \big([\mathrm{Rohr}], \text{12 Lemma}\big). \end{aligned}$$

Thus, at the center of the critical strip, we have again by [Rohr], 11 Prop., (iii)

$$\begin{aligned} \varepsilon(1/2, \varphi, \psi, dx_\psi) &= \varepsilon(\varphi, \psi, dx_\psi) q^{-(2d + f a(\varphi_0))/2} \\ &= \varepsilon(\varphi_0, \psi_E, dx_{\psi_E}) q_E^{-a(\varphi_0)/2} \omega_{E/F}(-1) \\ &= \varepsilon(1/2, \varphi_0, \psi_E, dx_{\psi_E}) \omega_{E/F}(-1), \end{aligned}$$

as desired.

In the case that π is non-supercuspidal the first equality in (7), (8) and (9) is known by [RS1] Theorems 7.5.3, 7.5.9 and Corollary 7.5.5.

Now consider the case that π is a supercuspidal representation. In this case all of the factors in (9) are 1. Let $\varepsilon(s, \pi)$ be the epsilon factor defined by the Novodvorsky zeta integrals of π as discussed in Section 5. By Corollary 7.5.5 of [RS1] we have that $\varepsilon(s, \pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}$. Further, as π is supercuspidal, the epsilon factor and gamma factor defined by the Novodvorsky zeta integrals coincide. By Propositions 5.3 and 5.6 we then have that

$$\varepsilon_\pi q^{-N_\pi(s-1/2)} = \begin{cases} \gamma(s, \pi_1, \psi)\gamma(s, \pi_2, \psi) & \text{split,} \\ \omega_{E/F}(-1)\gamma(s, \pi_0, \tilde{\psi}) & \text{non-split.} \end{cases}$$

Finally, the proof is completed by noting that we have

$$\begin{aligned} &\gamma(s, \pi_i, \psi) = \varepsilon(1/2, \pi_i, \psi, dx_\psi) q^{-a(\pi_i)(s-1/2)}, \quad i = 1, 2 \text{ split,} \\ &\gamma(s, \pi_0, \tilde{\psi}) = \varepsilon(1/2, \pi_0, \psi_E, dx_{\psi_E}) q^{-(2d(E/F) + f a(\pi_0))(s-1/2)} \quad \text{non-split.} \qquad \square \end{aligned}$$

Table 2
Hecke eigenvalues.

condition	N_π	ε_π	λ_π	μ_π
• $\pi_1=\alpha\times\alpha^{-1}$ $\pi_2=\alpha'\times\alpha'^{-1}$ $\pi=\alpha\alpha'\times\alpha\alpha'^{-1}\rtimes\alpha^{-1}$				I
α,α' unr	0	1	$q^{3/2}(\alpha_\varpi+\alpha_\varpi^{-1}+\alpha'_\varpi+\alpha'^{-1}_\varpi)$	$q^2(1-q^{-2}+$ $(\alpha_\varpi+\alpha_\varpi^{-1})(\alpha'_\varpi+\alpha'^{-1}_\varpi))$
α unr α' ram	$2a(\alpha')$	$\alpha'(-1)$	$q^{3/2}(\alpha_\varpi+\alpha_\varpi^{-1})$	0
α ram α' unr	$2a(\alpha)$	$\alpha(-1)$	$q^{3/2}(\alpha'_\varpi+\alpha'^{-1}_\varpi)$	0
α,α' ram	$2(a(\alpha)+a(\alpha'))$	$\alpha(-1)\alpha'(-1)$	0	$-q^2$
• $\pi_1=\alpha\times\alpha^{-1}$ $\pi_2=\alpha'\,\mathrm{St}_{\mathrm{GL}(2)}$ $\pi=\alpha\alpha'\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\alpha^{-1}$				IIa
α,α' unr	1	$-\alpha'_\varpi$	$q^{3/2}(\alpha_\varpi+\alpha_\varpi^{-1})+(q+1)\alpha'_\varpi$	$q^{3/2}(\alpha_\varpi\alpha'_\varpi+\alpha_\varpi^{-1}\alpha'^{-1}_\varpi)$
α unr α' ram	$2a(\alpha')$	$\alpha'(-1)$	$q^{3/2}(\alpha_\varpi+\alpha_\varpi^{-1})$	0
α ram α' unr	$2a(\alpha)+1$	$-\alpha^{-1}(-1)\alpha'_\varpi$	$q\alpha'_\varpi$	$-q^2$
α,α' ram	$2(a(\alpha)+a(\alpha'))$	$\alpha(-1)\alpha'(-1)$	0	$-q^2$
• $\pi_1=\alpha\times\alpha^{-1}$ π_2 supercuspidal $\pi=\alpha\pi_2\rtimes\alpha^{-1}$				X
α unr	$a(\pi_2)$	$\varepsilon(\frac{1}{2},\pi_2)$	$q^{3/2}(\alpha_\varpi+\alpha_\varpi^{-1})$	0
α ram	$a(\pi_2)+2a(\alpha)$	$\alpha(-1)\varepsilon(\frac{1}{2},\pi_2)$	0	$-q^2$
• $\pi_1=\alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ $\pi_2=\beta\,\mathrm{St}_{\mathrm{GL}(2)}$ $\alpha\neq\beta$ $\pi=\delta([\alpha^{-1}\beta,\nu\alpha^{-1}\beta],\nu^{-1/2}\alpha)$				Va
α,β unr	2	-1	0	$-q^2-q$
α unr β ram	$2a(\beta)+1$	$-\alpha_\varpi\beta(-1)$	$\alpha_\varpi q$	$-q^2$
α ram β unr	$2a(\alpha)+1$	$-\alpha(-1)\beta_\varpi$	$-\alpha_\varpi q$	$-q^2$
α,β ram	$2a(\alpha)+2a(\beta)$	$\alpha(-1)\beta(-1)$	0	$-q^2$
• $\pi_1=\alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ $\pi_2=\beta\,\mathrm{St}_{\mathrm{GL}(2)}$ $\alpha=\beta$ $\pi=\tau(S,\nu^{-1/2}\alpha)$				VIa
α unr	2	1	$2q\alpha_\varpi$	$-q(q-1)$
α ram	$4a(\alpha)$	1	0	$-q^2$
• $\pi_1=\alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ π_2 supercuspidal $\pi=\delta(\nu^{1/2}\alpha^{-1}\pi_2,\nu^{-1/2}\alpha)$				XIa
α unr	$a(\pi_2)+1$	$-\alpha_\varpi\varepsilon(\frac{1}{2},\pi_2)$	$q\alpha_\varpi$	$-q^2$
α ram	$a(\pi_2)+2a(\alpha)$	$\alpha(-1)\varepsilon(\frac{1}{2},\pi_2)$	0	$-q^2$
• π_1,π_2 supercuspidal $\pi_1\cong\pi_2$ $\pi=\tau(S,\pi_1)$				VIIIa
none	$2a(\pi_1)$	1	0	$-q^2$
• π_1,π_2 supercuspidal $\pi_1\ncong\pi_2$ π supercuspidal				XIIa*
none	$a(\pi_1)+a(\pi_2)$	$\varepsilon(\frac{1}{2},\pi_1)\varepsilon(\frac{1}{2},\pi_2)$	0	$-q^2$
• $\pi_0=\alpha\times\alpha^{-1}$ $\alpha^\sigma\neq\alpha$ $\alpha\vert_{F^\times}\neq\omega_{E/F}$ $\pi=\omega_{E/F}\,\alpha\vert_{F^\times}^{-1}\rtimes\pi(\alpha^{-1})$				VII
none	$2d+2fa(\alpha)$	$\omega_{E/F}(-1)\alpha(-1)$	0	$-q^2$
• $\pi_0=\alpha\times\alpha^{-1}$ $\alpha^\sigma\neq\alpha$ $\alpha\vert_{F^\times}=\omega_{E/F}$ $\pi=\tau(S,\pi(\alpha^{-1}))$				VIIIa
none	$2d+2fa(\alpha)$	1	0	$-q^2$
• $\pi_0=\alpha\times\alpha^{-1}$ $\alpha^\sigma=\alpha$ $\alpha=\hat{\alpha}\circ\mathrm{N}_F^E$ $\pi=\omega_{E/F}\hat{\alpha}^2\times\omega_{E/F}\rtimes\hat{\alpha}^{-1}$				I
α unr $\omega_{E/F}$ unr	0	1	0	$-q^2(\alpha_\varpi+\alpha_\varpi^{-1})$ $-q^2-1$

(*continued on next page*)

Table 2 (*continued*)

condition	N_π	ε_π	λ_π	μ_π
α unr	$2d$	$\omega_{E/F}(-1)$	$q^{3/2}(\hat\alpha_\varpi + \hat\alpha_\varpi^{-1})$	0
$\omega_{E/F}$ ram				
α ram	$2d+2fa(\alpha)$	$\omega_{E/F}(-1)$	0	$-q^2$
• $\pi_0 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ $\alpha^\sigma \neq \alpha$ $\pi = \delta(\nu\omega_{E/F}\alpha\|_{F^\times}, \nu^{-1/2}\pi(\alpha))$				IXa
none	$2d+2fa(\alpha)$	$\omega_{E/F}(-1)\alpha(-1)$	0	$-q^2$
• $\pi_0 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ $\alpha^\sigma = \alpha$ $\alpha = \hat\alpha \circ \mathrm{N}_F^E$ $\hat\alpha^2 = 1$ $\pi = \delta([\omega_{E/F}, \nu\omega_{E/F}], \nu^{-1/2}\hat\alpha)$				Va
α unr	2	-1	0	$-q^2-q$
$\omega_{E/F}$ unr				
α unr	$2d+1$	$-\alpha_{\varpi_E}\omega_{E/F}(-1)$	$\alpha_{\varpi_E} q$	$-q^2$
$\omega_{E/F}$ ram				
α ram	$2d+2fa(\alpha)$	$\omega_{E/F}(-1)$	0	$-q^2$
• $\pi_0 = \alpha\,\mathrm{St}_{\mathrm{GL}(2)}$ $\alpha^\sigma = \alpha$ $\alpha = \hat\alpha \circ \mathrm{N}_F^E$ $\hat\alpha^2 = \omega_{E/F}$ $\pi = \omega_{E/F} \rtimes \hat\alpha\,\mathrm{St}_{\mathrm{GSp}(2)}$				IIIa
α unr	2	1	$q(\hat\alpha_\varpi + \hat\alpha_\varpi^{-1})$	$-q^2+q$
α ram	$2d+2fa(\alpha) = 4a(\hat\alpha)$	1	0	$-q^2$
• π_0 supercuspidal $\pi_0^\sigma \ncong \pi_0$ π supercuspidal				XIII*
none	$2d+fa(\pi_0)$	$\varepsilon(\frac{1}{2}, \pi_0)\omega_{E/F}(-1)$	0	$-q^2$
• π_0 supercuspidal $\pi_0^\sigma \cong \pi_0$ $\mathrm{BC}(\hat\pi_0) = \pi_0$ $\omega_{\hat\pi_0} = 1$ π supercuspidal				XIIa*
none	$2d+fa(\pi_0)$	$\varepsilon(\frac{1}{2}, \pi_0)\omega_{E/F}(-1)$	0	$-q^2$
• π_0 supercuspidal $\pi_0^\sigma \cong \pi_0$ $\mathrm{BC}(\hat\pi_0) = \pi_0$ $\omega_{\hat\pi_0} = \omega_{E/F}$ $\pi = \omega_{E/F} \rtimes \hat\pi_0$				VII
none	$2d+fa(\pi_0) = 2a(\hat\pi_0)$	$\omega_{E/F}(-1)$	0	$-q^2$

5. Equality of gamma factors for supercuspidal representations

The main result of this section is the calculation of the Novodvorsky gamma factors of the supercuspidal representations of type XIIa* and XIII*.

Let π be a supercuspidal generic irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $s \in \mathbb{C}$. We say that π admits an s-Bessel model if π is isomorphic to a space of functions $B : \mathrm{GSp}(4, F) \to \mathbb{C}$ that satisfy

$$B\left(\begin{bmatrix} 1 & & y_1 & y_2 \\ & 1 & y_2 & y_3 \\ & & 1 & \\ & & & 1 \end{bmatrix} g\right) = \psi(y_2)B(g)$$

and

$$B\left(\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2 & \\ & & & t_1 \end{bmatrix} g\right) = |t_2/t_1|^{s-1/2}B(g)$$

for $y_1, y_2, y_3 \in F$, $t_1, t_2 \in F^\times$, and $g \in \mathrm{GSp}(4, F)$. If π admits an s-Bessel model, then this model is unique (see [RS1], Proposition 2.5.7), and we denote it by $B_s(\pi)$.

Now, let π be a representation of type XIIa* so that there exist two nonisomorphic, supercuspidal, irreducible, admissible representations π_1 and π_2 of $\mathrm{GL}(2, F)$ with trivial central characters such that $\pi = \theta_X(\sigma)$ where X is the four dimensional hyperbolic quadratic space over F, and σ

is a representation of $\mathrm{GO}(X)$ constructed from π_1 and π_2. Concretely, let $X = M_2(F)$, and equip X with the symmetric bilinear form defined by $(x, y) = \mathrm{Tr}(xy^*)/2$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Let $R := \{(g, h) \in \mathrm{GSp}(4, F) \times \mathrm{GO}(X)\colon\ \lambda(g) = \lambda(h)\}$, and let $\omega = \omega_\psi$ be the Weil representation of R on the Schwartz space $\mathcal{S}(X^2)$ with respect to ψ [R]. We have an exact sequence

$$1 \to F^\times \to \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) \xrightarrow{\rho} \mathrm{GSO}(X) \to 1$$

where $\rho(g_1, g_2)x = g_1 x g_2^*$. Let $W(\pi_i, \psi)$ be the ψ-Whittaker model for π_i, so that $W_i \in W(\pi_i, \psi)$ transforms according to the formula

$$W_i\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) = \psi(x) W_i(g)$$

for $g \in \mathrm{GL}(2, F)$ and $x \in F$. Let $x_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$ and let H be the stabilizer of x_1 and x_2 in $\mathrm{SO}(X)$. For $W_i \in W(\pi_i, \psi)$ and $\varphi \in S(X^2)$ we define

$$B(g, \varphi, W_1, W_2, s) := \int\limits_{H\backslash \mathrm{SO}(X)} \big(\omega\big(g, hh'\big)\varphi\big)(x_1, x_2) Z\big(s, \pi_1\big(h_1 h_1'\big) W_1\big) Z\big(s, \pi_2\big(h_2 h_2'\big) W_2\big)\, dh$$

where $h = \rho(h_1, h_2)$, h' is any element of $\mathrm{GSO}(X)$ such that $\lambda(h') = \lambda(g)$ and

$$Z(s, W_i) = \int\limits_{F^\times} W_i\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) |a|^{s-1/2} d^\times a, \quad W_i \in W(\pi_i, \psi).$$

Note that as π_i is supercuspidal, $Z(s, W_i)$ converges for all $s \in \mathbb{C}$ to a polynomial in $\mathbb{C}[q^{-s}, q^s]$. A similar statement is true for $B(g, \varphi, W_1, W_2, s)$. One can prove that each of the functions $B(\cdot, \varphi, W_1, W_2, s)$ is contained in the s-Bessel model for π and by extending linearly, we obtain a surjective map

$$\beta_s : S\big(X^2\big) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi) \to B_s(\pi)$$

with the property that for $g \in \mathrm{GSp}(4, F)$ and $h = \rho(h_1, h_2) \in \mathrm{GSO}(X)$ with $\lambda(h) = \lambda(g)$ we have

$$\beta_s\big(\omega(g, h)\varphi \otimes \pi_1(h_1) W_1 \otimes \pi_2(h_2) W_2\big) = g \cdot \beta_s(\varphi \otimes W_1 \otimes W_2). \tag{10}$$

On the other hand, for each $s \in \mathbb{C}$, there is another surjective map

$$\beta_s' : S\big(X^2\big) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi) \to B_s(\pi)$$

with the analogous transformation property. This map is constructed using the Weil representation and zeta integrals. Let $c_1, c_2 \in F^\times$ and let $W(\pi, \psi_{c_1,c_2})$ be the ψ_{c_1,c_2}-Whittaker model for π. If $W \in W(\pi, \psi_{c_1,c_2})$, then

$$W\left(\begin{bmatrix} 1 & x & * & * \\ & 1 & * & y \\ & & 1 & \\ & & -x & 1 \end{bmatrix} g\right) = \psi(c_1 x + c_2 y) W(g) \tag{11}$$

for $x, y \in F$ and $g \in \mathrm{GSp}(4, F)$. The map β'_s is defined to be the composition of $\mathrm{GSp}(4, F)$ maps

$$S(X^2) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi) \xrightarrow{\mathrm{id}\otimes S_{1/2}} S(X^2) \otimes W(\pi_1, \psi^{1/2}) \otimes W(\pi_2, \psi^{-1/2}) \xrightarrow{C} W(\pi, \psi_{1/2,1/2}) \xrightarrow{S_2} W(\pi, \psi_{1,1}) \xrightarrow{B_Z} B_s(\pi). \tag{12}$$

For the first map, we have

$$S_{1/2}(W_1 \otimes W_2)(g_1, g_2) = W_1\left(\begin{bmatrix} 1/2 & \\ & 1 \end{bmatrix} g_1\right) \otimes W_2\left(\begin{bmatrix} -1/2 & \\ & 1 \end{bmatrix} g_2\right).$$

For the second map, let $y_1 = \begin{bmatrix} & 1 \\ & \end{bmatrix}$ and $y_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ and let H' be the stabilizer of y_1 and y_2 in $\mathrm{SO}(X)$. Then the map C is given by

$$C(\varphi \otimes W_1 \otimes W_2)(g) = \int_{H'\backslash \mathrm{SO}(X)} \left(\omega_\psi(g, hh')\varphi\right)(y_1, y_2) W_1(h_1 h'_1) W_2(h_2 h'_2)\, dh,$$

where $h = \rho(h_1, h_2)$ and h' is any element of $\mathrm{GSO}(X)$ such that $\lambda(h') = \lambda(g)$. The map S_2 is defined by the formula

$$S_2(W)(g) = W\left(\begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1/4 & \\ & & & 1/2 \end{bmatrix} g\right).$$

To construct the final map, we recall that for $W \in W(\pi, \psi_{1,1})$, the zeta integral of W is given by

$$Z(s, W) = \int_{F^\times} \int_F W\left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & x & & 1 \end{bmatrix}\right) |a|^{s-3/2}\, dx\, d^\times a.$$

Since π is supercuspidal, $Z(s, W)$ converges for all $s \in \mathbb{C}$ to a polynomial in $\mathbb{C}[q^{-s}, q^s]$. We define the map $B_Z : W(\pi, \psi_{1,1}) \to B_s(\pi)$ by

$$B_Z(W)(g) := Z\left(s, \pi\left(\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix} g\right) W\right).$$

Lemma 5.1. *There exists a constant $c \in \mathbb{C}^\times$ such that*

$$\beta_s(x) = c|2|^s \beta'_s(x)$$

for all $x \in S(X^2) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi)$ and for all $s \in \mathbb{C}$.

Proof. It follows from Theorem 1.8 of [R] that for every $s \in \mathbb{C}$ the space of maps

$$S(X^2) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi) \to B_s(\pi)$$

satisfying the transformation property in (10) is one dimensional. Therefore, for every $s \in \mathbb{C}$ there exists a constant $c(s) \in \mathbb{C}^\times$ such that $\beta_s = c(s)\beta'_s$. To compute $c(s)$, let N_1 be a positive integer and for $i = 1, 2$ let W_i in the Whittaker model $W(\pi_i, \psi)$ of π_i correspond to the characteristic function $\chi_{1+\mathfrak{p}^{N_1}}$ in the Kirillov model of π_i with respect to ψ, so that

$$W_i\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix}\right) = \chi_{1+\mathfrak{p}^{N_1}}(x), \quad x \in F^\times.$$

Choose $N_2 > N_1$ such that

$$\pi_i(\Gamma_{N_2})W_i = W_i \quad \text{for } i = 1, 2,$$

where

$$\Gamma_{N_2} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o})\colon a, d \equiv 1 \left(\mathfrak{p}^{N_2}\right),\ b, c \equiv 0 \left(\mathfrak{p}^{N_2}\right)\right\}.$$

There is a homeomorphism

$$H \backslash \mathrm{SO}(X) \xrightarrow{\sim} \mathrm{SO}(X) \cdot (x_1, x_2).$$

Let p be the projection $p : \mathrm{SO}(X) \to H\backslash \mathrm{SO}(X)$. The set $p(\rho(\Gamma_{N_2} \times \Gamma_{N_2}) \cap \mathrm{SO}(X))$ is an open neighborhood of $H \cdot 1$. By applying the above homeomorphism to this set one obtains an open neighborhood of (x_1, x_2). Choose $N_3 > N_2$ such that this open neighborhood of (x_1, x_2) contains

$$\mathrm{SO}(X)(x_1, x_2) \cap \left(x_1 + \varpi^{N_3}\mathrm{M}(2, \mathfrak{o}), x_2 + \varpi^{N_3}\mathrm{M}(2, \mathfrak{o})\right).$$

Let $\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{S}(X^2)$ be the characteristic function of $(x_1 + \varpi^{N_3}\mathrm{M}(2, \mathfrak{o})) \times (x_2 + \varpi^{N_3}\mathrm{M}(2, \mathfrak{o}))$, where φ_i is the characteristic function of $x_i + \varpi^{N_3}\mathrm{M}(2, \mathfrak{o})$. It follows that $\beta_s(\varphi \otimes W_1 \otimes W_2)(1)$ is a non-zero constant C_1 independent of s. A lengthy computation shows that $\beta'_s(\varphi \otimes W_1 \otimes W_2)(1) = C_2|2|^{-s}$, where C_2 is a constant independent of s. Thus, $c(s) = C_1C_2^{-1}|2|^s$. □

Lemma 5.2. *For any $x \in S(X^2) \otimes W(\pi_1, \psi) \otimes W(\pi_2, \psi)$ and for any $g \in \mathrm{GSp}(4, F)$, there is a functional equation*

$$\beta_{1-s}(x)\left(\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} g\right) = |2|^{1-2s}\gamma(s, \pi_1)\gamma(s, \pi_2)\beta_s(x)(g),$$

where $\gamma(s, \pi_i)$ is the gamma factor for the $\mathrm{GL}(2, F)$ *representation π_i.*

Proof. By (10), we may assume that $g = 1$ and $x = \varphi \otimes W_1 \otimes W_2$. Note that

$$\rho\left(\begin{bmatrix} & 1/2 \\ -2 & \end{bmatrix}, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)x_1 = x_2, \qquad \rho\left(\begin{bmatrix} & 1/2 \\ -2 & \end{bmatrix}, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)x_2 = x_1. \tag{13}$$

Then,

$$\beta_{1-s}(x)\left(\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}\right)$$

$$= \int\limits_{H\backslash \mathrm{SO}(X)} \omega\big(1,\rho(h_1,h_2)\big)\varphi(x_2,x_1)Z\big(1-s,\pi_1(h_1)W_1\big)Z\big(1-s,\pi_2(h_2)W_2\big)\,dh$$

$$= \int\limits_{H\backslash \mathrm{SO}(X)} \omega\big(1,\rho(h_1,h_2)\big)\varphi(x_1,x_2)Z\left(1-s,\pi_1\left(\begin{bmatrix} & 1/2 \\ -2 & \end{bmatrix}^{-1}\right)\pi_1(h_1)W_1\right)$$

$$\times Z\left(1-s,\pi_2\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}\right)\pi_2(h_2)W_2\right)dh,$$

by applying the identity (13). For $i=1,2$, the zeta integral of π_i satisfies a functional equation

$$Z\left(1-s,\pi_i\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)W\right)=\gamma(s,\pi_i)Z(s,W).$$

This functional equation, together with the fact that these representations have trivial central character yields

$$Z\big(1-s,\pi_1\left(\begin{bmatrix} & 1/2 \\ -2 & \end{bmatrix}^{-1}\right)\pi_1(h_1)W_1\big)=|4|^{1-s-1/2}\gamma(s,\pi_1)Z\big(s,\pi_1(h_1)W_1\big),$$

and

$$Z\left(1-s,\pi_2\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}\right)\pi_2(h_2)W_2\right)=\gamma(s,\pi_2)Z\big(s,\pi_2(h_2)W_2\big).$$

Substituting these identities we obtain the lemma. □

We recall that the Novodvorsky zeta integrals for π satisfy a functional equation. In general if π' is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then there exists an element $\gamma(s,\pi')\in\mathbb{C}(q^{-s})$ such that

$$Z\left(1-s,\pi'\left(\begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}\right)W\right)=\gamma\big(s,\pi'\big)Z(s,W)$$

for all $W\in W(\pi',\psi_{1,1})$. See [RS1], Proposition 2.6.5. Moreover the zeta integrals also define a local L-factor, $L(s,\pi')$, and epsilon factor

$$\varepsilon\big(s,\pi'\big)=\gamma\big(s,\pi'\big)\frac{L(s,\pi')}{L(1-s,\pi')}.$$

If π' is supercuspidal, then $L(s,\pi')=1$ so that the epsilon and gamma factors coincide.

Proposition 5.3. *We have*

$$\gamma(s,\pi)=\gamma(s,\pi_1)\gamma(s,\pi_2).$$

Proof. The proposition is proved by applying the previous lemmas to the functional equation for Novodvorsky zeta integrals. Let $x\in S(X^2)\otimes W(\pi_1,\psi)\otimes W(\pi_2,\psi)$. Let $W'\in W(\pi,\psi_{1,1})$ be the image of x under the composition of the first three maps in (12). The functional equation implies that $g\in \mathrm{GSp}(4,F)$ and for all $s\in\mathbb{C}$

$$\begin{aligned}
&Z\left(1-s,\pi\left(\begin{bmatrix}&&&1\\&&-1&\\&-1&&\\1&&&\end{bmatrix}\right)\pi\left(\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}g\right)W'\right)\\
&=\gamma(s,\pi)Z\left(s,\pi\left(\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}g\right)W'\right).
\end{aligned}$$

Moreover, Lemma 5.1 asserts that for all $g\in\mathrm{GSp}(4,F)$ and for all $s\in\mathbb{C}$,

$$Z\left(s,\pi\left(\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}g\right)W'\right)=c^{-1}|2|^{-s}\beta_s(x)(g). \tag{14}$$

Now the left-hand side of the functional equation can be rewritten as

$$\begin{aligned}
&Z\left(1-s,\pi\left(\begin{bmatrix}&&&1\\&&-1&\\&-1&&\\1&&&\end{bmatrix}\right)\pi\left(\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}g\right)W'\right)\\
&=Z\left(1-s,\pi\left(\begin{bmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{bmatrix}\right)\pi\left(\begin{bmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{bmatrix}g\right)W'\right)\\
&=c^{-1}|2|^{s-1}\beta_{1-s}(x)\left(\begin{bmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{bmatrix}g\right).
\end{aligned}$$

Substituting (14) for the right-hand side of the functional equation we obtain

$$c^{-1}|2|^{s-1}\beta_{1-s}(x)\left(\begin{bmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{bmatrix}g\right)=\gamma(s,\pi)c^{-1}|2|^{-s}\beta_s(x)(g).$$

Applying Lemma 5.2, we get:

$$\begin{aligned}
c^{-1}|2|^{s-1}|2|^{1-2s}\gamma(s,\pi_1)\gamma(s,\pi_2)\beta_s(x)(g)&=\gamma(s,\pi)c^{-1}|2|^{-s}\beta_s(x)(g),\\
\gamma(s,\pi_1)\gamma(s,\pi_2)\beta_s(x)(g)&=\gamma(s,\pi)\beta_s(x)(g),
\end{aligned}$$

for all $g \in \mathrm{GSp}(4, F)$ and for all $s \in \mathbb{C}$. As x and g run through all possible values, the holomorphic functions in s, $\beta_s(x)(g)$, run through all possible zeta integrals of π by (14) and hence all functions of the form $P(q^{-s}, q^s)$ where $P \in \mathbb{C}[X, Y]$ (see [RS1], Proposition 2.6.4; recall that $L(s, \pi) = 1$). Thus, we obtain the statement of the proposition. □

Now, suppose that π is a representation of type XIII*. The shape of the argument is analogous to the previous case, but there are important differences which we will highlight. In this case, there exists a quadratic extension $E = F(\sqrt{d_0})$ such that $\pi = \theta_{X_0}(\sigma_0)$, where X_0 is the four dimensional quadratic space having anisotropic component (E, N_F^E) and σ_0 is the supercuspidal, irreducible, admissible representation of $\mathrm{GO}(X_0)$ with trivial central character associated to a supercuspidal, irreducible, admissible representation π_0 of $\mathrm{GL}(2, E)$ with trivial central character which is not Galois invariant. Concretely, we take X_0 to be the subspace of $M_2(E)$ such that for all $x \in X_0$, $\sigma(x) = x^*$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and equip X_0 with a symmetric bilinear form $\langle x, y\rangle = \mathrm{Tr}(xy^*)/2$. Let $R := \{(g, h) \in \mathrm{GSp}(4, F) \times \mathrm{GO}(X_0)\colon\ \lambda(g) = \lambda(h)\}$, and let $\omega = \omega_\psi$ be the Weil representation of R on the Schwartz space $\mathcal{S}(X_0^2)$ with respect to ψ. We have an exact sequence

$$1 \to E^\times \to F^\times \times \mathrm{GL}(2, E) \xrightarrow{\rho_0} \mathrm{GSO}(X_0) \to 1$$

where $\rho_0(t, g)x = t^{-1} g x \sigma(g)^*$ and the inclusion of $E^\times$ sends z to $(N_F^E(z), z)$. Set $\tilde{\psi} = \psi \circ \mathrm{Tr}_{E/F}$. Let $W(\pi_0, \tilde{\psi})$ be the $\tilde{\psi}$-Whittaker model for π_0 such that $W_0 \in W(\pi_0, \tilde{\psi})$ transforms according to the formula

$$W_0\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) = \tilde{\psi}(x) W_0(g)$$

for $g \in \mathrm{GL}(2, E)$ and $x \in E$. Let $x_1 = \begin{bmatrix} 0 & \sqrt{d_0} \\ 0 & 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 & 0 \\ -2/\sqrt{d_0} & 0 \end{bmatrix}$ and let H be the stabilizer of x_1 and x_2 in $\mathrm{SO}(X_0)$. For $g \in \mathrm{GSp}(4, F)^+$, $W_0 \in W(\pi_0, \tilde{\psi})$ and $\varphi \in S(X_0^2)$ we define

$$B(g, \varphi, W_0, s) := \int\limits_{H\backslash \mathrm{SO}(X)} \big(\omega\big(g, hh'\big)\varphi\big)(x_1, x_2) Z\big(s, \pi_0\big(h_0 h_0'\big) W_0\big)\, dh$$

where $h = \rho_0(t, h_0)$, h' is any element of $\mathrm{GSO}(X)$ such that $\lambda(h') = \lambda(g)$ and

$$Z(s, W_0) = \int\limits_{E^\times} W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) |a|_E^{s-1/2} d^\times a, \quad W_0 \in W(\pi_0, \tilde{\psi}).$$

Note that as π_0 is supercuspidal, $Z(s, W_0)$ converges for all $s \in \mathbb{C}$ to a polynomial in q_E^{-s} and q_E^s. A similar statement is true for $B(g, \varphi, W_0, s)$. We extend $B(\cdot, \varphi, W_0, s)$ to $\mathrm{GSp}(4, F)$ via the formula

$$B(g) = \big|\lambda(g)\big|^{-(s-1/2)} B(g_0)$$

for

$$g_0 = \begin{bmatrix} \lambda(g)^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \lambda(g)^{-1} \end{bmatrix} g$$

for $g \in \mathrm{GSp}(4, F)$. One can then prove that each of the functions $B(\cdot, \varphi, W_0, s)$ is contained in the s-Bessel model for π and by extending linearly, we obtain a surjective map

$$\beta_s : S(X^2) \otimes W(\pi_0, \tilde{\psi}) \to B_s(\pi)$$

with the property that for $g \in \mathrm{GSp}(4, F)$ and $h = \rho_0(t, h_0) \in \mathrm{GSO}(X_0)$ with $\lambda(h) = \lambda(g)$ we have

$$\beta_s\big(\omega(g, h)\varphi \otimes \pi_0(h_0)W_0\big) = g \cdot \beta_s(\varphi \otimes W_0). \tag{15}$$

For each $s \in \mathbb{C}$, there is another surjective map

$$\beta'_s : S(X^2) \otimes W(\pi_0, \tilde{\psi}) \to B_s(\pi)$$

with the analogous transformation property. This map is constructed using the Weil representation and zeta integrals. Let $c_1, c_2 \in F^\times$ and let $W(\pi, \psi_{c_1,c_2})$ be the ψ_{c_1,c_2}-Whittaker model for π as in (11). The map β'_s is defined to be the composition of $\mathrm{GSp}(4, F)$ maps

$$\begin{aligned} S(X^2) \otimes W(\pi_0, \tilde{\psi}) &\overset{\mathrm{id}\otimes S_{1/2\sqrt{d_0}}}{\longrightarrow} S(X^2) \otimes W\big(\pi_0, \tilde{\psi}^{1/2\sqrt{d_0}}\big) \\ &\overset{C}{\longrightarrow} W(\pi, \psi_{1/2,1/2}) \overset{S_2}{\longrightarrow} W(\pi, \psi_{1,1}) \overset{B_Z}{\longrightarrow} B_s(\pi). \end{aligned} \tag{16}$$

For the first map, we have

$$S_{1/2\sqrt{d_0}}(W_0)(g_0) = W_0\left(\begin{bmatrix} 1/2\sqrt{d_0} & \\ & 1 \end{bmatrix} g_0\right).$$

For the second map, let $y_1 = \begin{bmatrix} & \sqrt{d_0} \\ & \end{bmatrix}$ and $y_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ and let H' be the stabilize of y_1 and y_2 in $\mathrm{SO}(X)$. Then the map C is given by

$$C(\varphi \otimes W_0)(g) = \int\limits_{H'\backslash \mathrm{SO}(X)} \big(\omega_\psi\big(g, hh'\big)\varphi\big)(y_1, y_2) W_0\big(h_0 h'_0\big)\, dh,$$

where $g \in \mathrm{GSp}(4, F)^+$, $h = \rho(t, h_0)$ and h' is any element of $\mathrm{GSO}(X)$ such that $\lambda(h') = \lambda(g)$. We extend the function $C(\varphi \otimes W_0)$ to all of $\mathrm{GSp}(4, F)$ by zero. The maps S_2 and B_Z are as in the previous case.

Lemma 5.4. *There exists a constant $c \in \mathbb{C}^\times$ such that*

$$\beta_s(x) = c|2/d_0|^s \beta'_s(x)$$

for all $x \in S(X^2) \otimes W(\pi_0, \tilde{\psi})$ and for all $s \in \mathbb{C}$.

Proof. The proof is analogous to the proof of Lemma 5.1. □

Lemma 5.5. *For any $x \in S(X^2) \otimes W(\pi_0, \tilde{\psi})$ and for any $g \in \mathrm{GSp}(4, F)$, there is a functional equation*

$$\beta_{1-s}(x)\left(\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} g\right) = \omega_{E/F}(-1)|2/d_0|_E^{1/2-s} \gamma(s, \pi_0, \tilde{\psi}) \beta_s(x)(g),$$

where $\gamma(s, \pi_0, \tilde{\psi})$ is the gamma factor for π_0 with respect to $\tilde{\psi}$.

Proof. The proof is analogous to the proof of Lemma 5.2. □

Proposition 5.6. *We have*

$$\gamma(s,\pi) = \omega_{E/F}(-1)\gamma(s,\pi_0,\tilde{\psi}).$$

Proof. The proof is analogous to the proof of Proposition 5.3. □

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