ADVANCES IN MATHEMATICS 41, 209–280 (1981)

On the Necessary Use of Abstract Set Theory

HARVEY FRIEDMAN*

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

In this paper we present some independence results from the Zermelo–Frankel axioms of set theory with the axiom of choice (ZFC) which differ from earlier such independence results in three major respects.

Firstly, these new propositions that are shown to be independent of ZFC (i.e., neither provable nor refutable from ZFC) form mathematically natural assertions about Borel functions of several variables from the Hilbert cube I^ω into the unit interval, or back into the Hilbert cube. As such, they are of a level of abstraction significantly below that of the earlier independence results.

Secondly, these propositions are not only independent of ZFC, but also of ZFC together with the axiom of constructibility ($V = L$). The only earlier examples of intelligible statements independent of $ZFC + V = L$ either express properties of formal systems such as ZFC (e.g., the consistency of ZFC), or assert the existence of very large cardinalities (e.g., inaccessible cardinals). The great bulk of independence results from ZFC—the ones that involve standard mathematical concepts and constructions—are about sets of limited cardinality (most commonly, that of at most the continuum), and are obtained using the forcing method introduced by Paul J. Cohen (see [2]). It is now known in virtually every such case, that these independence results are eliminated if $V = L$ is added to ZFC.

Finally, some of our propositions can be proved in the theory of classes, as formalized by the Morse–Kelley class theory with the axiom of choice for sets (MKC), but not in ZFC. MKC still formalizes only commonly accepted principles of mathematical reasoning. Thus these propositions provide examples of interesting theorems whose proofs necessarily involve the outer limits of what is commonly accepted as valid principles of mathematical reasoning.

The starting point for the development of these new propositions was our consideration, in 1976, of certain aspects of Cantor's basic theorem that the set of all real numbers is not countable. Thus given any sequence of real numbers, there is a real number which is not a term of the sequence. By using nested sequence of closed intervals with rational endpoints, it is easy to

* This research was partially supported by NSF Grant MCS 78-02558.

0001-8708/81/090209–72$05.00/0

construct a Borel function $F: \mathbb{R}^\omega \to \mathbb{R}$ such that for all $x \in \mathbb{R}^\omega$, $F(x)$ is not a coordinate of x.

Observe that the way such an F is constructed, the value of F at an infinite sequence $x \in \mathbb{R}^\omega$ depends very much on the order in which x is given.

Let $\sim$ be the equivalence relation on $\mathbb{R}^\omega$ given by $x \sim y$ if and only if y is obtained from x by permuting finitely many coordinates of x. We also consider the weaker equivalence relation $\approx$ given by $x \approx y$ if and only if $\mathrm{rng}(x) = \mathrm{rng}(y)$.

In 1976 we proved what we call the basic Borel diagonalization theorem, which says that if $F: \mathbb{R}^\omega \to \mathbb{R}$ is any Borel function such that $x \sim y \to F(x) = F(y)$, then there is an x such that $F(x)$ is a coordinate of x. The remarkable feature of the proof of this theorem (Theorem 3.1 in the text) is that it relies on essential use of the uncountable. This remark can be made precise in the following two ways.

Firstly, there is a standard set theory for dealing with countable sets only. This "countable set theory" is obtained from ZFC by deleting the power set axiom, and is written as $ZFC - \mathscr{P}$. In $ZFC - \mathscr{P}$ we cannot, of course, prove the existence of $\mathbb{R}$, but since any Borel function can be built up in a countably transfinite construction by means of sequential limit processes, there is no trouble treating Borel functions in $ZFC - \mathscr{P}$ by means of what are called *Borel codes*. The main point is that the basic Borel diagonalization theorem, when suitably formalized in $ZFC - \mathscr{P}$, cannot be proved in $ZFC - \mathscr{P}$, even if we use $\approx$ instead of $\sim$. The basic Borel diagonalization theorem can, however, be proved well within ZFC; e.g., in $ZFC - \mathscr{P} +$ "$\mathbb{R}$ exists."

The other way to make this precise is to develop a formal system in which all of the usual constructions on complete separable metric spaces can be made directly. Such a formal system is presented in the Appendix (*ECST*), and is strong enough to carry out the great preponderance of mathematics in a natural and direct way. Yet the basic Borel diagonalization theorem is given in the Appendix in standard mathematical terminology, using a Baire category argument on the space $(\underset{\sim}{\mathbb{R}})^\omega$, where $\underset{\sim}{\mathbb{R}}$ is the *discrete* topology on the reals, following a suggestion of Donald A. Martin. Note that use of the (non-separable) discrete topology on $\mathbb{R}$ is necessary in light of the unprovability from *ECST*.

Let Q be the Hilbert cube I^ω, which is the countably infinite product of the unit interval I, where I is given the usual topology and I^ω is given the infinite product topology. We can obviously transfer the basic Borel diagonalization theorem over to Borel functions from Q into I: for all Borel functions $F: Q \to I$ such that $x \sim y \to F(x) = F(y)$, we have $(\exists x)(F(x)$ is a coordinate of $x)$.

In this form, the basic Borel diagonalization theorem appears closely related to the Hewitt–Savage theorem: for all measurable functions $F: Q \to I$

such that $x \sim y \rightarrow F(x) = F(y)$, F is constant almost everywhere (see [8]). However, the Hewitt–Savage theorem is proved by a rather standard measure theoretic argument, which can be proved well within separable mathematics (e.g., as formalized by *ECST*). Observe that since almost no elements of Q have any given $a \in I$ as one of its coordinates, the following is an immediate consequence of the Hewitt–Savage theorem: for all Borel (in fact, measurable) functions $F: Q \rightarrow I$ such that $x \sim y \rightarrow F(x) = F(y)$, we have $(\exists x)(F(x)$ is *not* a coordinate of $x)$.

The new propositions independent of *ZFC* which are presented in Section 5, are far reaching extensions of the basic Borel diagonalization theorem, and can be motivated from basic Borel diagonalization by a few simple steps as follows.

First of all, there is the parameter form of Borel diagonalization which asserts the following. Let $F: Q \times Q \rightarrow I$ be a Borel function such that $y \sim z \rightarrow F(x, y) = F(x, z)$. Then for each x there is a y such that $F(x, y)$ is a coordinate of y. This is an immediate consequence of the basic Borel diagonalization theorem.

We next consider an iterated form of the above. Let $F: Q \times Q \rightarrow I$ be a Borel function such that $y \sim z \rightarrow F(x, y) = F(x, z)$. Then there exists an infinite sequence $\{x_k\}$ from Q such that $F(x_k, x_{k+1})$ is a coordinate of x_{k+1}. This can also be proved straightforwardly from basic Borel diagonalization.

Now consider the following variant of the iterated form. Let $F: Q \times Q \rightarrow I$ be a Borel function such that $y \sim z \rightarrow F(x, y) = F(x, z)$. Then there exists an infinite sequence $\{x_k\}$ from Q such that for $s < t$, $F(x_s, x_t)$ is a coordinate of x_{s+1}.

Now consider a multivariable form of the above. For $y, z \in Q^n$ let $y \sim z$ mean that there is a permutation ρ of ω which is the identity almost everywhere, such that each $z_i = y_i \circ \rho$. Thus we are using the diagonal action of the group of finite permutations of ω on Q^n. Let $F: Q \times Q^n \rightarrow I$ be a Borel function such that $y \sim z \rightarrow F(x, y) = F(x, z)$. Then there exists an infinite sequence $\{x_k\}$ from Q such that for $s < t_1 < \cdots < t_n$, $F(x_s, x_{t_1}, \ldots, x_{t_n})$ is a coordinate of x_{s+1}.[1]

This is almost our independent Proposition P. In Proposition P, we specify the coordinates as follows: let $F: Q \times Q^n \rightarrow I$ be a Borel function such that $y \sim z \rightarrow F(x, y) = F(x, z)$. Then there exists an infinite sequence $\{x_k\}$ from Q such that for $s < t_1 < \cdots < t_n$, $F(x_s, x_{t_1}, \ldots, x_{t_n})$ is the first coordinate of x_{s+1}.

We call Proposition P a *Borel Ramsey theorem* because for each fixed s, $F(x_s, x_{t_1}, \ldots, x_{t_n})$ is independent of the choice of $s < t_1 < \cdots < t_n$. This is similar to the situation in the usual Ramsey theorem for ω, which can be formulated as follows: Let $G: \omega^n \rightarrow [0, k]$. Then there is an infinite sequence

[1] It is necessary and sufficient to use $\omega + \omega$ iterations of the power set operation to prove this. This result will appear elsewhere.

$\{a_m\}$ such that for $t_1 < \cdots < t_n$, $r_1 < \cdots < r_n$, we have $G(a_{t_1},..., a_{t_n}) = G(a_{r_1},..., a_{r_n})$ (see [13]).

The first example of a Borel Ramsey theorem is due to Galvin. The following weak form of his theorem fits into the present exposition. Let J be a finite set, and let $F: \mathbb{R}^2 \to J$ be a Borel function. Then there is an infinite sequence $\{x_m\}$ from $\mathbb{R}$ such that for $p < q$, $s < t$, we have $F(x_p, x_q) = F(x_s, x_t)$. This result was extended to all finite exponents in an interesting way in [1]. These results are proved in separable mathematics, well within *ZFC*.

Propositions Q, R are variants of Proposition P. To prove P, Q, R we must go beyond *ZFC* and use Mahlo cardinals of arbitrarily high finite order. The Mahlo cardinals of order 0 are just the inaccessible cardinals, and these already go beyond *ZFC*. The Mahlo cardinals of order $n+1$ are those cardinals every closed and unbounded subset of which contains a Mahlo cardinal of order n. As shown in Section 5, Propositions P–R can be proved in $ZFC + (\forall n)(\exists \kappa)$ (κ is a Mahlo cardinal of order n), but cannot be proved in $ZFC + (\exists \kappa)$ (κ is a Mahlo cardinal of order $\bar{n}$), for any specific n. For a more penetrating discussion of what it means to "require Mahlo cardinals of arbitrarily high finite order," see the discussion in Section 1.

We now return to the beginning of this Introduction where we listed three ways these independence results differ from earlier ones.

With regard to the first point, we take the point of view that non-set-theoretic mathematics is characterized by the use of sequential limit processes to construct objects. Set theoretic mathematics is characterized by the use of other means for constructing objects such as quantification over an uncountable domain (as in the construction of analytic sets), or by the consideration of arbitrary subsets of an uncountable set regardless of how they are constructed.

From this point of view, probably the construction of arbitrary Borel sets of reals would be regarded as on the set theoretic side, but Borel sets of reals of *finite rank* are definitely on the non-set-theoretic side. (The Borel sets of rank $\leqslant 0$ are the open and closed sets. The Borel sets of rank $\leqslant n+1$ are the countable unions and countable intersections of Borel sets of rank $\leqslant n$.) Similarly, arbitrary Borel functions on $\mathbb{R}$ would probably be regarded as on the set theoretic side, but functions on $\mathbb{R}$ obtained at a *finite level* of the Baire hierarchy are definitely on the non-set-theoretic side. (The functions in Baire class 0 are the continuous functions. The functions in Baire class $n+1$ consist of the pointwise limits of pointwise convergent sequences of functions in Baire class n.) We mostly consider Borel sets and Borel functions on spaces other than $\mathbb{R}$ such as $Q = I^\omega$, but this discussion carries over to such spaces without change. We use "finitely Borel set" and "finitely Borel function" for restrictions to the finite levels of the Borel hierarchy of sets and the Baire hierarchy of functions.

Propositions P–R provide examples of propositions from non-set-theoretic mathematics (provided they are restricted to finitely Borel functions) whose proofs require not only use of set theoretic mathematics, but substantial extensions of *ZFC* by means of Mahlo cardinals of finite order.

Even the first example of a mathematically interesting proposition from non-set-theoretic mathematics whose proof requires use of some set theoretic mathematics, is relatively recent. This example is from infinite game theory, and is referred to as Borel determinacy. In 1967, Martin [9] proved Borel determinacy using some "large cardinals" going well beyond *ZFC*, and in fact well beyond the Mahlo cardinals that we use here (he used what are called Ramsey cardinals). In 1968, we [3] showed that any proof of Borel determinacy requires use of uncountably many iterations of the power set operation. ($V(0)=\varnothing$, $V(\alpha+1)=\mathscr{P}(V(\alpha))$, $V(\lambda)=\bigcup_{\alpha<\lambda} V(\alpha)$. $V(\alpha)$ is αth iterate of the power set operation.) In 1974, Martin [10] proved Borel determinacy using exactly uncountably many iterations of the power set operation. In Section 1, there is a detailed discussion of the meaning of "necessary use of ω_1 iterations of the power set operation."

In light of our emphasis on finitely Borel sets, we state that Martin proved finitely Borel determinacy using exactly $\omega+\omega$ iterations of the power set operation, and we proved that $\omega+\omega$ iterations are necessary.

In Section 2 we obtain these same results for some new propositions that are closely related to Borel determinacy, but whose formulation does not involve game theory. The statements are entirely natural and straightforward (e.g., every symmetric Borel set in $I \times I$ contains or is disjoint from the graph of a Borel function.)

In Section 3 we prove the Borel diagonalization theorem for arbitrary Borel equivalence relations using ω_1 iterations of the power set operation, as an application of both Borel determinacy and the methods of [3]. We also show that ω_1 iterations are necessary. If everything is restricted to the finitely Borel, then it is necessary and sufficient to use $\omega+\omega$ iterations of the power set operation. Thus the Borel diagonalization theorem for Borel equivalence relations shares the same basic metamathematical properties as the earlier Borel determinacy. One difference is that Borel diagonalization for Borel equivalence relations is π^1_2, whereas Borel determinacy is π^1_3.

In Section 4 we present some more non-set-theoretic propositions which share the same basic metamathematical properties as the above. These propositions are certain fixed point type theorems for Borel functions defined on a Borel quasi order. Again these are proved as an application of Borel determinacy and the methods in [3].

The use of (finitely) Borel sets and (finitely) Borel functions as a working model for non-set-theoretic mathematics has interesting consequences for many of the existing set theoretic independence results. We consider two examples: the continuum hypothesis, and Souslin's hypothesis.

The continuum hypothesis asserts that every set of real numbers either can be mapped into the natural numbers or mapped one-one onto the set of all real numbers. The use of arbitrary sets of real numbers regardless of how they are constructed, and also the use of arbitrary one-one mappings onto the set of all real numbers, regardless of how they are constructed, are characteristic of set theoretic statements. Now we can put the continuum hypothesis into the Borel world as follows: Every Borel set of real numbers either can be mapped into the natural numbers by a Borel function, or can be mapped one-one onto the set of all real numbers by a Borel function. However, this new form of the continuum hypothesis obtained by relativizing to the Borel world, is a classical theorem of descriptive set theory proved well within ZFC (it follows from the theorem that every uncountable Borel set of reals contains a perfect subset). Furthermore, this is true if "finitely Borel" is used throughout.

The Souslin hypothesis asserts that for every dense linear ordering, either there is a countable dense subset or there is an uncountable set of points no element of which is a limit point of the remaining elements. We put Souslin's hypothesis into the Borel world as follows. A *Borel linear* ordering is a linear ordering whose field of points is $\mathbb{R}$ and whose relation is a Borel subset of $\mathbb{R} \times \mathbb{R}$. The new form of Souslin's hypothesis asserts that for every dense Borel linear ordering, either there is a countable dense subset or there is an uncountable Borel set of reals (in fact, a perfect set of reals) no element of which is a limit point (under the ordering) of the remaining elements. This is a joint result with S. Shelah (see [4]), and is proved using a technique of L. Harrington introduced in [6].

With regard to the second point, we regard $V = L$ not so much as an axiom, but rather as coming out of a specialization of the set concept. L consists of the class of all sets which can be built up in a transfinite hierarchy in which at any stage, only those sets which can be explicitly defined over the class of sets introduced at earlier stages, are introduced. Specialization of the set concept to L is attractive for several reasons.

Firstly, all of the usual axioms of set theory (i.e., ZFC) can be proved to hold in L. So nothing really is lost in terms of ordinary mathematical activity. Even the axiom of choice can be proved to hold in L, just using ZF.

Secondly, the specialization to L is based on a natural and coherent idea, and is not simply some ad hoc or artificial restriction.

Thirdly, there has been no convincing proof of the existence of a set that is not constructible (i.e., not in L). The only proposals involve hypotheses like the existence of measurable cardinals, which certainly are not evident. The consistency of ZFC with the existence of a measurable cardinal is perhaps more compelling, but this does not suffice to prove the existence of non-constructible sets.

Fourthly, it now appears that all of the independence results from ZFC

obtained through forcing are eliminated if one specializes to L. For example, the continuum hypothesis becomes a theorem of ZFC when relativized to L, and the Souslin hypothesis becomes refutable in ZFC when relativized to L. As far as the earlier independence results are concerned, specialization to L leaves only statements about formal systems of set theory (such as the consistency of ZFC), or the existence of large cardinals such as inaccessible or Mahlo cardinals (but not the much larger ones such as measurable cardinals), or artificial statements, as independent of ZFC.

Since our Borel Ramsey propositions remain independent of ZFC even when relativized to L, the advantages of specializing to L are thus weakened.

There is a further specialization of the set concept which also makes sense. The idea is to restrict to those sets which are "forced to exist by the axioms of ZFC." This is the so called *minimum model* of ZFC, and is defined as the least transitive class satisfying the axioms of ZFC. The corresponding axiom would assert that the universe, V, is the minimum model of ZFC. Let us denote the minimum model by L^*.

Strictly speaking, the stated definition of L^* cannot be given in ZFC. We modify it for use in ZFC as follows. L^* is the least transitive *set* satisfying ZFC if there is one; L otherwise. (This is appropriate since L is the least transitive class containing all ordinals and satisfying ZFC.)

If we specialize to L^*, then the fourth point above can be strengthened: Even the existence of large cardinals such as inaccessible or Mahlo cardinals, when relativized to L^*, do not remain independent of ZFC (they become refutable). However, the independence results in Section 5 remain in force when specialized to L^*, or even other more severe specializations.

With regard to the third point made at the beginning of this Introduction, Proposition R for $n = 4$, $m < \omega$ can be proved in MKC but not in ZFC (or even $ZFC + V = L$). We do not know if Proposition P for $n = 3$ or 4, $m < \omega$, has this property, or whether any natural restriction of Proposition P or Q has this property.

This result has an advantage over the independence results for unrestricted P, Q, and R. We can prove in MKC that Proposition R for $n = 4$, $m < \omega$ is independent of ZFC. However, in order to prove that Propositions P, Q, R are independent of ZFC, we need to assume that $ZFC + (\forall n)(\exists \kappa)$ (κ is n-Mahlo) is consistent (in order to show nonrefutability of P, Q, R in ZFC; for non-provability, we need only Con(ZFC)).

There have been earlier results which give natural mathematical examples where it is necessary and sufficient to use large cardinals to give a proof. The most concrete example of such is analytic determinacy, which is proved from Ramsey cardinals in [9]. In [7] analytic determinacy is shown to be equivalent to $(\forall x \subset \omega)(x^{\#}$ exists), which establishes the necessity of using substantial large cardinals to prove it. Note that analytic determinacy differs from our Borel Ramsey theorems in that (a) it is on the set theoretic side of mathematics because of its use of analytic sets, (b) it is refutable when specialized to L, and (c) unlike Proposition R for $n = 4$, $m < \omega$, we cannot prove it to be true or prove it to be nonrefutable in ZFC using the commonly accepted principles of mathematical reasoning.

The major shortcoming of this work is that, despite the elegance and simplicity of our Borel Ramsey propositions, they had not actually existed in the mathematical literature up to now. It would, of course, be more striking to have examples of this new incompleteness phenomena come from the existing mathematical literature. The well known examples of set theoretic independence results did come from the existing mathematical literature.

However, there are many reasons to believe that the situation with regard to non-set-theoretic independence results is quite different from the situation with regard to set theoretic independence results. For example, one of the very first questions raised in abstract set theory—the continuum hypothesis—turns out to be independent of ZFC. Also, there were many set theoretic candidates for independence results, where the obstacles to settling them seemed similar or related. In addition, for some time before the invention of forcing by P. J. Cohen in 1962, there was some conceptual basis for approaching set theoretic independence results. Specifically, it was recognized that one should start with a countable transitive model of $ZFC + V = L$ (or a fragment of $ZFC + V = L$) and add new objects in such a manner as to keep the same ordinals and preserve the axioms of ZFC. Thus even without knowing how to add the new elements, one could conceptualize having thrown in many new real numbers so that the continuum hypothesis might fail in the extension. No remotely similar conceptual basis seems to exist for nonset-theoretic independence results, partly because the ordinals cannot be preserved. One seems forced to consider non-standard models (one must if the independent statement is, like ours, π^1_2). In addition, no one to my knowledge has suggested that they have any intuition which would point to a candidate for a non-set-theoretic independence result from the existing literature.

Nevertheless, we feel that the examples given here are of sufficient simplicity and are based on such fundamental theorems as the uncountability of the reals and Ramsey's theorem, that they suggest the probability of a diverse collection of further non-set-theoretic independence results, some of which should be at least very closely tied to the existing mathematical literature.

In 1977, Jeffrey Paris and Leo Harrington gave an example of a mathematical statement in finite set theory which cannot be proved in finite set theory. Their example is a new finite form of the usual infinite Ramsey theorem. The finite form is π^0_2 and the infinite form is π^1_2 (see [12]). By building on their work, we have constructed π^0_2 statements which bear the same relationship to Propositions C and P as their π^0_2 statements have to the infinite Ramsey theorem. This provides mathematical π_2^0 statements equivalent, respectively, to the 1-consistency of Z_2, and of $ZFC + \{(\exists\kappa)$ (κ is $\bar{n}$-Mahlo)$\}_n$. These statements are somewhat more complicated than those of [12]. This work will appear elsewhere.

1. Some Basic Formal Systems

The axiom system ZFC is the usual formulation of the commonly accepted principles of mathematical reasoning in terms of set theory. It is the one-sorted theory based on $\in$, $=$ among sets, whose nonlogical axioms are:

a. Extensionality. $(\forall a)(a \in x \leftrightarrow a \in y) \to x = y$.

b. Pairing. $\{a, b\}$ exists.

c. Union. $\{a: (\exists b \in x)(a \in b)\}$ exists.

d. Infinity. There is a set ω such that $\varnothing \in \omega \,\&\, (\forall x \in \omega)(x \cup \{x\} \in \omega)$.

e. Power set. $\{a: a \subset x\}$ exists.

f. Separation. $\{a \in x: \varphi(a)\}$ exists for any formula φ with possibly additional free variables.

g. Foundation. Every nonempty set has an $\in$-minimal element.

h. Replacement. $(\forall a \in x)(\exists! b)(\varphi(a, b)) \to (\exists f)(f$ is a function $\&$ $(\forall a \in x)(\varphi(a, f(a))))$, where φ is any formula in which f is not free. (Here functions are treated as being reduced to sets.)

i. Choice. Every set of nonempty sets has a choice function.

The fragment consisting of axioms a–f is referred to as the Zermelo set theory, written Z. ZF consists of a–h. ZC is Z with choice.

The constructible hierarchy of sets is defined by transfinite recursion on ordinals as follows: $L(0) = \varnothing$; $L(\alpha + 1) = \{x \subset L(\alpha): x$ is definable over $L(\alpha)$ by a formula of set theory with parameters allowed from $L(\alpha)\}$; and for limit ordinals λ, $L(\lambda) = \bigcup_{\alpha > \lambda} L(\alpha)$. L is the class of constructible sets; i.e., $L = \bigcup_\alpha L(\alpha)$. V is the class of all sets. $V = L$ is the so called axiom of constructibility, which asserts that every set is constructible.

We also consider the axiom system MKC (the Morse–Kelley theory of classes with choice for sets) which is the usual formulation of the commonly accepted principles of mathematical reasoning in terms of class theory. This is the one-sorted theory, with variables ranging over classes (of sets), $\in$, $=$ between classes, and the special unary predicated symbol $S(x)$ meaning "x is a set." Every object is a class.

The nonlogical axioms are as follows.

a. Extensionality.

b. Every element of a class is a set, every subset of a set is a set.

c. Pairing for sets. $\{a, b\}$ exists as a set for sets a, b.

d. Union for sets. $\{a: (\exists b \in x)(a \in b)\}$ exists as a set for sets x.

e. Infinity. There is a *set* ω as in axiom d of *ZFC*.

f. Power set for sets. $\{a: a \subset x\}$ exists as a set for sets x.

g. Class separation. $\{a: S(a) \& \varphi(a)\}$ exists as a class, for any formula φ with possibly additional free variables.

h. Foundation for sets. Every nonempty set has an $\in$-minimal element.

i. Replacement for classes. Every class function (i.e., a class of ordered pairs of sets) whose domain is a set, has the property that its range is a set.

j. Choice for sets. Every set of nonempty sets has a choice function.

This system *MKC* is stronger than another theory of classes called *VBC*, (the Von Neumann–Bernays theory of classes with choice for sets), where in class separation, all quantifiers ocurring in φ must be restricted to sets.

At various points in the paper, we wish to consider systems such as the above with the power set axiom, $\mathscr{P}$, removed; e.g., we write $ZF - \mathscr{P}$ for *ZF* without the power set axiom.

By weak *Z* we mean the system which is the same as *Z* except that in the separation axiom scheme, all quantifiers in the formula φ must be bounded to sets. These are the so called Δ_0-formulas, and are defined inductively by (a) all atomic formulas are Δ_0, (b) Δ_0-formulas are closed under propositional combinations, and (c) if φ is Δ_0 then so are $(\forall x)(x \in y \rightarrow \varphi)$, $(\exists x)(x \in y \& \varphi)$.

One way of classifying fragments of *ZF* is according to how much of the cumulative hierarchy of sets can be proved to exist. The cumulative hierarchy of sets is defined by transfinite recursion on ordinals as follows: $V(0) = \varnothing$, $V(\alpha + 1) = \mathscr{P}(V(\alpha))$, $V(\lambda) = \bigcup_{\alpha<\lambda} V(\alpha)$, for limit ordinals λ. $V(\alpha)$ is also called the αth iteration of the power set operation.

On this criterion, *Z* proves that for all n, $V(\omega + n)$ exists. However, weak *Z* proves only that each specific $V(\omega + n)$ exists, but not $(\forall n)(V(\omega + n)$ exists).

One way of classifying the inherent set theoretic content of theorems of *ZFC* is according to how much of the cumulative hierarchy of sets is needed in order to prove them. There are several difficulties involved in making this precise, which we address now.

First of all, the ordinal α such that all of the $V(\beta)$, $\beta < \alpha$, are needed to prove the theorem, may not be definable. Even if it is, whether or not the $V(\beta)$, $\beta < \alpha$, are sufficient to prove the theorem may depend on the choice of definition of α.

Even if this first problem is solved, because of a canonical definition of α, how can we ever say that the $V(\beta)$, $\beta < \alpha$, are needed to prove the theorem? There may be some other axiomatization which contains principles not directly connected with lengths of iterations of the power set operation, which prove the theorem, yet does not prove the existence of many iterations of the power set operation.

There are two cases that come up here for theorems of ZFC. When is it necessary and sufficient to use $\omega + \omega$ iterations of the power set operation? And when is it necessary and sufficient to use ω_1 iterations of the power set operation? For our purposes, we will assume that the theorem, φ, is about Borel sets and Borel functions (so that it can be formalized in the language of second order arithmetic).

Clearly if φ is provable in ZC then it is sufficient to use $\omega + \omega$ iterations of the power set operation. We say that $\omega + \omega$ iterations of the power set operation are necessary to prove φ, if weak Z can be translated into $ZFC - \mathscr{P} + \varphi$.

The point of this condition on φ is that any "reasonable" system which proves φ will have any specified $\omega + n$ iterations of the power set operation present, disguised in a translation. Thus even though the "reasonable" formal system which proves φ may not directly have these iterations of the power set operation, it must have them indirectly.

The case of ω_1 iterations of the power set operation is more delicate. For sufficiency, we use the following countable rank axiom, CRA: Let (ω, R) be a well ordering. Then there is a function f on ω such that $f(n) = (x\colon (\forall y \in x)(\exists m)(R(m, n) \;\&\; y \in f(m))\}$. Thus we say that ω_1 iterations of the power set operation are sufficient to prove φ if φ is provable in $ZC + CRA$.

However, since obviously only countable many countable ordinals can be defined in any way, there is a new problem in making sense out of saying that ω_1 iterations of the power set operation are necessary.

It is tempting to weaken the CRA to the scheme which asserts that for each explicitly set theoretically definable well ordering (ω, R), the cumulation function f exists. The difficulty with this is that this "weakened" CRA is actually equivalent to CRA over $V = L$ (take the first well ordering in the constructible hierarchy on which there is no cumulation function). Because of the absoluteness properties of the examples given here, the ones which are provable using CRA are also provable using this weakened form of CRA.

Instead we weaken CRA to assert the existence of cumulation functions only for (ω, R) which are definable in certain restricted ways. Actually, it turns out that sharp formulations require use of rules of inference rather than axiom schemes.

In the first rule of inference that we use, if $(\exists! R)((\omega, R)$ is a well ordering and $\psi(R))$ has been derived, where ψ is a Σ^1_2 formula with no other free variables, then derive $(\exists f)$ (f is a cumulation function on (ω, R)). Call this rule $\mathscr{R}_1$.

$\mathscr{R}_1$ is more than enough to cover natural descriptions of countable ordinals in our context. However, in the case of Section 2 we can be yet more general. Rule $\mathscr{R}_2$ is the same as $\mathscr{R}_1$ except that we allow Σ_3^1 formulas. For set theories T, $T + \mathscr{R}_1$ and $T + \mathscr{R}_2$ represent the closure of T under the rules $\mathscr{R}_1$, $\mathscr{R}_2$.

Finally, necessity of use of ω_1 iterations of the power set operation will mean the following for our purposes: that there is a translation of $ZC + \mathscr{R}_1$ (or $ZC + \mathscr{R}_2$) into $ZFC - \mathscr{P} + \varphi$.

To verify this condition for our examples, we use T_α-models, $\alpha < \omega_1$. A T_α-model is a transitive set A in $V(\alpha + 1)$ such that for any formula φ with parameters in A and ordinal $\beta < \alpha$, $\{x \in V(\beta) : A \models \varphi(x)\} \in A$. It is easily verified that any $T_{\omega+\omega}$-model satisfies Z. Obviously, we can define T_R-models in an entirely analogous manner for well orderings (ω, R). This must be done in systems which do not prove that every countable well ordering is isomorphic to an ordinal (such as ZC).

A T_α-model is called *well founded absolute* if every linear ordering on ω that is satisfied to be a well ordering, really is a well ordering.

THEOREM 1. *The following is provable in $ZF - \mathscr{P}$.* (a) *If for every $\alpha < \omega_1$ there is a T_α-model, then $ZC + V = L + \mathscr{R}_1$ is consistent (in fact, has an ω-model)*; (b) *if for every $\alpha < \omega_1$ there is a well founded absolute T_α-model, then $ZC + V = L + \mathscr{R}_2$ is consistent (in fact, has an ω-model).*

Proof. We argue inside L, using $ZF - \mathscr{P}$. Let x be the complete Σ_3^1 set of integers, and let α be the union of all Δ_3^1 ordinals.

For (a), let M be a $T_{\alpha+\omega}$-model. By developing L within M, we pass to another $T_{\alpha+\omega}$-model A, which satisfies $ZC + V = L$. It is clear that $x \in A$. We show by induction on the number of applications of $\mathscr{R}_1$ that every theorem of $ZC + \mathscr{R}_1$ holds in A. Assume that $ZC + V = L + \mathscr{R}_1 \vdash (\exists! R)((\omega, R)$ is a well ordering & $\psi(R))$, where ψ is Σ_2^1. Rewrite this as $ZC + V = L + \mathscr{R}_1 \vdash (\exists! R)(\exists y)(\rho(R, y))$, where ρ is π_1^1, and also assume that $A \models (\exists! R)(\exists y)(\rho(R, y))$. Now $(\exists! R)(\exists y)(\rho(R, y))$ is true. By the choice of x, there are $(R, y) \leqslant_T x$ such that $\rho(R, y)$. Since the length of R is below α, $A \models$ "a cumulation hierarchy exists on R."

For (b), let M be a well founded absolute $T_{\alpha+\omega}$-model. By developing L within M, we pass to another well founded absolute $T_{\alpha+\omega}$-model A, which satisfies $ZC + V = L$. The remainder of the argument is the same as for (a), except ρ is π_2^1. So we must invoke well founded absoluteness to obtain $A \models \rho(R, y)$. This completes the proof of the theorem.

This also completes our discussion of "necessary and sufficient use of ω_1 iterations of the power set operation."

In Section 5 we present the Borel Ramsey theorems, which are independent of $ZFC + V = L$. We say that "it is necessary to go beyond ZFC in order to prove φ," for these propositions φ. This is made precise as follows: there is a translation from ZFC into $ZFC - \mathscr{P} + \varphi$, yet no translation exists from $ZFC - \mathscr{P} + \varphi$ into ZFC.

We use so called Mahlo cardinals of finite order (n-Mahlo) to prove some of the Borel Ramsey theorems. We say that "it is necessary and sufficient to use Mahlo cardinals of arbitrarily high finite order to prove φ" if φ can be proved in $ZFC + (\forall n)(\exists \kappa)$ (κ is n-*Mahlo*), yet for all n, $ZFC + (\exists \kappa)$ (κ is $\bar{n}$-Mahlo) can be translated into $ZFC - \mathscr{P} + \varphi$.

Another system that frequently arises is that of second order arithmetic, written Z_2. This is the two-sorted theory with numerical variables and set (of natural number) variables, with $=$ among numerical variables, as well as 0, 1, $+$, $\cdot$, and with $\in$ between numerical and set variables. The axioms consist of the usual axioms for 0, 1, $+$, $\cdot$, the induction scheme for all formulas in the language, and the comprehension scheme $(\exists x)(\forall n)(n \in x \to \varphi)$, for all formulas φ in the language without x free. Z_2 is intertranslatable with $ZF - \mathscr{P}$, and any π^1_3 sentence provable in $ZF - \mathscr{P}$ (in fact even in $ZFC + V = L - \mathscr{P}$) is provable in Z_2.

We now come to the formal treatment of Borel functions $F\colon Q \to I$, where I is the closed unit interval and Q is the Hilbert cube I^ω. The other cases that come up here ($F\colon Q^n \to I$, $F\colon Q^n \to Q$) are obviously reducible to this case. In systems containing at least Z, we can use the most set theoretic treatment: Define the Borel subsets of Q as comprising the least σ-algebra of subsets of Q generated by the open subsets of Q (under the separable product topology of I^ω). The Borel functions $F\colon Q \to I$ are those functions where inverse images of open sets in I are Borel. Alternatively, we can adopt the Baire approach and define the Borel functions $F\colon Q \to I$ as comprising the least set Y of functions from Q into I which include all continuous functions, and which are closed under pointwise limits of sequences of such functions (i.e., if $\lim_n F_n = F$ and each $F_n \in Y$, then $F \in Y$). These definitions come out to be the same.

In order to prove standard facts about Borel functions and Borel sets defined in this manner, some form of axiom of choice must be used. Otherwise, it is known that one may have a decomposition of $\mathbb{R}$ into countably many countable sets, in which case every set and every function would be Borel. However, the choice needed to make sense out of the set theoretic definition of Borel is very weak and natural, and is just what is needed in certain other contexts. Namely, the countable axiom of choice, which asserts that any countable set of nonempty sets has a choice function, and is written AC_ω. For instance, AC_ω is good enough to prove the Lebesgue measurability of Borel functions. It is also good enough to prove the equivalence of the set theoretic definition with the less set theoretic definition given in the Appendix in terms of *Borel codes.*

In systems which do not discuss or fully discuss sets of sets of reals, such as the system *ECST* of the Appendix, we cannot use the above definitions of Borel sets and Borel functions. Furthermore, in systems as $ZF - \mathscr{P}$, we do not even have $\mathbb{R}$ as an object, and so even more care has to be taken.

In the case of $ZF - \mathscr{P}$, we identify Borel sets (in Q) with their recipes for membership which, as discussed in the Appendix, are given by labelled well found trees of finite sequences of natural numbers. Actually in $ZF - \mathscr{P}$ it is somewhat cumbersome to define Borel functions from Borel sets, and so we develop codes for Borel functions directly, again by means of labelled well founded trees of finite sequences of natural numbers. We require that topmost nodes be labelled with the restriction of a continuous function from Q to a countable dense set, and that any node which is not topmost has infinitely many immediate successors.

Given such a labelled well founded tree T, we must indicate how we use it to define a function from Q into I; i.e., how to produce a value in I given $x \in Q$. Naturally, this is done by transfinite recursion on T. To the topmost nodes of T, assign the value of the unique uniformly continuous extension of the label at x. At other nodes of T, assign the lim sup of the numbers assigned to the immediate successors (arranged from left to right). Finally, the value that we want is the value assigned to the root of T.

Such labelled well founded trees are called *Borel codes*, and are used extensively here. Under this treatment, no use of the axiom of choice is needed to argue about Borel sets and Borel functions.

A Borel set of rank $\leqslant\alpha$ is a Borel set with a Borel code whose ordinal is $\leqslant\alpha$. A Borel function of rank $\leqslant\alpha$ is a Borel function with a Borel code of ordinal $\leqslant\alpha$. The rank of a Borel set of Borel function is the least α such that it is of rank $\leqslant\alpha$.

We use the phrase *finitely Borel* whenever the Borel rank is finite. The finitely Borel functions under this definition are the same as those present in the Baire classes n, $n < \omega$, as given in the Introduction.

2. Borel Selection Theorems

In 1967 Martin proved Borel determinacy (in fact, analytic determinacy) from certain large cardinal hypotheses (see [9]). In 1968 we proved that uncountably many iterations of the power set operation are needed in order to prove Borel determinacy (see [3]). In 1974 Martin proved Borel determinacy using exactly uncountably many iterations of the power set operation (see [10]).

In this section we present selection theorems which are consequences of Borel determinacy, and which are significantly closer to ordinary analysis.

Here are four forms of these selection theorems, in two propositions.

A set E of ordered pairs is *symmetric* if $(x, y) \in E$ if and only if $(y, x) \in E$. Let I be the unit interval $[0, 1]$. Let K be the Cantor set (as a subset of I).

PROPOSITION A. *Every symmetric Borel subset of $I \times I$ contains or is disjoint from the graph of a Borel function on I (left continuous function on I; closed set which meets every vertical line in $I \times I$).*

PROPOSITION B. *Every symmetric Borel subset of $K \times K$ contains or is disjoint from the graph of a continuous function.*

Proposition B is an immediate consequence of Borel determinacy as follows. We can assume that K is represented as the space of infinite sequences of 0's and 1's. If $E \subset K \times K$ is a symmetric Borel set then play the infinite game where player II wins if and only if the ordered pair of completed plays is in E. If player II has a winning strategy then that strategy defines a continuous function F such that $(\forall x \in K)((x, F(x)) \in E)$. If player I has a winning strategy then that strategy defines a continuous function G such that $(\forall y \in K)((G(y), y) \notin E)$, and so $(\forall x \in K)((x, G(x)) \notin E)$.

For Proposition A, let $E \subset I \times I$ be a symmetric Borel set. It is immediate from Proposition B, using a Borel correspondence between I and K, that E contains or is disjoint from the graph of a Borel function. To obtain the two stronger conclusions of Proposition A, we actually show that E contains or is disjoint from the topological closure of the graph of a left continuous function whose right limits exist.

We apply Borel determinacy as follows. Players I and II alternately play 0's and 1's, and their infinite sequences of plays are viewed as infinite decimal expansions in base 2 notation. Player II wins if and only if the resulting pair of points in I is in E. If player II has the winning strategy then let $F: I \to I$ be defined by taking $F(x)$ to be the result of II's strategy if I plays the base 2 expansion of x for x not of the form $p/2^q$; otherwise use the base 2 expansion ending in an infinite sequence of 1's (for $x = 0$ use $\bar{0}$). It is easy to see that $\lim_{x \to a-} F(x) = F(a)$, and $\lim_{x \to a+} F(x)$ is the result of II's strategy if I plays the base 2 expansion of a for a not of the form $p/2^q$; otherwise use the base 2 expansion ending in an infinite sequence of 0's (for $a = 1$ use $\bar{1}$). Since F is left continuous and right limits exist, the topological closure of the graph of F is $\{(a, f(a)) : a \in I\} \cup \{(a, \lim_{x \to a+} f(x)) : a \in I\}$, which is included in E. Argue analogously if player I has the winning strategy.

We now wish to prove that these propositions require uncountably many iterations of the power set operation to prove, in the sense of Section 1.

We first work in $ZF - \mathscr{P}$ and show that for each countable limit ordinal λ, if every symmetric Borel subset of $I \times I$ of Borel rank $< \lambda$ contains or is disjoint from the graph of a Borel function, then there is a countable T_α-model for every $\alpha < \lambda$. The proof is a modification of the proof given in [3] that for each countable limit ordinal λ, if all Borel sets of rank $< \lambda$ are determined, then there is a countable T_α-model for every $\alpha < \lambda$. (Actually, the proof is given in detail for only the case $\alpha = \omega + \omega$ in [3].) It is like Martin's modification of [3] where he proves the independence of Σ_4^0-determinacy from second order arithmetic (improving on our Σ_5^0) by applying determinacy directly to models, instead of using Turing degree determinacy as in [3] (see [11]).

Within $ZF - \mathscr{P}$ build the hierarchy of constructible sets $L(0) = \varnothing$, $L(\alpha + 1) = \{x \subset L(\alpha) : x$ is first order definable over $(L(\alpha), \in)$ with parameters$\}$, $L(\lambda) = \bigcup_{\alpha < \lambda} L(\alpha)$. We can also build the cumulative hierarchy $V(0) = \varnothing$, $V(\alpha + 1) = \mathscr{P}(V(\alpha))$, $V(\lambda) = \bigcup_{\alpha < \lambda} V(\alpha)$, but we cannot even prove that $V(\omega + 1)$ exists in $ZF - \mathscr{P}$. We say that β is *α-small* if and only if for all $\gamma \leqslant \beta$, $L(\gamma + 1) \cap V(\alpha) \neq L(\gamma) \cap V(\alpha)$. Note that we cannot even prove the existence of a non-$(\omega + 1)$–small ordinal, since the least such β would immediately give rise to a $T_{\omega+1}$-model.

Actually, we will use the following relativized form: an ordinal β is α–small relative to $u \subset \omega$ if for all $\gamma \leqslant \beta$, $L(\gamma + 1, u) \cap V(\alpha) \neq L(\gamma, u) \cap V(\alpha)$. (Here $L(\gamma, u)$ is the $(\gamma + 1)$st level of the constructible hierarchy relative to u.)

Let $\omega_1^{L[u]}$ be the first ordinal which is uncountable according to $L[u]$. Note that even ω_1^L may be a proper class in $ZF - \mathscr{P}$.

Until further notice, fix $\lambda < \omega_1$ and assume that every symmetric Borel subset of $I \times I$ of rank $< \lambda$ contains or is disjoint from the graph of a Borel function. Fix $\alpha < \lambda$, and a code $u \subset \omega$ for α. We wish to prove (in $ZF - \mathscr{P}$) that there is a countable T_α-model. Since the existence of a non-β–small ordinal relative to u implies the existence of a countable T_β-model within $ZF - \mathscr{P}$, without loss of generality we may assume that all ordinals are $(\omega + \alpha + 1)$–small relative to u.

LEMMA 1. *If $\beta < \omega_1^{L[u]}$ then there is a $\beta < \gamma < \omega_1^{L[u]}$ such that γ is a u-admissible limit of u-admissibles, and $L(\gamma, u)$ has every element definable from u.*

Proof. Suppose this is false and let β be the sup of all u-admissible limits of u-admissibles $\gamma < \omega_1^{L[u]}$ such that $L(\gamma, u)$ has every element definable from u. By hypothesis, $\beta < \omega_1^{L[u]}$. Note that β is Δ_3 in $L(\omega_1^{L[u]}, u)$ relative to u. Let γ be the sup of all ordinals Δ_3 in $L(\omega_1^{L[u]}, u)$ relative to u. Then it is easily seen that γ is a u-admissible limit of u-admissibles, and every element of $L(\gamma, u)$ is definable from u, which is a contradiction.

LEMMA 2. *If $\beta < \omega_1^{L[u]}$ then there is a $\beta < \delta < \omega_1^{L[u]}$ such that δ is a successor u-admissible, and $L(\delta, u)$ has every element definable from u.*

Proof. Choose γ as in Lemma 1. Observe that γ is definable in $L(\delta, u)$ relative to u, where δ is the next u-admissible after γ, as the largest u-admissible ordinal. By a Skolem hull argument, every element of $L(\delta, u)$ is definable over $L(\delta, u)$ from γ and elements of $L(\gamma, u)$. Hence $L(\delta, u)$ has every element definable from u.

We now define the class K of all models (ω, R) of $KP + V = L[u]$ whose well founded part has ordinal $>\omega + \alpha$, and which satisfies "every ordinal is $(\omega + \alpha + 1)$–small relative to u." Let (ω, R), $(\omega, S) \in K$. We wish to compare them. Consider the set of all pairs (a, b) such that a is an ordinal (ω, R), b is an ordinal in (ω, S), and every element of the $L(a, u) \cap V(\omega + \alpha + 1)$ of (ω, R) is (equivalent to) an element of the $L(b, u) \cap V(\omega + \alpha + 1)$ of (ω, S), and vice versa. We can cross identify objects of rank $\leqslant \omega + \alpha$ because $\omega + \alpha$ is standard.

It is easily seen that this set of ordered pairs is a partial order preserving function f from the ordinals of (ω, R) into the ordinals of (ω, S). Let g be the largest restriction of f which maps an initial segment of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S).

We say that (ω, S) is *longer* than (ω, R) if and only if the above g maps all of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S) determined by an ordinal of (ω, S), or the domain of g is a proper initial segment of the ordinals of (ω, R) that is not determined by an ordinal of (ω, R) and the range of g is either all of the ordinals of (ω, S) or is an initial segment of the ordinals determined by an ordinal of (ω, S).

We are now prepared to apply the weak form of Proposition A for symmetric Borel subsets of $I \times I$ of rank $< \lambda$.

LEMMA 3. *If $E \subset I \times I$ is a Borel set of rank $< \lambda$ then either there is a Borel function on I whose graph is contained in E or there is a Borel function on I whose graph is disjoint from the converse of E.*

Proof. Let $E \subset I \times I$ be given. We define a symmetric subset $E^* \subset ([0, 1] \cup [2, 3])^2$ as follows. Let $E^* = \{(x, y + 2) : (x, y) \in E\} \cup \{(x + 2, y) : (y, x) \in E\} \cup ([2, 3] \times [2, 3])$. We know that E^* either contains or is disjoint from the graph of a Borel function on $[0, 1] \cup [2, 3]$. In the first case E must contain the graph of a Borel function on $[0, 1]$. In the second case the converse of E must be disjoint from the graph of a Borel function. The identification of $[0, 1] \cup [2, 3]$ with I costs at most one level in the Borel hierarchy.

We now let $Y = \{((\omega, R), (\omega, S))$: if $(\omega, R) \in K$ and $(\omega, R) \models$ "there are arbitrarily large u-admissible ordinals," then $(\omega, S) \in K$, $(\omega, S) \models$ "there is a largest u-admissible ordinal," and (ω, S) is longer than $(\omega, R)\}$. We view Y as a subset of the square of the Cantor space $\mathscr{P}(\omega \times \omega)$. There is no difficulty for us in identifying this square with the unit square, since they are in one–one correspondence by a Borel function of finite rank. Note that Y is a Borel set of rank $< \lambda$ which has a code in $L[u]$. (An exception is the case $\lambda = \omega$. In this case, use a suitably large finite fragment of the axioms of *KP*, so that it will be true.)

LEMMA 4. *Either there is a u-constructibly coded Borel function* $F: \mathscr{P}(\omega \times \omega) \to \mathscr{P}(\omega \times \omega)$ *whose graph is included in Y, or there is a u-constructibly coded Borel function* $G: \mathscr{P}(\omega \times \omega) \to \mathscr{P}(\omega \times \omega)$ *whose graph is disjoint from Y.*

Proof. By Lemma 3, there is such an F or G which is Borel. By absoluteness, there is a u-constructibly coded Borel F or G.

LEMMA 5. *Y contains no u-constructibly coded Borel function F, and the converse of Y is not disjoint from any u-constructibly coded Borel function G.*

Proof. In the first case, suppose v is a u-constructible code for the F in Lemma 4 and $v \in L(\beta, u)$, where $\beta < \omega_1^{L[u]}$. Then by Lemma 1, let γ be a u-admissible limit of u-admissibles, $\beta < \gamma < \omega_1^{L[u]}$, $\alpha < \gamma$, and $L(\gamma, u)$ has every element definable from u. Let $(\omega, R) \approx (L(\gamma, u), \varepsilon)$ be such that R is arithmetic in the theory of $(L(\gamma, u), u, \varepsilon)$. Then $(\omega, R) \in K$ and $(\omega, R) \models$ "there are arbitrarily large u-admissible ordinals." Hence $F((\omega, R)) \in K$, and $F((\omega, R))$ is longer than (ω, R). From this it follows that the ordinal of the standard part of $F((\omega, R)) = (\omega, S)$ is at least $\gamma + 1$, and hence at least the next u-admissible γ^+ after γ. Therefore every element of $L(\gamma^+, u) \cap \mathscr{P}(\omega)$ is arithmetic in S. Note that the theory of $(L(\gamma, u), u, \varepsilon)$ is in $L(\gamma^+, u)$, and hence $R \in L(\gamma^+, u)$. Since F is coded by v, $F((\omega, R)) \in L(\gamma^+, u)$. Hence the ωth jump of $F((\omega, R)) =$ the ωth jump of (ω, S), is arithmetic in S. This is a contradiction.

In the second case, suppose v is a u-constructible code for the G in Lemma 4 and $v \in L(\beta, u)$. By Lemma 2, let γ be a successor u-admissible, $\beta < \gamma < \omega_1^{L[u]}$, $\alpha < \gamma$. and $L(\gamma, u)$ has every element definable from u. Let $(\omega, S) \approx (L(\gamma, u), \varepsilon)$ be such that S is arithmetic in the theory of $(L(\gamma, u), u, \varepsilon)$. Then $(\omega, S) \in K$ and $(\omega, S) \models$ "there is a largest u-admissible ordinal." Then $G((\omega, S)) \in K$, $G((\omega, S)) \models$ "there are arbitrarily large u-admissible ordinals," and (ω, S) is not longer than $G((\omega, S))$. It is clear that $G((\omega, S))$ is not isomorphic to $(L(\gamma, u), \varepsilon)$, and so that the ordinal of the standard part of $G((\omega, S))$ is at least $\gamma + 1$, and hence at least the next u-admissible γ^+ after γ. As above, this is a contradiction.

Observe that in the proof of Lemma 5, which contradicts Lemma 4, we needed our hypothesis that all ordinals are $(\omega + \alpha + 1)$–small relative to u, in order to conclude that $(\omega, R) \in K$ (or $(\omega, S) \in K$).

Recall that a T_α-model is a transitive set $A \in V(\alpha + 1)$ such that $\{x \in V(\beta): A \models \varphi(x)\} \in A$, where φ is a first order formula which may contain parameters from A, and $\beta < \alpha$. Thus we have proved the following.

LEMMA 6. *There is an ordinal which is not $(\omega + \alpha + 1)$–small relative to u. Hence there is a $T_{\omega+\alpha+1}$-model containing u.*

We summarize what we have proved.

LEMMA 7. *The following is provable in $ZF - \mathscr{P}$. Let λ be a countable limit ordinal. Suppose that every symmetric Borel subset of $I \times I$ of rank $< \lambda$ contains or is disjoint from the graph of a Borel function. Then for all $\alpha < \omega + \lambda$ and $u \subset \omega$, there is a T_α-model containing u as an element.*

In order to obtain the sharpest result, namely, the existence of well founded absolute T_α-models, we need a few lemmas.

LEMMA 8. *Let λ be a limit ordinal. Then for all $n \in \omega$, $(L(\lambda + n + 1) - L(\lambda + n)) \cap \mathscr{P}(L(\lambda)) \neq \varnothing$. If $x \in L(\lambda + 1) - L(\lambda)$ then there is a surjective map $F: TC(x)^{\angle\omega} \to L(\lambda)$ which is in $L(\lambda + 1)$.*

Proof. From the standard theory of the constructible hierarchy.

LEMMA 9. *Let λ be a limit ordinal, and suppose that $\omega \leqslant \alpha < \lambda$ and λ is $(\alpha + 1)$–small. Then there is a surjective function $G: L(\lambda) \cap V(\alpha) \to L(\lambda)$ which is in $L(\lambda + 1)$.*

Proof. Let $x \in (L(\lambda + 1) - L(\lambda)) \cap V(\alpha + 1)$. By Lemma 8, let $F: TC(x)^{\angle\omega} \to L(\lambda)$ be surjective and in $L(\lambda + 1)$. Note that $TC(x) \subset V(\alpha)$. Code finite sequences from $L(\lambda) \cap V(\alpha)$ as elements of $L(\lambda) \cap V(\alpha)$ to obtain G.

LEMMA 10. *Let β be the least non-$(\alpha + 1)$–small ordinal, $\omega \leqslant \alpha$. Then β is a limit ordinal.*

Proof. Suppose $\beta = \lambda + n + 1$, $n \in \omega$. Then since λ is $(\alpha + 1)$–small, let $G: L(\lambda) \cap V(\alpha) \to L(\lambda)$ be surjective and in $L(\lambda + 1)$. By Lemma 1, let $y \in L(\lambda + n + 2) - L(\lambda + n + 1), y \subset L(\lambda)$. Then obviously $G^{-1}[y] \in L(\lambda + n + 2)$. If $G^{-1}[y] \in L(\lambda + n + 1)$ then $G[G^{-1}[y]]$ is first order definable over $L(\lambda + n)$. So $G[G^{-1}[y]] = y \in L(\lambda + n + 1)$. Hence $G^{-1}[y] \in L(\lambda + n + 2) - L(\lambda + n + 1)$, and $G^{-1}[y] \in V(\alpha + 1)$. This contradicts the assumption that $\lambda + n + 1$ is not $(\alpha + 1)$–small.

LEMMA 11. *Let λ be the least non-$(\alpha+1)$–small ordinal, $\omega \leqslant \alpha$. Then $L(\lambda) \models$ "$V(\alpha)$ exists." Furthermore, $L(\lambda) \models$ "every set can be injectively mapped into $V(\alpha)$."*

Proof. Observe that $L(\lambda) \cap V(\alpha) \in L(\lambda+1) \cap V(\alpha+1)$, and so $L(\lambda) \cap V(\alpha) \in L(\lambda)$. Hence $L(\lambda) \models$ "$V(\alpha) = x$," where $x = L(\lambda) \cap V(\alpha)$. For the second claim it is enough to prove that $L(\lambda) \models$ "every ordinal can be injectively mapped into $V(\alpha)$." Let $\beta < \lambda$ be a limit ordinal. Since β is $(\alpha+1)$–small, there is a surjective $G: L(\beta) \cap V(\alpha) \to L(\beta)$ in $L(\lambda)$. This provides the required injection of $L(\beta)$ into $V(\alpha)$ within $L(\lambda)$.

LEMMA 12. *Let λ be the least non-$(\alpha+1)$–small ordinal, $\omega \leqslant \alpha$. Then $L(\lambda) \models ZF - \mathscr{P}$.*

Proof. Let $L(\lambda) \models V(\alpha) = A$. To verify replacement in $L(\lambda)$ it suffices to consider the case of $(\forall x \in A)(\exists y)(L(\lambda) \models \varphi(x,y)) \to (\exists z)(\forall x \in A)(\exists y \in z)$ $(L(\lambda) \models \varphi(x,y))$, where parameters are allowed (by Lemma 4). Assume the hypotheses. For each $x \in A$ let f_x be the least constructed function from A onto $L(\gamma_x) \cap V(\alpha+1)$, where γ_x is the least ordinal such that $(\exists y \in L(\gamma_x))(L(\lambda) \models \varphi(x,y))$. Let f be given by $f(x,y) = f_x(y)$. Observe that $f \in L(\lambda+1)$. Using a pairing function on A in $L(\lambda)$, we can view f as an element of $V(\alpha+1)$. Since λ is not $(\alpha+1)$–small, we see that $f \in L(\lambda)$. Let $f \in L(\beta)$, $\alpha+1 < \beta < \lambda$. If the γ_x, $x \in A$, are unbounded in λ, then $L(\lambda) \cap V(\alpha+1) \subset L(\beta+1)$. Since $\beta+1$ is $(\alpha+1)$–small, this is a contradiction. Hence the γ_x, $x \in A$, are bounded in λ. The conclusion of the given instance of replacement now follows.

THEOREM 2.1. *The following is provable in $ZF - \mathscr{P}$. Let λ be a countable limit ordinal. Then every symmetric Borel subset of $I \times I$ of rank $< \lambda$ contains or is disjoint from the graph of a Borel function if and only if for every $x \subset \omega$ and $\alpha < \omega + \lambda$ there is a well founded absolute T_α–model containing x as an element.*

Proof. For the forward direction, use Lemma 7, Lemma 12, and the fact that every transitive set satisfying $ZF - \mathscr{P}$ is well founded absolute.

For the converse, the proof in [10] establishes that for any Borel set E of rank $\alpha < \lambda < \omega_1$ with code $x \subset \omega$, if M is a $T_{\omega+\alpha+c}$-model containing x, then M satisfies that E is determined (here $c < \omega$ is a constant independent of α, λ). Hence if M is also well founded absolute then E is really determined. Therefore every symmetric Borel subset of $I \times I$ of rank $< \lambda$ contains or is disjoint from the graph of a Borel function.

The following are from Theorem 2.1, Theorem 1 of Section 1, and the general discussion in Section 1.

COROLLARY 2.2. *All four forms of Propositions* A, B *can be proved in* $Z + AC_\omega + CRA$, *but none of them can be proved in* $ZC + V = L + \mathscr{R}_2$. *If we restrict these forms to finitely Borel sets* E *then they can be proved in* $Z + AC_\omega$, *but none in weak* $ZC + V = L$.

COROLLARY 2.3. *For any one of the four forms of Propositions* A, B, *it is necessary and sufficient to use uncountably many iterations of the power set operation to give a proof. If we restrict to symmetric sets of finite Borel rank then it is necessary and sufficient to use* $\omega + \omega$ *iterations of the power set operation to give a proof.*

3. BOREL DIAGONALIZATION THEOREMS

Cantor proved that given any sequence of reals there is always a real outside the sequence. In fact, by using nested rational intervals one easily sees that there is a Borel function $F: \mathbb{R}^N \to \mathbb{R}$ (of finite rank) such that for all $x \in \mathbb{R}^N$, $F(x)$ is not a coordinate of x.

Observe that the value of F at $x \in \mathbb{R}^N$ depends not only on the set of coordinates of x but also on the order in which they are given. Can one pass from a sequence of reals to a real outside the sequence independently of the order in which the reals are given, in a Borelian way?

The answer is no, and this is the most basic of the Borel diagonalization theorems.

More precisely, let $\mathbb{R}^N$ be given the usual infinite product topology (where $\mathbb{R}$ is given the usual topology). Note that the group of permutations of N that leave all but at most finitely many numbers fixed, acts on $\mathbb{R}^N$ by permutation of coordinates. Write $x \sim y$ if x and y are in the same orbit under this group action. Write $x \simeq y$ if x and y have the same image (i.e., $\mathrm{Rng}(x) = \mathrm{Rng}(y)$). We give a proof of the following proposition using a forcing argument. However in the Appendix we give a proof using the Baire category theorem applied to $\underset{\sim}{\mathbb{R}}^N$, where $\underset{\sim}{\mathbb{R}}$ is the reals with discrete topology. The two proofs are essentially equivalent.

PROPOSITION C. *There is no Borel function* $F: \mathbb{R}^N \to \mathbb{R}$ *such that* (a) *if* $x \sim y$ (*if* $x \approx y$) *then* $F(x) = F(y)$, *and* (b) *for all* x, $F(x)$ *is not a coordinate of* x.

THEOREM 3.1. *Proposition* C *is provable in* $MK + AC_\omega - \mathscr{P}$, *but not in* $ZFC + V = L - \mathscr{P}$, *even for finitely Borel functions* F. *Proposition* C *is also provable in* $ZF + AC_\omega - \mathscr{P} +$ "$\mathscr{P}(\omega)$ *exists.*"

We first give a proof of Proposition C in $MK + AC_\omega - \mathscr{P}$. Let $F: \mathbb{R}^N \to \mathbb{R}$

be a Borel function such that $x \sim y$ implies $F(x) = F(y)$. Let A be a countable admissible set which contains a code u for F. By a Skolem hull argument, we can produce a countable elementary substructure of $L[u]$ with respect to $\mathscr{L}_A$, which can be isomorphically collapsed onto some $L_\alpha[u]$. It is clear that $L_\alpha[u]$ satisfies $\mathscr{L}_A - ZFC$ without power set. We then consider functions $f\colon N \to \mathbb{R} \cap L_\alpha[u]$ which are generic over $L_\alpha[u]$, using conditions which are finite sequences from $\mathbb{R} \cap L_\alpha[u]$ and are ordered by extension. We claim that for each rational q, ϕ decides $F(f) < q$, where f is a symbol for the generic object. (Here we view this statement as a formula in $\mathscr{L}_A$.) To see this, suppose that p_1 forces $F(f) < q$ and p_2 forces $F(f) \geqslant q$. Let $p_1 \subset g$ be generic. Then $F(g) < q$, and there is an $n \in \omega$ such that $(\forall i \in \operatorname{dom}(p_2))(p_2(i) = g(i+n))$. Hence there is an $h \sim g$ such that $p_2 \subset h$. Now such an h must also be generic and hence $F(h) \geqslant q$. This contradicts the fact that $F(h) = F(g)$.

We now know that ϕ decides $F(f) < q$ for all rationals q. Fix g to be generic. Observe that forcing over $L_\alpha[u]$ for formulas in $\mathscr{L}_A$ is definable over $L_\alpha[u]$ by a formula in $\mathscr{L}_A$. Hence by $\mathscr{L}_A$-separation, $\{q\colon \phi \text{ forces } F(f) < q\}$ is in $L_\alpha[u]$. Hence $F(g) \in L_\alpha[u]$, which immediately implies that $F(g)$ is a coordinate of g.

The proof from $ZF + AC_\omega - \mathscr{P} +$ "$\mathscr{P}(\omega)$ exists" is similar and a little simpler.

The idea of using forcing rather than Borel determinacy to prove Proposition C goes back to L. Harrington. We had an earlier version of Proposition C involving Borel functions on Turing degrees, which we proved using Borel determinacy. Harrington later gave a forcing argument.

We now wish to prove that neither form of Proposition C can be proved in $ZFC + V = L - \mathscr{P}$. (We obviously need only consider the formulation of Proposition C in terms of Borel codes.) We will do this by constructing an ω-model of Z_2 (second order arithmetic) in the theory Z_2 + Proposition C. Thus if Z_2 proves Proposition C then Z_2 proves the existence of an ω-model of Z_2, which contradicts the second incompleteness theorem. Hence this will establish that Proposition C cannot be proved in Z_2. This is enough by the following lemma:

LEMMA 3.1.1. *If $ZFC + V = L - \mathscr{P}$ proves Proposition* C *(formulated in terms of Borel codes and using $\approx$) then Z_2 proves Proposition* C.

Proof. Observe that Proposition C can be put in π_2^1 form. It is well known that $ZFC + V = L - \mathscr{P}$ is conservative over Z_2 for π_2^1 sentences.

Let pZ_2 (parameterless Z_2) consist of the usual axioms for 0, $'$, $+$, $\cdot$, induction for all formulas, and the axioms $(\exists x)(\forall n)(n \in x \leftrightarrow \varphi)$, where φ has no free variables other than "n."

LEMMA 3.1.2. *Z_2 + Proposition* C *(formulated in terms of Borel codes and using $\approx$) proves the existence of a countable ω-model of pZ_2.*

Proof. Argue in Z_2 + Proposition C. Let $F: \mathbb{R}^N \to \mathbb{R}$ be the following Borel function. Let $x \in \mathbb{R}^N$. In viewing real numbers as sets of natural numbers, the range of x yields a countable ω-structure—namely, $(\omega, 0, ', +, \cdot, \in, \text{rng}(x))$. If this ω-structure does not satisfy pZ_2 then let $\varphi(n)$ be the formula with the least Gödel number with only "n" free such that $(\exists y)(\forall n)(n \in y \leftrightarrow \varphi(n))$ fails in this structure. Set $F(x) = \{n: (\omega, 0, ', +, \cdot, \in, \text{rng}(x)) \models \varphi(n)\}$, as a real number. If this ω-structure does satisfy pZ_2 then set $F(x) = 0$. It is clear that $x \approx y$ implies $F(x) = F(y)$, and that for x such that $F(x)$ is a coordinate of x, $(\omega, 0, ', +, \cdot, \in, \text{rng}(x))$ is an ω-model of pZ_2. This proof takes place in a weak fragment of $ZF - \mathscr{P}$.

LEMMA 3.1.3. *There are formulas $\varphi_1(x)$ and $\varphi_2(x, y)$, with only the free variables shown, such that* (a) *pZ_2 proves that $\varphi_2(x, y)$ defines a linear ordering on the x with $\varphi_1(x)$, and $(\exists x)(\varphi_1(x))$,* (b) *for arithmetic formulas $\psi(x_1, \ldots, x_n, k)$ with no other free set variables, pZ_2 proves $(\varphi_1(x_1) \,\&\, \cdots \,\&\, \varphi_1(x_n)) \to (\exists y)(\varphi_1(y) \,\&\, (\forall k)(k \in y \leftrightarrow \psi(x_1, \ldots, x_n, k)))$,* (c) *for formulas $\psi(x)$ and $\rho(x)$ with no other free set variables, pZ_2 proves $[(\exists! x)(\psi(x) \,\&\, \varphi_1(x)) \,\&\, (\exists x)(\psi(x) \,\&\, \varphi_1(x) \,\&\, (\exists y)(\rho(y) \,\&\, \varphi_2(y, x)))] \to (\exists x)(\psi(x) \,\&\, \varphi_1(x) \,\&\, (\exists y)(\rho(y) \,\&\, \varphi_2(y, x) \,\&\, \sim(\exists z)(\rho(z) \,\&\, \varphi_2(z, y))))$,* (d) *$pZ_2$ proves "there exists an ω-model of Z_2" or $(\forall n)(\exists! x)(\varphi_1(x) \,\&\, \psi(n, x)) \to (\exists y)(\varphi_1(y) \,\&\, (\forall n)(\exists x)(\exists m)(x = (y)_m \,\&\, \psi(n, x)))$, where ψ is any formula with no free set variables other than x.*

Proof. Let ACA (arithmetic comprehension axiom schema) consist of axioms for $0, ', +, \cdot$; induction for all formulas; extensionality; and comprehension for all arithmetic formulas (with parameters). Let K be the theory of second order arithmetic obtained from ACA by adding the two binary relation symbols $\prec$, $\sim$ among sets of natural numbers, together with the axioms: (a) $\sim$ is an equivalence relation on all sets, (b) $\prec$ forms a linear ordering on the equivalence classes under $\sim$, such that every point has an immediate successor, and there is a first element, (c) each $\{z: z \prec x\}$ is countable, and each $(\omega, 0, ', +, \cdot, \in, \sim, \prec, \{z: z \prec x\})$ has a satisfaction relation, (d) $x \sim y$ if and only if y is first order definable over $(\omega, 0, ', +, \cdot, \in, \sim, \prec, \{z: z \prec x\})$ with parameters allowed for elements, and not $y \prec x$. In the next paragraph, we will be considering models of K of the form $(\omega, 0, ', +, \cdot, E, R, S, y)$, where the number variables range over ω, the set variables range over the elements of $y \subset \omega$, E interprets $\in$, R interprets $\sim$ and S interprets $\prec$.

Within pZ_2, we make the following definitions. We let $(x)_n = \{k: 2^n 3^k \in x\}$, if it exists. A *K-structure* consists of x, R, S, where x is a subset of ω, and R, S are binary relations on ω, such that (1) each $(x)_m$

exists, $E(n, m) \leftrightarrow n \in (x)_m$ exists, and $(\omega, 0, ', +, \cdot, E, R, S, \omega) = |x, R, S|$ exists and has a satisfaction predicate, and satisfies K, (2) every element in $|x, R, S|$ is definable, and (3) for all m the initial segment of $|x, R, S|$ up till the level of m,

$$(\omega, 0, ', +, \cdot, E \restriction \omega \cdot \{\omega \cdot \{n: S(n, m)\}^2, R \restriction \{n: S(n, m)\}^2,$$
$$S \restriction \{n: S(n, m)\}^2, \{n: S(n, m)\}),$$

exists, its satisfaction predicate exists, and has every element definable.

A *well founded K-structure* is a K-structure (x, R, S) such that for any y with (1) each $(y)_n$ exists, and (2) $(\exists k)(\forall n)(\forall m)((y)_n = (x)_m \rightarrow S(m, k))$, there is a k which is S-least such that (2) holds. (Intuitively, this asserts that any subset of the reals internal to $|x, R, S|$ that is S-bounded, has an S-least upper bound.)

A *strong K-structure* is a well founded K-structure (x, R, S) such that for any other well founded K-structure (x^*, R^*, S^*), (x, R, S) and (x^*, R^*, S^*) are comparable; i.e., either

(1) $(\forall n)(\exists m)((x)_n = (x^*)_m)$, and

(2) $(\forall m)(\exists n)((x)_n = (x^*)_m)$, and

(3) $(\forall n)(\forall m)(\forall p)(\forall q)(((x)_n = (x^*)_m \;\&\; (x)_p = (x^*)_q) \rightarrow ((R(n, p) \leftrightarrow R^*(m, q)) \;\&\; (S(n, p) \leftrightarrow S^*(m, q))))$,

or for some k,

(1) $(\forall n)(\exists m)(S^*(m, k) \;\&\; (x)_n = (x^*)_m)$, and

(2) $(\forall m)(S^*(m, k) \rightarrow (\exists n)((x)_n = (x^*)_m))$, and

(3) same as above,

or for some k,

(1) $(\forall m)(\exists n)(S(n, k) \;\&\; (x)_n = (x^*)_m)$, and

(2) $(\forall n)(S(n, k) \rightarrow (\exists m)((x)_n = (x^*)_m))$, and

(3) same as above.

We now define $\varphi_1(x)$ if and only if for some strong K-structure (y, R, S), we have $(\exists n)((y)_n = x)$. We define $\varphi_2(x, y)$ if and only if there is a strong K-structure (z, R, S) and some n, m such that $(z)_n = x$, $(z)_m = y$, and either (i) $S(n, m)$, or (ii) $R(n, m)$, and in the initial segment of $|z, R, S|$ given by going up through the level of n, m, the least Gödel number of a definition of n (as a set) is smaller than the least Gödel number of a definition of m (as a set).

Conclusions (a) and (b) of the lemma are obvious, except for $(\exists x)(\varphi_1(x))$. Here, we merely need to show the existence of a strong K-structure. The usual K-structure of height ω can be proved to exist, using induction and parameterless comprehension. By using induction, it can be shown to be well founded. Also using induction, we can prove that the first n levels of any K-structure, well founded or not, agree with the first n levels of the usual K-structure of height ω. Therefore the usual K-structure of height ω is comparable with any well founded K-structure.

For conclusion (c), let x be a definable set with $\varphi_1(x)$, and let $\varphi_2(y, x)$ be such that $\rho(y)$. Let (z, R, S) be a strong K-structure, where $x = (z)_k$. By using the definability of x and the fact that the initial segment of $|z, R, S|$ given by going up through the level of k has every element definable, and its set of true sentences is independent of the choice of (z, R, S), we can construct a definable set u such that $(\forall n)(\varphi_2((u)_n, x))$ & $(\forall y)(\varphi_2(y, x) \rightarrow (\exists n)(y = (u)_n))$. By using u and the well foundedness of (z, R, S), we can find a φ_2-least y such that $\rho(y)$.

For (d), assume $(\forall n)(\exists! x)(\varphi_1(x)$ & $\psi(n, x))$. Then there is a set z such that each $(z)_n$ is the (unique) set of true sentences of the initial segment of some strong K-structure given by going up through the level of the unique x such that $\varphi_1(x)$ & $\psi(n, x)$. We can assume without loss of generality that there is no φ_2-greatest x with $(\exists n)(\varphi_1(x)$ & $\psi(n, x))$.

We can piece the $(z)_n$ together into a limit, since z is definable. This will result in a structure $|u, R, S|$ which obviously obeys all conditions for being a K-structure except possibly that every element of $|u, R, S|$ is definable. We now wish to show this.

By using the construction of $|u, R, S|$, the definability of u, R, S, and conclusion (c) of the lemma, we see that the definable elements in $|u, R, S|$ form an elementary substructure of $|u, R, S|$. Similar considerations also show that the definable elements in $|u, R, S|$ either include all elements, or are the initial segment determined by an element. If the latter holds, then that determining element is definable over the initial segment with parameters. Hence the determining element is definable over $|u, R, S|$ with parameters from the initial segment. But then the determining element is in the initial segment it determines, which is a contradiction.

We have thus shown that (u, R, S) is a K-structure. It is easily seen by construction that (u, R, S) is a well founded K-structure.

To see that (u, R, S) is a strong K-structure, let (u^*, R^*, S^*) be another well founded K-structure. Since (u^*, R^*, S^*) is comparable with every initial segment of (u, R, S) determined by an element, the only case we have to handle is if (u, R, S) is a proper initial segment of (u^*, R^*, S^*). In this case, we must show that it is determined by an element. But this follows from the well foundedness of (u^*, R^*, S^*).

To complete the proof of conclusion (d), we argue by cases. If (u, R, S) is not the greatest strong K-structure, then using a strong K-structure greater than (u, R, S), we can find the desired upper bound.

If (u, R, S) is the greatest strong K-structure, then we claim that (u, R, S)

satisfies the full comprehension axiom. If this were not the case we could extend (u, R, S) by ω steps to obtain a larger strong K-structure (here we again use the definability of u, R, S).

LEMMA 3.1.4. *Every ω-model of pZ_2 contains an ω-model of Z_2.*

Proof. Let $\mathcal{O}\!\mathcal{l}$ be an ω-model of pZ_2. If $\mathcal{O}\!\mathcal{l} \models$ "there exists an ω-model of Z_2," then we obviously can find an ω-submodel of α which satisfies Z_2. We assume not.

Let $\mathcal{B}$ be the ω-model of $\mathcal{O}\!\mathcal{l}$ consisting of all x such that $\mathcal{O}\!\mathcal{l} \models \varphi_1(x)$, and x is definable in $\mathcal{O}\!\mathcal{l}$.

We claim that for every formula $\varphi(\mathbf{x}, \mathbf{n})$ there is a formula $\varphi^*(\mathbf{x}, \mathbf{n})$ with the same free variables, such that for $\mathbf{x}$ in $\mathcal{B}$, $\mathbf{n}$ in ω, $\mathcal{B} \models \varphi(\mathbf{x}, \mathbf{n})$ if and only if $\mathcal{O}\!\mathcal{l} \models \varphi^*(\mathbf{x}, \mathbf{n})$. This is proved by induction on the complexity of φ. The nontrivial case is the existential quantifier, $(\exists y)(\varphi(y, \mathbf{x}, \mathbf{n}))$. But note that $\mathcal{B} \models (\exists y)(\varphi(y, \mathbf{x}, \mathbf{n}))$ if and only if $\mathcal{O}\!\mathcal{l} \models (\exists y)(\varphi_1(y)$ & $\varphi^*(y, \mathbf{x}, \mathbf{n})$ & $\sim(\exists z)(\varphi_1(z)$ & $\varphi^*(z, \mathbf{x}, \mathbf{n})$ & $\varphi_2(z, y)))$, using (c) of Lemma 3.1.3.

Next, note that an axiomatization which yields Z_2 is given by the axioms of ACA together with the scheme $(\forall n)(\exists x)(\psi(n, x, \mathbf{m}, \mathbf{y})) \rightarrow (\exists z)(\forall n)(\exists x)(\exists m)(x = (z)_m$ & $\psi(n, x, \mathbf{m}, \mathbf{y}))$. We know that $\mathcal{B} \models ACA$ by (b) of Lemma 3.1.3. Suppose $(\forall n)(\exists x)(\psi(n, x, \mathbf{m}, \mathbf{y}))$ holds in $\mathcal{B}$. Then $\mathcal{O}\!\mathcal{l} \models (\forall n)(\exists! x)(\varphi_1(x)$ & $\psi^*(n, x, \mathbf{m}, \mathbf{y})$ & $\sim(\exists w)(\varphi_1(w)$ & $\psi^*(n, w, \mathbf{m}, \mathbf{y})$ & $\varphi_2(w, x)))$. Now we can replace each of $\mathbf{y}$ by their definitions in $\mathcal{O}\!\mathcal{l}$. By (c) and (d) of Lemma 3.1.3, we can find a z in $\mathcal{B}$ such that for all n there is an x in $\mathcal{B}$ such that for some m, $\mathcal{O}\!\mathcal{l} \models x = (z)_m$ & $\psi^*(n, x, \mathbf{m}, \mathbf{y})$. We are done, since $\mathcal{O}\!\mathcal{l} \models x = (z)_m$ & $\psi^*(n, x, \mathbf{m}, \mathbf{y})$ if and only[2] if $\mathcal{B} \models x = (z)_m$ & $\psi(n, x, \mathbf{m}, \mathbf{y})$. This completes the proof of the lemma.

The following is evident from the proof of Lemma 3.1.4.

LEMMA 3.1.5. *Z_2 proves that every countable ω-model of pZ_2 contains an ω-submodel of Z_2.*

We are now prepared to complete the proof of Theorem 3.1. By Lemma 3.1.1 it suffices to show that Z_2 does not prove Proposition C (formulated in terms of Borel codes and using $\approx$). If Z_2 does prove Proposition C then by Lemmas 3.1.2 and 3.1.5, Z_2 proves the existence of a countable ω-model of Z_2. This contradicts the Gödel second incompleteness theorem.

We indicate the modifications necessary to show that the restriction of Proposition C to finitely Borel functions cannot be proved in $ZFC + V = L - \mathcal{P}$.

[2] We can find such a Z in $\mathcal{B}$ since the second disjunct in Lemma 3.1.3(d), can be strengthened to assert that y is φ_2-least with $(\forall n)$ $(\exists x)$ $(\exists m)$ $(x = (y)_m$ & $\psi(n, x))$. Our proof of Lemma 3.1.3 establishes this.

LEMMA 3.1.6. *Z_2 + Proposition* C *restricted to finitely Borel functions (formulated in terms of Borel codes and using $\approx$) proves the existence of a countable model of pZ_2.*

Proof. The proof of Lemma 3.1.2 shows that we can obtain ω-models of any given finite fragment of pZ_2. Then apply the compactness theorem.

LEMMA 3.1.7. *Z_2 proves that every countable model of pZ_2 contains a submodel of Z_2 with the same arithmetic part.*

Proof. Minor modifications of the proof of Lemma 3.1.4 are needed. Firstly, note that pZ_2 proves comprehension for all formulas with no set parameters (number parameters are allowed). So in Lemma 3.1.3, number parameters can be used. Instead of taking $\mathscr{B}$ to be the submodel of definable elements, take $\mathscr{B}$ to be submodel of elements definable from finitely many number parameters. The reaxiomatization of Z_2 used there works in the context of arbitrary models if we add the axiom that every nonempty set has a least element. This is obviously true in $\mathscr{B}$.

This completes the proof of Theorem 3.1 by the second incompleteness theorem.

We now wish to consider a more general form of Borel diagonalization. Let $\mathscr{S}$ be a topological space, and let E be an equivalence relation on $\mathscr{S}$. We use [ ] in connection with E. For sets $A \subset |\mathscr{S}|$ we write $[A] = \{[x] : x \in A\}$.

We say that the Borel diagonalization theorem holds for $(\mathscr{S}, E)$ if there is no Borel function $F: \mathscr{S}^N \to \mathscr{S}$ such that (a) if $[\mathrm{rng}(x)] = [\mathrm{rng}(y)]$ then $[F(x)] = [F(y)]$, and (b) $[F(x)] \notin [\mathrm{rng}(x)]$.

PROPOSITION D. *The Borel diagonalization theorem holds for all Borel equivalence relations. That is, for all Borel equivalence relations E on $\mathbb{R}$, there is no Borel function $F: \mathbb{R}^N \to \mathbb{R}$ such that* (a) *if* $[\mathrm{rng}(x)] = [\mathrm{rng}(y)]$ *then* $[F(x)] = [F(y)]$, *and* (b) *for all x,* $[F(x) \notin [\mathrm{rng}(x)]$.

THEOREM 3.2. *The following is provable in $ZF - \mathscr{P}$. Let $\lambda < \omega_1$ be a limit ordinal. Then Proposition* D *holds for all Borel equivalence relations of rank $< \lambda$ if and only if for all $\alpha < \omega + \lambda$, $x \subset \omega$, there is a T_α-model in which x is present.*

We now wish to prove the Borel diagonalization theorem for all Borel equivalence relations (in fact, the reverse direction of Theorem 3.2). This proof uses Borel determinacy. In Section 4 we extend this technique to prove certain Borel fixed point theorems, and in [5] we extend this technique to prove the Borel diagonalization theorem for analytic equivalence relations. Kechris and the author have independently seen how Borel determinacy can be eliminated in favor of some recent work of Jacques Stern in the case of Borel diagonalization for Borel equivalence relations. However, it remains unclear how Stern's work can be used for our other applications of Borel determinacy.

Fix a limit ordinal $\lambda < \omega_1$, and let E be a Borel equivalence relation of rank $< \lambda$ with Borel code $u \subset \omega$. Let $F\colon \mathbb{R}^N \to \mathbb{R}$ be a Borel diagonalizer for E (i.e., (a), (b) above hold for F). Let v be a Borel code for F.

Let Y be the set of all structures (ω, R) satisfying $KP + V = L[u, v]$ + "every set is countable $+F$ is a diagonalizer for E," where (ω, R) is an ω-model in which u, v are internal and E, F are described in terms of u, v.

We wish to compare pairs $(\omega, R), (\omega, S)$ of elements of Y. Within any $(\omega, R) \in Y$, make the following definition by transfinite recursion on the ordinals of (ω, R). Firstly, define $G_R(0) = 0$. For $\alpha > 0$, let g be the first surjective mapping from ω onto α in the constructible hierarchy relative to (u, v), and define $G_R(\alpha) = F(G_R \circ g)$. Observe that in (ω, R) the various $G_R(\alpha)$ are inequivalent according to E. By absoluteness, the various $G_R(\alpha)$ really are inequivalent according to E.

Let H' be the set of ordered pairs (a, b) of ordinals a from (ω, R) and ordinals b from (ω, S) such that $E(G_R(a), G_S(b))$. Note that H' is a partial function. Finally, let H be the largest restriction of H' which maps an initial segment of the ordinals of (ω, R) in an order preserving way onto an initial segment of the ordinals of (ω, S).

We say that (ω, S) is *longer* than (ω, R) if and only if the above H maps all of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S), determined by an ordinal of (ω, S), or the domain of H is a proper initial segment of the ordinals of (ω, R) that is not determined by any ordinal of (ω, R) and the range of H is either all of the ordinals of (ω, S) or is an initial segment of the ordinals determined by an ordinal of (ω, S).

LEMMA 3.2.1. *$\{((\omega, R), (\omega, S))\colon (\omega, R), (\omega, S) \in Y$ and (ω, S) is longer than $(\omega, R)\}$ is a Borel set of rank $< \lambda$. (If $\lambda = \omega$ then we use a suitably large finite fragment of KP).*

Proof. Observe that Y is a Borel set of low rank. In comparing (ω, R), (ω, S), note that E is responsible for the entire Borel complexity.

LEMMA 3.2.2. *For any ordinal $\alpha < \omega_1^{L[u,v]}$ there is an ordinal $\alpha < \gamma < \omega_1^{L[u,v]}$ such that $L_\gamma[u, v] \models KP$ + "every set is countable" $+ F$ is a diagonalizer for E + "there are arbitrarily large (u, v)-admissible ordinals," and such that every element of $L_\gamma[u, v]$ is definable over $L_\gamma[u, v]$ relative to u, v.*

Proof. Suppose that the lemma is false and let $\beta < \omega_1^{L[u,v]}$ be the supremum of all the ordinals in question. Then β is Δ_2 relative to (u, v) in $L_{\omega_1}[u, v]$. Let γ be the first ordinal that is not Δ_2 relative to (u, v) in

$L_{\omega_1}[u, v]$. Then $L_\gamma[u, v] \models KP +$ "every set is countable" + "there are arbitrarily large (u, v)-admissible ordinals," and every element of $L_\gamma[u, v]$ is definable over $L_\gamma[u, v]$ relative to u, v. Since "F is a diagonalizer for E" is a true π^1_1 sentence, it also holds in $L_\gamma[u, v]$. Thus γ is one of the ordinals in question, which is a contradiction. (The ω_1 in the subscripts, which occurs twice, refers to $\omega_1^{L[u,v]}$.)

LEMMA 3.2.3. *For any ordinal* $\alpha < \omega_1^{L[u,v]}$ *there is an ordinal* $\alpha < \gamma < \omega_1^{L[u,v]}$ *such that* $L_\gamma[u, v] \models KP + V = L[u, v] +$ *"every set is countable"* $+ F$ *is a diagonalizer for* E + *"there is a largest* (u, v)*-admissible ordinal," and such that every element of* $L_\gamma[u, v]$ *is definable over* $L_\gamma[u, v]$ *relative to* u, v.

Proof. Let γ' be given as in Lemma 3.2.2, and let γ be the next (u, v)-admissible after γ'. We have merely to show that every element of $L_\gamma[u, v]$ is definable over $L_\gamma[u, v]$ relative to u, v. By a Skolem hull argument, it is easy to see that every element of $L_\gamma[u, v]$ is definable in $L_\gamma[u, v]$ from u, v and elements of $L_{\gamma'}[u, v]$. Since γ' is definable in $L_\gamma[u, v]$ from (u, v), we see that every element of $L_\gamma[u, v]$ is definable over $L_\gamma[u, v]$ relative to u, v.

Now let $Z = \{((\omega, R), (\omega, S))$: if $(\omega, R) \in Y$ and $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible ordinals," then $(\omega, S) \in Y$, $(\omega, S) \models$ "there is a largest (u, v)-admissible ordinal," and (ω, S) is longer than $(\omega, R)\}$. We can view Z as a subset of the Cantor square, and we can play the game where II wins iff the pair of plays is in Z.

By absoluteness, the game has a winning strategy J in $L[u, y]$. (Observe that Z is a Borel set of rank $< \lambda$, and we are using determinacy for Borel sets of rank $< \lambda$. In the case $\lambda = \omega$, use a suitably large finite fragment of KP.)

LEMMA 3.2.4. *J is not a winning strategy for* II. *J is not a winning strategy for* I.

Proof. Let $\alpha < \omega_1^{L[u,v]}$ be such that J has a Borel code in $L_\alpha[u, v]$. First suppose that J is a winning strategy for II. Let γ be chosen as in Lemma 3.2.2. Let (ω, R) be isomorphic to $L_\gamma[u, v]$, where R is arithmetic in the theory of $L_\gamma[u, v]$ relative to (u, v). Observe that $(\omega, R) \in Y$ and $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible ordinals." Let (ω, S) be J applied to (ω, R). Then $(\omega, S) \in Y$ and (ω, S) is longer than (ω, R). Hence the ordinal of the standard part of (ω, S) is at least $\lambda + 1$, and therefore at least the next (u, v)-admissible μ after γ. Hence the ωth jump of S is internal to (ω, S). This is a contradiction.

Now suppose that J is a winning strategy for player I. Let μ be chosen as in Lemma 3.2.3. Let (ω, S) be isomorphic to $L_\mu[u, v]$, where S is arithmetic in the theory of $L_\mu[u, v]$, relative to (u, v). Observe that $(\omega, S) \in Y$ and $(\omega, S) \models$ "there is a largest (u, v)-admissible ordinal." Let (ω, R) be J applied to (ω, S). Then $(\omega, R) \in Y$, $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible ordinals," and (ω, S) is not longer than (ω, R).

Let H be as given in the definition of "(ω, S) is longer than (ω, R)." By the invariance property of F, we see that either $\mathrm{dom}(H)$ has no sup, or $\mathrm{rng}(H)$ has no sup. First assume that $\mathrm{rng}(H)$ has no sup. Then H is onto, since (ω, S) is well founded. Since (ω, S) is not longer than (ω, R) and not isomorphic to (ω, R), we see that the ordinal of the standard part of (ω, R) is at least $\mu + 1$, and hence at least the next (u, v)-admissible after μ. This results in a contradiction as above.

Secondly assume that $\mathrm{dom}(H)$ has no sup. Then it is easy to see that H is total. But then $(\omega, R) \approx (\omega, S)$, which is impossible.

The above argument can be formalized in Z_2, and thus we have the following.

LEMMA 3.2.5. *There is a constant $c < \omega$ such that the following is provable in Z_2. For all $\alpha < \omega_1$, if all Borel sets of rank $\leqslant \alpha + c$ are determined then the Borel diagonalization theorem holds for all Borel equivalence relations of rank $\leqslant \alpha$.*

LEMMA 3.2.6. *The following is provable in $ZF - \mathscr{P}$. Let $\lambda < \omega_1$ be a limit ordinal. If for all $\alpha < \lambda$ and $x \subset \omega$, there is a $T_{\omega+\alpha}$-model containing x, then the Borel diagonalization theorem holds for all Borel equivalence relations of rank $< \lambda$.*

Proof. Let $\alpha < \lambda$, and let E be a Borel equivalence relation of rank α. Let F be a Borel diagonalizer for E. Choose $n \in \omega$ so that from [10], we see that any $T_{\omega+\alpha+n}$-model satisfies "all Borel sets of rank $\leqslant \alpha + c$ are determined." Under the hypothesis of this lemma, there is a $T_{\omega+\alpha+n}$-model containing Borel codes for E and F. By Lemma 3.2.5, this model satisfies the Borel diagonalization theorem for E. This is a contradiction.

This completes the "if" part of Theorem 3.2. For the "only if" part, let $\lambda < \omega_1$ be a limit ordinal. Let B be the set of all pairs (x, R), where $x \subset \omega$, $R \subset x \cdot x$. We define the function $H_\alpha((x, R)) = 0$ if $x = 0$ or (x, R) is not isomorphic to any $(A, \in)$ for transitive $A \in V(\alpha + 1)$; otherwise, $H_\alpha((x, R)) =$ the unique transitive $A \in V(\alpha + 1)$ with $(x, R) \approx (A, \in)$. Let E_α be the equivalence relation on B given by $E_\alpha(s, t) \leftrightarrow H_\alpha(s) = H_\alpha(t)$. It is easily seen that E_α is a Borel equivalence relation on B of rank $< \lambda$, for $\alpha < \omega + \lambda$.

Proof of Theorem 3.2. Arguing in $ZF - \mathscr{P}$, assume that the Borel diagonalization theorem holds for all E_α, $\alpha < \omega + \lambda$ (where B has been identified with $\mathbb{R}$), and fix $\alpha < \omega + \lambda$. We define the Borel function $G: B^N \to B$ as follows. Let $x \in B^N$ be given. Let A be the union of all $H_\alpha(x(n))$, $n < \omega$. Then $A \in V(\alpha + 1)$ and A is transitive. Let A^* be A together with all sets in

$V(\alpha)$ which are first order definable with parameters over $(A, \in)$. Finally, let $G(x)$ be a canonically chosen code for A^*; i.e., $H_\alpha(G(x)) = A^*$. Obviously G is a Borel function which has the invariance property for Borel diagonalization. By hypothesis let $E_\alpha(x(n), G(x))$. Then $H_\alpha(x(n)) = H_\alpha(G(x)) = A^*$. Hence $A^* \subset A$, and so A is a T_α-model. This completes the proof.

The following are from Theorem 3.2 and the discussion in Section 1.

COROLLARY 3.3. *Proposition* D *can be proved in* $Z + AC_\omega + CRA$, *but not in* $ZC + V = L + \mathscr{R}_1$. *If we restrict Proposition* D *to finitely Borel relations E, then it is provable in* $Z + AC_\omega$, *but not in weak* $ZC + V = L$.

COROLLARY 3.4. *It is necessary and sufficient to use* ω_1 *iterations of the power set operation to prove Proposition* D. *It is necessary and sufficient to use* $\omega + \omega$ *iterations of the power set operation to prove Proposition* D *for finitely Borel relations E. The latter holds even if everything involved is restricted to the finitely Borel.*

We now consider the Baire space N^N under the equivalence relation of conjugation.

PROPOSITION E. *The Borel diagonalization theorem holds for* N^N *under conjugation.*

THEOREM 3.5. *The following is provable in* $ZF - \mathscr{P}$. *Proposition* E *is equivalent to "for all* $\alpha < \omega_1$ *and* $x \subset \omega$, *there is a* T_α*-model containing* x."

We first prove the backwards direction of Theorem 3.5. Actually, N^N under conjugation is a special case of a wider situation. Looked at model theoretically, N^N under conjugation is the same as the structures with domain ω with one unary function under isomorphism.

More generally, let σ be any at most countably infinite relational type. Let $\mathscr{S}(\sigma)$ be the space of all structures with domain ω of relational type σ, with the usual Baire topology (the basic open sets are given by finite pieces of information about the structures). We wish to prove the Borel diagonalization theorem for $\mathscr{S}(\sigma)$ under isomorphism. This theorem follows from Borel diagonalization for analytic equivalence relations proved in [5]. However, the proof given here is of independent interest.

Assume that for all $\alpha < \omega_1$ and $x \subset \omega$, there is a T_α-model.

We first review some concepts used in the construction of "Scott sentences." Let $\mathscr{B}$ be a structure. The 0-type of a finite (possibly empty) sequence of elements from $\mathscr{B}$ is the set of all formulas of ordinary first order predicate calculus with equality, $\varphi(x_1, \ldots, x_n)$, which are true of the sequence (of length n). The $(\alpha + 1)$-type of a finite sequence is the set of all α-types of extensions of that finite sequence. The λ-type of a finite sequence is the function f with domain λ such that $f(\alpha)$ is the α-type of the sequence. The α-type of $\mathscr{B}$ is the α-type of the empty sequence.

Without referring to any structure, we can define *possible α-types* as follows. A possible 0-type of a finite sequence is any set of formulas $\varphi(x_1,\ldots,x_n)$ in $\mathscr{L}_{\omega\omega}$ with $=$, where n is the length of the sequence. A possible $(\alpha+1)$-type of a finite sequence is any set of possible α-types of longer finite sequences. A possible λ-type of a finite sequence is any function f with domain λ such that $f(\alpha)$ is a possible α-type. A possible α-type (of a structure) is a possible α-type of $\langle\ \rangle$.

There is a canonical way to associate to each possible α-type, an infinitary sentence of rank $< \alpha+\omega$ whose models are exactly the structures with the given possible α-type. Now a given possible α-type may or may not be realizable, in the sense that there may or may not be a structure with that possible α-type. The possible α-type is realizable if and only if the corresponding infinitary sentence is consistent. We thus often identify a possible α-type with its corresponding sentence.

Let $F\colon \mathscr{S}(\sigma)^\infty \to \mathscr{S}(\sigma)$ be Borel, and assume that $[\mathrm{Rng}(\mathfrak{A})] = [\mathrm{Rng}(\mathscr{B})] \to [F(\mathfrak{A})] = [(F(\mathscr{B})]$, where $[\ ]$ refers to isomorphism.

LEMMA 3.5.1. *For every $x \subset \omega$ there is a countable power admissible set E with $x \in E$, and hence a countable non-standard power admissible set with standard part E.*

Proof. Let $x \subset \omega$, $x \in A$, where A is a countable admissible set. Let c be a constant symbol, and consider the axioms in the infinitary language $\mathscr{L}_A$ which assert that c is an ordinal greater than each $\alpha \in A$, transfinite induction holds up to c, and that the universe is isomorphic to a T_{c+1}-model which contains x. Then every A-finite subset of these axioms is consistent, since for all $\alpha \in A$, there is a T_α-model containing x. Hence the theory is consistent. The standard part of any countable model of the theory is a countable power admissible set containing x.

Now let A be a countable power admissible set such that the rank of F is in A, $\sigma \in A$, and some code for F is in A. Let (B,R) be a countable non-standard power admissible set whose standard part is A. Let β be any non-standard ordinal in B. In (B,R), let K be the set of all possible β-types which according to (B,R), are consistent (with the axioms "there are infinitely many") in the sense of $\mathscr{L}_{\infty\omega}$. We collapse K to ω by forcing over (B,R) with finite conditions. Let f be any such generic collapse, where $f\colon \omega \to K$ is one-one onto, and let $B[f]$ be the resulting structure, with a predicate symbol added for B. Then $B[f]$ will remain a power admissible structure.[3]

[3] In the definition of K, consistency in $\mathscr{L}_{\infty\omega}$ refers to all proof figures coded as well founded trees. The completeness theorem for countable admissible fragments is used for fragments which are coded by well founded relations (or trees).

Observe that in $B[f]$, $V(\beta)^B$ is countable. Hence by the completeness theorem for countable admissible fragments, we see that every φ in K has a model in $B[f]$ with domain ω.

Let $\tau(f)$ be a forcing term which is forced by ϕ to produce a function with domain K such that $\tau(f)(\varphi)$ is a model of φ with domain ω.

LEMMA 3.5.2. *Suppose that $g: \omega \to K$ is generic over B. Then $\tau(f)$ and $\tau(g)$ are termwise isomorphic, $\tau(f) \circ f$, $\tau(g) \circ g$ represent the same isomorphism types up to a permutation, and $F(\tau(g) \circ g)$, $F(\tau(f) \circ f)$ are isomorphic.*

Proof. The last two claims follow immediately from the first. Let $\varphi \in K$. It suffices to show that any model of φ in $B[f]$ is externally isomorphic to any model of φ in $B[g]$. This follows by a standard back and forth argument, since these models must have the same γ-type for all $\gamma \in A$.

LEMMA 3.5.3. *Let $\gamma \in A$. Then the γ-type of $F(\tau(f) \circ f)$ lies in A. Furthermore, it is forced by ϕ to be some particular element of A.*

Proof. By transfinite induction on γ. Assume that for all $\alpha < \gamma$, the α-type of every finite sequence in $F(\tau(f) \circ f)$ is in A. Then the γ-type of every finite sequence in $F(\tau(f) \circ f)$ is a bounded subset of A, that is present in $B[f]$. Now the set of all γ-types of finite sequences in $F(\tau(f) \circ f)$ is independent of the choice of the generic object f, by Lemma 3.5.2. Thus all of these γ-types for $F(\tau(f) \circ f)$ are present in every generic extension $B[g]$, and so they must be in B. Hence they are in A. The Lemma then follows by another use of genericity.

LEMMA 3.5.4. *There is a greatest non-standard ordinal $\beta^* \leqslant \beta$ such that for some $\varphi \in B$, ϕ forces the β^*-type of $F(\tau(f) \circ f)$ to be φ.*

Proof. By Lemma 3.5.3, there is such a non-standard ordinal $\beta^* \leqslant \beta$. To see that there is a largest, let $\lambda \leqslant \beta$ be such that for all $\alpha < \lambda$ there is a $\varphi_\alpha \in B$ such that ϕ forces the α-type of $F(\tau(f) \circ f)$ to be φ_α. Then these φ_α can be put together to produce a φ such that ϕ forces the λ-type of $F(\tau(f) \circ f)$ to be φ.

We now observe that we have passed from any non-standard ordinal β to a non-standard ordinal $\beta^* \leqslant \beta$ in a manner which is definable in (B, R). By the axiom of foundation in (B, R), we can assume without loss of generality that $\beta^* = \beta$.

LEMMA 3.5.5. *The β-type φ of $F(\tau(f) \circ f)$ is in K (according to $B[f]$). $F(\tau(f) \circ f)$ is isomorphic to $\tau(f)(\varphi)$.*

Proof. The first part is immediate from $\beta = \beta^*$. The second part follows from the fact that $\tau(f)(\varphi)$ and $F(\tau(f) \circ f)$ have the same γ-types for $\gamma \in A$, and both sit in $B[f]$.

This completes the proof of the reverse direction of Theorem 3.5. We now prove the forward direction of Theorem 3.5. Assume that the Borel diagonalization theorem holds for N^N under conjugation.

A *full well founded tree* is a partial ordering $\leqslant$ with field N such that (a) there is a least element, (b) the set of predecessors of any number is linearly ordered and finite, and (c) there is no infinite strictly increasing sequence. We associate hereditarily countable sets to full well founded trees by $|\leqslant, n| = \{|\leqslant, m| : n < m\}$; $|\leqslant| = |\leqslant, t|$, where t is the root of $\leqslant$.

Let $F: N \to N$. We say that F is *setlike* if (1) there is a unique n such that $F(n) = n$, (2) for every m there is a $k \geqslant 0$ such that $F^{(k)}(m) = n$, (3) there is no infinite sequence $n = n_0, n_1, n_2, \ldots$, such that each $F(n_{i+1}) = n_i$. For setlike F, we let Tree(F) be the full well founded tree with root n given by $m \leqslant_F r \leftrightarrow (\exists k \geqslant 0)(F^{(k)}(r) = m)$.

The following is left to the reader.

LEMMA 3.5.6. *For setlike F, G, F and G are conjugate if and only if* Tree(F) *and* Tree(G) *are isomorphic. Every full well founded tree is* Tree(F) *for some unique setlike F. There is a Borel function which sends setlike F to* Tree(F). *For each $\alpha < \omega_1$, $W(\alpha) = \{\leqslant : |\leqslant|$ is a transitive element of $V(\alpha + 1)\}$ is Borel. There is a Borel function H: $W(\alpha)^N \to W(\alpha)$ such that* (a) $|H(\mathbf{x})|$ *is the set of all subsets of* $(\bigcup_N \mathbf{x}(n))$ *in $V(\alpha)$ which are first order definable over $\bigcup_N \mathbf{x}(n)$ with parameters, and* (b) *if* $[\mathrm{rng}(\mathbf{x})] = [\mathrm{rng}(\mathbf{y})]$ *then* $[H(\mathbf{x})] = [H(\mathbf{y})]$, *where* [ ] *refers to tree isomorphism.*

Now let $\alpha < \omega_1$, and define the Borel function Φ: $(N^N)^N \to N^N$ as follows. Let $\mathbf{F}: N \to N^N$ be given. Let $\mathbf{x} \in W(\alpha)^N$ be such that $\mathrm{Rng}(\mathbf{x}) = \{\mathrm{Tree}(\boldsymbol{F}(n)) \in W(\alpha) : n \in N\}$. Let $\Phi(\mathbf{F})$ be such that $\mathrm{Tree}(\Phi(\mathbf{F})) = H(\mathbf{x})$.

The hypothesis of Borel diagonalization for N^N under conjugation hold for Φ by Lemma 3.5.6. Hence let $\Phi(\mathbf{F})$ be conjugate to $\mathbf{F}(n)$. Then $H(\mathbf{x})$ is isomorphic to Tree $(\mathbf{F}(n))$, and so $H(\mathbf{x})$ is isomorphic to a term of $\mathbf{x}$. Therefore $|H(\mathbf{x})|$ is a T_α-model.

This completes the proof of Theorem 3.5.

The following is from Theorem 3.5. and the discussion in Section 1.

COROLLARY 3.6. *Proposition* E *can be proved in $Z + AC_\omega + CRA$, but not in $ZC + V = L + \mathscr{R}_1$. It is necessary and sufficient to use ω_1 iterations of the power set operation in order to prove Proposition* E.

Observe that we considered two forms of the basic Borel diagonalization theorem (Proposition C), and one form involved finite invariance ($\sim$). The following theorem shows that there is no such form of general Borel diagonalization.

THEOREM 3.7. *There is a Borel equivalence relation E on $\mathbb{R}$ of finite rank and a Borel function F: $\mathbb{R}^N \to \mathbb{R}$ such that* (a) *if x and y are coor-*

dinatewise equivalent, or if $x \sim y$, *then* $F(x) = F(y)$, *and* (b) *for all* x, $F(x)$ *is not equivalent to any coordinate of* x.

To prove this, we actually construct a Borel function $F\colon (HC_{\omega+2})^{\omega} \to HC_{\omega+2}$ by cases, such that F is invariant under finite permutations of the arguments, yet $F(\mathbf{x}) \notin \mathrm{Rng}(\mathbf{x})$.

Let $\mathbf{x} \in (HC_{\omega+2})^{\omega}$, and set $A = \bigcup \mathrm{Rng}(\mathbf{x})$. Let $\mathfrak{A}$ be the structure $(A \cup \mathrm{Rng}(\mathbf{x}), \in)$.

1. $\mathfrak{A}$ does not satisfy the scheme $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in HC_{\omega}\ \&\ \varphi(y)))$, where φ may have parameters. Choose $\varphi(y, z_1, \ldots, z_n)$ to be the least formula such that the scheme fails for some $z_1, \ldots, z_n \in |\mathfrak{A}|$, and set $F(\mathbf{x}) = \{\{y \in HC_{\omega}\colon \mathfrak{A} \models \varphi(y, z_1, \ldots, z_n)\}\colon z_1, \ldots, z_n \in |\mathfrak{A}|\}$. Obviously $F(x) \notin \mathrm{Rng}(\mathbf{x})$.

2. $\mathfrak{A}$ does satisfy the above scheme, yet $\mathfrak{A}$ does not satisfy the scheme $(\exists x)(\forall y)(y \in x \leftrightarrow (y \subset HC_{\omega}\ \&\ \varphi(y)))$, where this time φ does not have any parameters. Then choose φ to be the least formula such that the scheme fails, and set $F(\mathbf{x}) = \{y \subset HC_{\omega}\colon \mathfrak{A} \models \varphi(y)\}$. Then $F(\mathbf{x}) \notin \mathrm{Rng}(\mathbf{x})$.

3. In this case, cases 1 and 2 fail, and another condition holds, which we give after making the following definitions. An *ordinal* of $\mathfrak{A}$ is a $y \in |\mathfrak{A}|$ such that for some $z \subset HC_{\omega}$ in $|\mathfrak{A}|$, $\mathfrak{A} \models$ "z is a well ordering on a subset of HC_{ω} and y is the set of all well orderings on subsets of HC_{ω} which are isomorphic to z." By using the two schemes in $\mathfrak{A}$, it is easily seen that in $\mathfrak{A}$, every well ordering of HC_{ω} is in some (unique) ordinal.

Let $\gamma(\mathbf{x}) = \{\langle n, m\rangle\colon \mathbf{x}(n)$ and $\mathbf{x}(m)$ are ordinals of $\mathfrak{A}$, and the elements of $\mathbf{x}(n)$ are shorter than the elements of $\mathbf{x}(m)\}$. Let G be the set of all permutations of ω which are the identity almost everywhere.

In this case, we assume $\{\gamma(\mathbf{x} \circ f)\colon f \in G\} \notin \mathrm{Rng}(\mathbf{x})$. Then set $F(\mathbf{x}) = \{\gamma(\mathbf{x} \circ f)\colon f \in G\}$. Again $F(\mathbf{x}) \notin \mathrm{Rng}(\mathbf{x})$.

4. Cases 1, 2 and 3 fail, and $\gamma(\mathbf{x})$ is not a prewell ordering in $\mathfrak{A}$. Then no $\gamma(\mathbf{x} \circ f)$ is a well ordering in $\mathfrak{A}$. Let $S = \{z \subset HC_{\omega}\colon (\exists n)(n$ is in the non-well-founded part of $\gamma(\mathbf{x})$ according to $\mathfrak{A}$, and $z \in \mathbf{x}(n))\}$. Observe that S remains the same if in its definition, $\mathbf{x}$ is replaced by $\mathbf{x} \circ f$, $f \in G$. Now if $S \in |\mathfrak{A}|$, then S is a nonempty set of well orderings on subsets of HC_{ω} with no shortest element, which contradicts the first scheme in $\mathfrak{A}$. Hence $S \notin |\mathfrak{A}|$. Set $F(\mathbf{x}) = S$, and so $F(\mathbf{x}) \notin \mathrm{Rng}(\mathbf{x})$.

5. In this case, cases 1–4 fail, and another condition holds. Note that each $\gamma(\mathbf{x} \circ f)$ is a well ordering in $\mathfrak{A}$. For each n such that $\mathbf{x}(n)$ is an ordinal in $\mathfrak{A}$, let $\gamma(\mathbf{x})_n$ be the initial segment of $\gamma(\mathbf{x})$ up to but not including n. In this case, we assume that the following fails: for every $f \in G$ and n such that $\mathbf{x}(f(n))$ is an ordinal in $\mathfrak{A}$, $\gamma(\mathbf{x} \circ f)_n \in \mathbf{x}(f(n))$.

Fix $f \in G$ such that $\sigma(f) = \{n\colon (\mathbf{x} \circ f)(n)$ is an ordinal in $\mathfrak{A}$, yet $\gamma(\mathbf{x} \circ f)_n \notin (\mathbf{x} \circ f)(n)\}$ is nonempty. Then $\sigma(f)$ is a nonempty subset of

$\mathrm{Fld}(\gamma(\mathbf{x} \circ f))$ with no $\gamma(\mathbf{x} \circ f)$-least element. Hence $\sigma(f) \notin |\mathcal{O}|$. Since each $\sigma(f) \subset \omega$, we can set $F(\mathbf{x}) = \{\sigma(f) : f \in G\}$. Again $F(\mathbf{x}) \notin \mathrm{Rng}(\mathbf{x})$.

6. Cases 1–5 fail. But this case is impossible. For, we know that every $\mathbf{x}(n)$ that is an ordinal, is the ordinal of $\gamma(\mathbf{x})_n$. But there must be a k such that $\mathbf{x}(k)$ is the ordinal of the well ordering $\gamma(\mathbf{x})$. Hence the ordinal of $\gamma(\mathbf{x})_k$ is the same as the ordinal of $\gamma(\mathbf{x})$, which is a contradiction.

This completes the proof of Theorem 3.7. By using bounded complexities in the argument, we can even arrange for F to be finitely Borel.

4. Borel Fixed Point Theorems

We say that $(\mathbb{R}, \leqslant)$ is a *quasi order* if $\leqslant$ is transitive and reflexive (i.e., $(a \leqslant b \;\&\; b \leqslant c) \to a \leqslant c$, $a \leqslant a$). We let $a \simeq b$ mean $(a \leqslant b \;\&\; b \leqslant a)$, $a < b$ mean $(a \leqslant b \;\&$ not $b \leqslant a)$. We say that $(\mathbb{R}, \leqslant)$ is *ω-closed* if every strictly increasing sequence has a (unique up to $\simeq$) least upper bound, and *ω-complete* if every at most countable set has a least upper bound. $F: \mathbb{R} \to \mathbb{R}$ is *invariant* if $a \simeq b \to F(a) \simeq F(b)$. A *fixed point* for F is an x such that $F(x) \simeq x$.

Proposition F. *Let $(\mathbb{R}, \leqslant)$ be an ω-closed (ω-complete) Borel quasi order. Let $F: \mathbb{R} \to \mathbb{R}$ be an invariant Borel function such that for all x, $F(x) \geqslant x$. Then F has a fixed point.*

This follows immediately from (and is in fact equivalent to) Proposition G.

Proposition G. *Let $(\mathbb{R}, \leqslant)$ be an ω-closed (ω-complete) Borel quasi order. Then there is no invariant Borel function such that for all x, $F(x) > x$.*

Proposition H. *Let $(\mathbb{R}, \leqslant)$ be an ω-complete Borel quasi order. Let $F: \mathbb{R} \to \mathbb{R}$ be an invariant Borel function. Then for some x, $F(x) \leqslant x$.*

Theorem 4.1. *The following is provable in $ZF - \mathscr{P}$. Let $\lambda < \omega_1$ be a limit ordinal. If $(\forall \alpha < \omega + \lambda)(V(\alpha)$ exists) then Propositions* F–H *(all five forms) hold for all Borel quasi orders of rank $<\lambda$. If any one of the five forms of Propositions* F–H *holds for all Borel quasi orders of rank $<\lambda$, then for all $\alpha < \lambda$ and $x \subset \omega$ there is a $T_{\omega+\alpha}$-model containing x.*

We now prove the first part of Theorem 4.1. Let $\lambda < \omega_1$, and assume that $(\forall \alpha < \omega + \lambda)(V(\alpha)$ exists). Let $(\mathbb{R}, \leqslant)$ be an ω-closed Borel quasi order of rank α, where $\alpha < \omega + \lambda$.

The proof is closely related to the proof of Borel diagonalization for Borel equivalence relations, but is more delicate. Let $F: \mathbb{R} \to \mathbb{R}$ be an invariant

Borel function such that for all x, $F(x) > x$. We will obtain a contradiction, thus establishing Proposition G. Let $u \subset \omega$ be a Borel code for $\leqslant$, and $v \subset \omega$ be a Borel code for F.

Let Y be the set of all structures (ω, R) satisfying $KP + V = L[u, v] +$ "every set is countable" + "there are arbitrarily large (u, v) − admissible ordinals" + "$(\mathbb{R}, \leqslant)$ is an ω-closed quasi order" + "F is an invariant function with $F(x) > x$," where (ω, R) is an ω-model in which u, v are internal and $\leqslant$, F are described in terms of u, v.

We wish to compare pairs (ω, R), (ω, S) of elements of Y. Within any $(\omega, R) \in Y$, make the following definition by transfinite recursion on the ordinals of (ω, R). Firstly define $G_R(0) = 0$. Define $G_R(\alpha + 1) = F(G_R(\alpha))$, and $G_R(\lambda)$ to be the least upper bound which occurs first in the constructible hierarchy relative to (u, v). Observe that according to (ω, R), the range of G_R forms a strictly increasing sequence of length ω_1. By absoluteness, the range of G_R is really strictly increasing. However, at limits G_R may not give a real least upper bound.

Let H' be the set of ordered pairs (a, b) of ordinals a from (ω, R) and ordinals b from (ω, S) such that $G_R(a) \simeq G_S(b)$. Note that H' is an order preserving partial function. Finally, let H be the largest restriction of H' which maps an initial segment of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S).

We say that (ω, S) is *longer* than (ω, R) if and only if one of the following holds: (i) H maps all of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S) determined by an ordinal of (ω, S), (ii) the domain of H is a proper initial segment of the ordinals of (ω, R) that is not determined by any ordinal of (ω, R) and the range of H is either all of the ordinals of (ω, S) or is an initial segment of the ordinals determined by an ordinal of (ω, S), (iii) the domain and range of H have, respectively, sups a, b which are limit ordinals, and $G_S(b) < G_R(a)$.

LEMMA 4.1.1. *$\{((\omega, R), (\omega, S)): (\omega, R), (\omega, S) \in Y$ and (ω, S) is longer than $(\omega, R)\}$ is a Borel set of rank $<\lambda$. (In case $\lambda = \omega$, we use a suitably large finite fragment of the axioms of KP.)*

Proof. Like Lemma 3.2.1.

LEMMA 4.1.2. *For any ordinal $\alpha < \omega_1^{L[u,v]}$ there is an ordinal $\alpha < \mu < \omega_1^{L[u,v]}$ such that $L_\mu[u, v] \models KP +$ "every set is countable" + "there are arbitrarily large (u, v)-admissible limits of (u, v)-admissible ordinals" + "$(\mathbb{R}, \leqslant)$ is an ω-closed quasi order" + "F is an invariant function with $F(x) > x$," and such that every element of $L_\mu[u, v]$ is definable over $L_\mu[u, v]$ relative to u, v.*

Proof. Suppose that the lemma is false and let $\beta < \omega_1^{L[u,v]}$ be the supremum of all the ordinals in question. Observe that β is definable over $L_{\omega_1}[u, v]$ relative to u, v. Let μ be the first ordinal that is not definable over $L_{\omega_1}[u, v]$ relative to u, v. Then $L_\mu[u, v] \prec L_{\omega_1}[u, v]$. Since "$(\mathbb{R}, \leqslant)$ is an ω-closed quasi order" is π^1_3 relative to u, it holds in $L_{\omega_1}[u, v]$, and therefore in $L_\mu[u, v]$. Hence $L_\mu[u, v] \models KP +$ "every set is countable" $+$ "there are arbitrarily large (u, v)-admissible limits of (u, v)-admissible ordinals" $+$ "$(\mathbb{R}, \leqslant)$ is an ω-closed quasi order" $+$ "F is an invariant function with $F(x) > x$." It is also clear that every element of $L_\mu[u, v]$ is definable over $L_\mu[u, v]$ relative to u, v. This is one of the ordinals in question, which is a contradiction.

LEMMA 4.1.3. *For any ordinal $\alpha < \omega_1^{L[u,v]}$ there is an ordinal $\alpha < \mu < \omega_1^{L[u,v]}$ such that $L_\mu[u, v] \models KP +$ "every set is countable" $+$ "$(\mathbb{R}, \leqslant)$ is an ω-closed quasi order" $+$ "F is an invariant function with $F(x) > x$" $+$ "there is a largest (u, v)-admissible limit of (u, v)-admissible ordinals," and such that every element of $L_\mu[u, v]$ is definable over $L_\mu[u, v]$ relative to u, v.*

Proof. See the proof of Lemma 3.2.3 from Lemma 3.2.2.

LEMMA 4.1.4. *Let (ω, R) be a well founded element of Y of ordinal μ, and let $(\omega, S) \in Y$. If (ω, S) is longer than (ω, R) then the ordinal of the standard part of (ω, S) is at least $\mu + 1$.*

Proof. Assume hypotheses. If the comparison map H is totally defined then we are done. Otherwise let α be the (order type of the) first ordinal at which H is undefined. Then α is obviously a limit ordinal. Since (ω, S) is longer than (ω, R), there must be an ordinal of (ω, S) of type α, and $G_S(\alpha) < G_R(\alpha)$. But since (ω, R) is well founded absolute we see that $G_R(\alpha)$ is a least upper bound for $\{G_R(\beta): \beta < \alpha\}$, and hence for $\{G_S(\beta): \beta < \alpha\}$. This contradicts $G_S(\alpha) < G_R(\alpha)$, since $G_S(\alpha)$ is also an upper bound for $\{G_S(\beta): \beta < \alpha\}$.

LEMMA 4.1.5. *Let (ω, S) be a well founded element of Y of ordinal μ, and let $(\omega, R) \in Y$. If the ordinal of the standard part of (ω, R) is $\leqslant \mu$ and (ω, R) is not isomorphic to (ω, S), then (ω, S) is longer than (ω, R).*

Proof. Assume hypotheses. Firstly, suppose that the comparison map H is defined at all the standard ordinals of (ω, R). Then obviously (ω, S) is longer than (ω, R). Secondly, suppose that the comparison map H is not defined at all the standard ordinals of (ω, R). Let $\alpha < \mu$ be the (order type of the) first standard ordinal of (ω, R) at which H is not defined. Then α is a limit ordinal. Observe that since (ω, S) is well founded absolute, $G_S(\alpha)$ is really a least upper bound for $\{G_S(\beta): \beta < \alpha\}$. Because H is not defined at

$G_R(\alpha)$, $G_R(\alpha)$ must not be a least upper bound for $\{G_R(\beta): \beta < \alpha\}$. Hence $G_S(\alpha) < G_R(\alpha)$, and so (ω, S) is longer than (ω, R).

Now let $Z = \{((\omega, R), (\omega, S))$: if $(\omega, R) \in Y$ and $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible limits of (u, v)-admissible ordinals," then $(\omega, S) \in Y$, $(\omega, S) \models$ "there is a largest (u, v)-admissible limit of (u, v)-admissible ordinals," and (ω, S) is longer than $(\omega, R)\}$. We can view Z as a subset of the Cantor square, and we can play the game where II wins if and only if the pair of plays is in Z. Observe that Z is a Borel set of rank $<\lambda$, and we can use determinacy for such Borel sets (by $(\forall \alpha < \omega + \lambda)(V(\alpha)$ exists)).

By absoluteness, the game has a winning strategy J in $L[u, v]$.

LEMMA 4.1.6. *J is not a winning strategy for* II. *J is not a winning strategy for* I.

Proof. Let $\alpha < \omega_1^{L[u,v]}$ be such that J has a Borel code in $L_\alpha[u, v]$. Firstly, suppose that J is a winning strategy for II. Choose μ according to Lemma 4.1.2. Let (ω, R) be isomorphic to $L_\mu[u, v]$, where R is arithmetic in the theory of $L_\mu[u, v]$ relative to (u, v). Observe that $(\omega, R) \in Y$ and $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible limits of (u, v)-admissible ordinals." Let (ω, S) be J applied to (ω, R). Then $(\omega, S) \in Y$ and (ω, S) is longer than (ω, R). By Lemma 4.1.4, the ordinal of the standard part of (ω, S) is at least $\mu + 1$. Hence the ordinal of the standard part of (ω, S) is at least the next (u, v)-admissible after μ. This is a contradiction by recursion theoretic considerations.

Secondly, suppose that J is a winning strategy for I. Choose μ according to Lemma 4.1.3. Let (ω, S) be isomorphic to $L_\mu[u, v]$, where S is arithmetic in the theory of $L_\mu[u, v]$ relative to (u, v). Observe that $(\omega, S) \in Y$ and $(\omega, S) \models$ "there is a largest (u, v)-admissible limit of (u, v)-admissible ordinals." Let (ω, R) be J applied to (ω, S). Then $(\omega, R) \in Y$, $(\omega, R) \models$ "there are arbitrarily large (u, v)-admissible limits of (u, v)-admissible ordinals," and (ω, S) is not longer than (ω, R). By Lemma 4.1.5, the ordinal of the standard part of (ω, R) is at least $\mu + 1$. Hence the ordinal of the standard part of (ω, R) is at least the next (u, v)-admissible after μ. This is a contradiction for recursion theoretic reasons.

This completes our proof of Proposition G for ω-closed Borel quasi orders. The proof of Proposition H is virtually identical, and we omit it. Thus the proof of the first half of Theorem 4.1 is complete. For the second half, the following is sufficient by Theorem 3.2.

LEMMA 4.1.7. *The following is provable in $ZF - \mathscr{P}$. Let $\lambda < \omega_1$ be a limit ordinal. If any one of Propositions* F–H *holds for all Borel quasi orders of rank $<\lambda$, then Proposition* D *holds for all Borel equivalence relations of rank $<\lambda$.*

Proof. Observe that the weakest of the five forms is Proposition F (or G) with ω-completeness. Let E be a Borel equivalence relation on $\mathbb{R}$ of rank $<\lambda$. Consider the Borel quasi order $(\mathbb{R}^N, \leqslant)$ given by: $x \leqslant y$ if and only if every coordinate of x is equivalent to some coordinate of y (under E). We can obviously view $(\mathbb{R}^N, \leqslant)$ as a Borel quasi order on $\mathbb{R}$ of rank $<\lambda$. It is clear that $\leqslant$ is ω-complete.

Suppose $F: \mathbb{R}^N \to \mathbb{R}$ is a counterexample to the Borel diagonalization theorem for E. Let $G: \mathbb{R}^N \to \mathbb{R}^N$ be given by: $G(x)(1) = F(x)$; $G(x)(n+1) = x(n)$. Then clearly G is invariant and $G(x) \geqslant x$ for all x. However, clearly G has no fixed point. This completes the proof of the lemma, and hence of Theorem 4.1.

The following are from Theorem 4.1 and the discussion in Section 1.

COROLLARY 4.2. *Propositions* F–H *(all five forms) can be proved in* $Z + AC_\omega + CRA$, *but not in* $ZC + V = L + \mathscr{R}_1$. *If we restrict to finitely Borel quasi orders, then they are provable in* $Z + AC_\omega$, *but not in weak* $ZC + V = L$.

COROLLARY 4.3. *For any one of the five forms of Propositions* F–H, *it is necessary and sufficient to use* ω_1 *iterations of the power set operation in order to give a proof. If we restrict to finitely Borel quasi orders, then it is necessary and sufficient to use* $\omega + \omega$ *iterations of the power set operation in order to give a proof. The later also holds true if everything involved is restricted to the finitely Borel.*

Notice that in the hypotheses on the quasi orders in Propositions F–H, the least upper bounds that are hypothesized may not be given explicitly. It is natural to ask whether the logical strength of these propositions is due to this lack of explicitness in the hypotheses. The answer is no by the following.

Let $(\mathbb{R}, \leqslant)$ be a quasi order. We say that it is *explicitly ω-closed* if there is a Borel function $H: \mathbb{R}^N \to \mathbb{R}$ such that for all strictly increasing $x \in \mathbb{R}^N$, $H(x)$ is a least upper bound for $\operatorname{rng}(x)$. We say that $(\mathbb{R}, \leqslant)$ is *explicitly ω-complete* if there is a Borel function $H: \mathbb{R}^N \to \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $H(x)$ is a least upper bound for $\operatorname{rng}(x)$.

We rephrase Propositions F–H using these stronger hypotheses.

PROPOSITION I. *Let $(\mathbb{R}, \leqslant)$ be an explicitly ω-closed (explicitly ω-complete) Borel quasi order. Let $F: \mathbb{R} \to \mathbb{R}$ be an invariant Borel function such that for all x, $F(x) \geqslant x$. Then F has a fixed point.*

PROPOSITION J. *Let $(\mathbb{R}, \leqslant)$ be an explicitly ω-closed (explicitly ω-complete) Borel quasi order. Then there is no invariant Borel function such that for all x, $F(x) > x$.*

PROPOSITION K. *Let* $(\mathbb{R}, \leqslant)$ *be an explicitly* ω*-complete Borel quasi order. Let* $F: \mathbb{R} \to \mathbb{R}$ *be an invariant Borel function. Then for some* x, $F(x) \leqslant x$.

THEOREM 4.4. *The following is provable in* $ZF - \mathscr{P}$. *Let* $\lambda < \omega_1$, *be a limit ordinal. Then each one of the five forms of Propositions* I–K *for Borel quasi orders of rank* $<\lambda$ *is equivalent to "for every* $\alpha < \omega + \lambda$ *and* $x \subset \omega$ *there is a* T_α*-model containing* x."

Proof. The forward direction is immediate from the proof of Lemma 4.1.7; the Borel quasi orders constructed there are explicitly ω-complete. For the reverse, let $\alpha < \lambda$ and assume that for all $\beta < \omega + \lambda$, $x \subset \omega$, there is a T_β-model containing x. It follows from the proof of Theorem 4.1 that for some $\beta < \lambda$, every T_β-model satisfies Propositions F–H for Borel quasi orders of rank α, and hence Propositions I–K for Borel quasi orders of rank α. Now observe that Propositions I–K for Borel quasi orders of rank α are π^1_2 (in a code for α). Since we are assuming that there are T_β-models containing any given real, we see that Propositions I–K hold for Borel quasi orders of rank α.

We now consider Borel partial orders. A *Borel partial order* is a Borel quasi order $(\mathbb{R}, \leqslant)$ in which $a \simeq b \to a = b$.

PROPOSITION L. *Every* ω*-closed* (ω*-complete*) *Borel partial order has a maximal element.*

PROPOSITION M. *Every* ω*-complete Borel partial order has a maximum element.*

THEOREM 4.5. *Propositions* L, M (*all three forms*) *are provable in* $Z + AC_\omega + CRA$.

The proof of this theorem is closely related to that of Propositions F–H. Let $(\mathbb{R}, \leqslant)$ be an ω-closed Borel partial order with no maximal element. Let $u \subset \omega$ be a Borel code for $\leqslant$.

Let Y be the set of all structures (ω, R) satisfying $KP + V = L[u]$ + "every set is countable" + "there are arbitrarily large u-admissible ordinals" + "$(\mathbb{R}, \leqslant)$ is an ω-closed partial order with no maximal element," where (ω, R) is an ω-model in which u is internal and $(\mathbb{R}, \leqslant)$ is described using u.

We wish to compare pairs (ω, R), (ω, S) of elements of Y. Within any $(\omega, R) \in Y$, make the following definition by transfinite recursion on the ordinals of (ω, R). Firstly define $G_R(0) = 0$. Define $G_R(\lambda)$ for limit ordinals λ as the least upper bound of $\{G_R(\alpha): \alpha < \lambda\}$. Define $G_R(\alpha + 1)$ as follows. In (ω, R) let x be the hyperjump of $(G_R(\alpha), u)$; i.e., the set of all indices e of $(G_R(\alpha), u)$-recursive well orderings on ω. Take $G_R(\alpha + 1)$ to be the real number y recursive in x with least recursive index such that $G_R(a) < y$. Of course, the hyperjump according to (ω, R) may not be the actual hyperjump, and in the limit case, the least upper bound according to (ω, R) may not be the actual least upper bound.

Let H' be the set of ordered pairs (a, b) of ordinals a from (ω, R) and ordinals b from (ω, S) such that $G_R(a) = G_S(b)$. Note that H' is an order preserving partial function. Finally, let H be the largest restriction of H' which maps an initial segment of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S).

We say that (ω, S) is *longer* than (ω, R) if and only if one of the following holds: (i) H maps all of the ordinals of (ω, R) onto an initial segment of the ordinals of (ω, S) determined by an ordinal of (ω, S), (ii) the domain of H is a proper initial segment of the ordinals of (ω, R) that is not determined by any ordinal of (ω, R) and the range of H is either all of the ordinals of (ω, S) or is an initial segment of the ordinals determined by an ordinal of (ω, S), (iii) the domain and range of H have, respectively, sups a, b which are limit ordinals and $G_S(b) < G_R(a)$, (iv) the domain and range of H have, respectively, largest elements a, b, and the hyperjump of $G_S(b)$ in (ω, S) is properly included in the hyperjump of $G_R(a)$ in (ω, R).

LEMMA 4.5.1. *$\{((\omega, R), (\omega, S))$: $(\omega, R), (\omega, S) \in Y$ and (ω, S) is longer than $(\omega, R)\}$ is a Borel set.*

Proof. Left to the reader.

LEMMA 4.5.2. *For any ordinal $\alpha < \omega_1^{L[u]}$ there is an ordinal $\alpha < \lambda < \omega_1^{L[u]}$ such that $L_\lambda[u] \models KP +$ "every set is countable" $+$ "there are arbitrarily large u-admissible limits of u-admissible ordinals" $+$ "$(\mathbb{R}, \leqslant)$ is an ω-closed partial order with no maximal element," and such that every element of $L_\lambda[u]$ is definable over $L_\lambda[u]$ relative to u. Furthermore, this is true if "there are arbitrarily large" is replaced by "there is a largest."*

Proof. See the proofs of Lemmas 4.1.2 and 4.1.3.

LEMMA 4.5.3. *Let (ω, R) be a well founded element of Y of ordinal λ, and let $(\omega, S) \in Y$. If (ω, S) is longer than (ω, R) then the ordinal of the standard part of (ω, S) is at least $\lambda + 1$.*

Proof. Assume hypotheses. If the comparison map H is defined at all of the ordinals of (ω, R) then we are done. Otherwise let α be the (order type of the) first ordinal at which H is undefined. Let us first suppose that α is a limit ordinal. Since (ω, S) is longer then (ω, R), there must be an ordinal of (ω, S) of type α, and $G_S(\alpha) < G_R(\alpha)$. But since (ω, R) is well founded absolute, $G_R(\alpha)$ is really the appropriate least upper bound, and hence we have a contradiction. Now suppose that $\alpha = \beta + 1$. Then the hyperjump in (ω, S) of $G_S(\beta)$ is properly included in the hyperjump in (ω, R) of $G_R(\beta) = G_S(\beta)$. But the hyperjump in (ω, R) of $G_R(\beta) = G_S(\beta)$ is the actual hyperjump. This is a contradiction.

LEMMA 4.5.4. *Let (ω, S) be a well founded element of Y of ordinal λ, and let $(\omega, R) \in Y$. If the ordinal of the standard part of (ω, R) is $\leqslant \lambda$ and (ω, R) is not isomorphic to (ω, S), then (ω, S) is longer than (ω, R).*

Proof. Assume hypotheses. If the comparison map H is defined at all the standard ordinals of (ω, R), then obviously (ω, S) is longer than (ω, R). Now suppose that H is not defined at all the standard ordinals of (ω, R). Let $\alpha < \lambda$ be the (order type of the) first standard ordinal of (ω, R) at which H is not defined. Firstly, suppose that α is a limit ordinal. Since (ω, S) is well founded absolute, $G_S(\alpha)$ is really the least upper bound for $\{G_S(\beta) : \beta < \alpha\}$. Since $G_S(\alpha) \neq G_R(\alpha)$, we have $G_S(\alpha) < G_R(\alpha)$, and so (ω, S) is longer than (ω, R). Secondly, suppose that $\alpha = \beta + 1$. Since H is not defined at α, clearly the hyperjump of $G_S(\beta)$ in (ω, R) is not the same as the hyperjump of $G_S(\beta) = G_R(\beta)$ in (ω, S). By the well founded absoluteness of (ω, S), the hyperjump in (ω, S) is properly included in the hyperjump in (ω, R). Hence (ω, S) is longer than (ω, R).

Now let $Z = \{((\omega, R), (\omega, S))$: if $(\omega, R) \in Y$ and $(\omega, R) \models$ "there are arbitrarily large u-admissible limits of u-admissible ordinals" then $(\omega, S) \in Y$, $(\omega, S) \models$ "there is a largest u-admissible limit of u-admissible ordinals," and (ω, S) is longer than $(\omega, R)\}$. We can view Z as a Borel subset of the Cantor square, and we can play the game where II wins if and only if the pair of plays is in Z. By Borel determinacy, this game has a winning strategy. By absoluteness, this game has a winning strategy J in $L[u]$.

LEMMA 4.5.5. *J is not a winning strategy for* II. *J is not a winning strategy for* J.

Proof. See the proof of Lemma 4.1.6.

This completes our proof of Proposition L. Proposition M follows immediately from Proposition L. This completes the proof of Theorem 4.5. If we just want L, M for finitely Borel partial orders, then $Z + AC_\omega$ suffices.

We now give proofs of the explicit forms of Propositions L, M. Here we use substantially less set theory.

PROPOSITION N. *Every explicitly ω-closed Borel partial order has a maximal element.*

PROPOSITION O. *Every explicitly ω-complete Borel partial order has a maximum element.*

THEOREM 4.6. *Propositions* N, O *are provable in* $MK + AC_\omega - \mathscr{P}$, *and* $ZF + AC_\omega - \mathscr{P} + $ "$\mathscr{P}(\omega)$ *exists.*"

Proof. It is easy to see that these two systems prove the same π^1_2 sentences. We now prove Proposition N in $ZF + AC_\omega - \mathscr{P} + $ "$\mathscr{P}(\omega)$ exists."

Let $(\mathbb{R}, \leqslant)$ be a Borel partial order, and let $H: \mathbb{R}^N \to \mathbb{R}$ be a Borel function such that for strictly increasing $x \in \mathbb{R}^N$, $H(x)$ is the least upper bound of $\mathrm{rng}(x)$. Let $u \subset \omega$ code $\leqslant$ and $v \subset \omega$ code H. By a Skolem hull argument, we can produce a countable Σ_3-elementary substructure of $L[u, v]$, which can be isomorphically collapsed onto some $L_\lambda[u, v]$. It is then clear that $L_\lambda[u, v]$ satisfies Σ_2-replacement + "ω_1 exists," and $\lambda < \omega_1$.

Inside $L_\lambda[u, v]$ we can define a partial function $G: On \to \mathbb{R}$ by $G(0) = 0$, $G(\alpha + 1) =$ the first element in the constructible hierarchy relative to (u, v) which is greater than $G(\alpha)$, and $G(\lambda) =$ the least upper bound of $\{G(\alpha): \alpha < \lambda\}$. This definition is made as far as possible.

Observe that by the axioms of Σ_2-replacement + "ω_1 exists," we see that in $L_\lambda[u, v]$, G must not be total. We now assume that $(\mathbb{R}, \leqslant)$ has no maximal element. Then this also holds in $L_\lambda[u, v]$. Hence the range of G is an unbounded countable strictly increasing transfinite sequence. Let x be its least upper bound. Using H, observe that x must be present in every generic extension of $L_\lambda[u, v]$ obtained by collapsing ω_1 to ω (because the range of G has cardinality at most ω_1 in $L_\lambda[u, v]$). Hence $x \in L_\lambda[u, v]$, which contradicts the fact that G is defined as far as possible.

We prove Proposition O in $MK + AC_\omega - \mathscr{P}$ as follows.

Let $(\mathbb{R}, \leqslant)$ be a Borel partial order with code $u \subset \omega$, and let $H: \mathbb{R}^N \to \mathbb{R}$ be a Borel function with code $v \subset \omega$ such that for $x \in \mathbb{R}^N$, $H(x)$ is the least upper bound of $\mathrm{rng}(x)$. Let A be a countable admissible set which contains u, v. By a Skolem hull argument, we can produce a countable elementary substructure of $L[u, v]$ with respect to $\mathscr{L}_A$, which can be isomorphically collapsed onto some $L_\lambda[u, v]$. It is then clear that $L_\lambda[u, v]$ satisfies $\mathscr{L}_A - ZFC$ without power set.

If $j: \omega \to L_\lambda[u, v]$ is any generic enumeration of $L_\lambda[u, v]$, we can define the least upper bound x of $L_\lambda[u, v] \cap \mathbb{R}$ by a formula in $\mathscr{L}_A$ over $(L_\lambda[u, v], \in, j)$. The definition used is uniform, and so x is defined by a formula in $\mathscr{L}_A$ over $L_\lambda[u, v]$. By $\mathscr{L}_A$-replacement, $x \in L_\lambda[u, v]$. Therefore $L_\lambda[u, v] \models$ "x is the maximum element of $(\mathbb{R}, \leqslant)$." Hence x really is the maximum element of $(\mathbb{R}, \leqslant)$.

Of course, Proposition O immediately follows from Proposition N, but the above proof of Proposition O uses a weaker fragment of $MK + AC_\omega - \mathscr{P}$ than does the above proof of Proposition N. In particular, as is the case with

Proposition C, Proposition O is proved in $VB + AC_\omega - \mathscr{P} +$ "transfinite class recursion on countable well orderings." And if we restrict everything to any fixed finite level of the Borel hierarchy, then Propositions C, O are provable in second order arithmetic (or $ZF + AC_\omega - \mathscr{P}$).

5. Borel Ramsey Theory

We let $Q = I^N$ be the space of infinite sequences from I, with the infinite product topology (Hilbert Cube). The group H of all permutations of N which fix all but finitely many natural numbers acts on Q by permuting coordinates. This group also acts diagonally on any Q^n by $g \cdot (x_1,\dots, x_n) = (g \cdot x_1,\dots, g \cdot x_n)$. For $x, y \in Q^n$ we use $x \sim y$ to indicate that x and y are in the same orbit under this diagonal action.

In this language, the basic Borel diagonalization theorem asserts the following: if $F: Q \to I$ is a Borel function such that $x \sim y \to F(x) = F(y)$, then for some x, $F(x)$ is the first coordinate of x.

An immediate consequence of the above is: if $F: Q \times Q \to I$ is a Borel function such that $y \sim z \to F(x, y) = F(x, z)$, then for some x, y, $F(x, y)$ is the first coordinate of y.

An extension of the above is: if $F: Q \times Q \to I$ is a Borel function such that $y \sim z \to F(x, y) = F(x, z)$, then there exists an infinite sequence $\{x_k\}$ from Q such that for all indices s, $F(x_s, x_{s+1})$ is the first coordinate of x_{s+1}.

A further extension of the above is: if $F: Q \times Q \to I$ is a Borel function such that $y \sim z \to F(x, y) = F(x, z)$, then there exists an infinite sequence $\{x_k\}$ from Q such that for all indices $s < t$, $F(x_s, x_t)$ is the first coordinate of x_{s+1}.

Finally, we come to an even further extension of the above.

Proposition P. *Let $F: Q \times Q^n \to I$ be a Borel function such that if $x \in Q$, $y, z \in Q^n$, and $y \sim z$, then $F(x, y) = F(x, z)$. Then there is a sequence $\{x_k\}$ from Q of length $m \leqslant \omega$ such that for all indices $s < t_1 < \cdots < t_n \leqslant m$, $F(x_s, x_{t_1},\dots, x_{t_n})$ is the first coordinate of x_{s+1}.*

A 0-*Mahlo* cardinal is a strongly inaccessible cardinal. An $(n + 1)$-*Mahlo* cardinal is a cardinal in which every closed and unbounded subset contains an n-Mahlo cardinal. A *Mahlo* cardinal is a 1-Mahlo cardinal.

Theorem 5.1. *$ZFC + (\forall n)(\exists\kappa)(\kappa$ is n-Mahlo$)$ proves Proposition* P. *However for every n, $ZFC + (\exists\kappa)(\kappa$ is $\bar{n}$-Mahlo$) + V = L$ does not prove Proposition* P.

We first prove Proposition P in $ZFC + (\forall n)(\exists\kappa)(\kappa$ is n-Mahlo$)$. Actually for expositional purposes, it is convenient to first prove the following weaker form of Proposition P—the forcing lemmas needed are much easier in this case. For $x, y \in Q$ let $x \approx y$ mean that x, y have the same range. For

$x, y \in Q^n$, let $x \approx y$ mean that for all $1 \leqslant i \leqslant n$, $x(i) \approx y(i)$. The weakened form of Proposition P that we prove first is the same as Proposition P except that $\sim$ is replaced by $\approx$.

We begin by fixing $n \geqslant 1$ and a Borel function $F: Q \times Q^n \to I$ such that $y \approx z \to F(x, y) = F(x, z)$. Let $u \subset \omega$ be a Borel code for F. By a Skolem hull argument, we can fix a countable transitive set M satisfying $ZFC + (\exists\kappa)(\kappa$ is $(n-1)$-Mahlo), where $u \in M$. We fix κ to be an $(n-1)$-Mahlo cardinal in M.

It will of course be more convenient to switch to a more set theoretic mode, and identify I with $\mathscr{P}(\omega)$ and Q with $\mathscr{P}(\omega)^\omega$.

In M, we use the following notion of forcing. The set C of conditions consists of all finite partial functions $f: \kappa \times \omega \to V(\kappa)^M$ such that $f(\alpha, m) \in V(\alpha)^M$. The partial ordering $\leqslant$ on C is of course inclusion. Let C_α be the set of conditions whose domain is included in $\alpha \times \omega$.

Observe that C is the standard notion of forcing for adding a system $\{f_\alpha\}_{\alpha<\kappa}$ of mutually generic enumerations $f_\alpha: \omega \to V(\alpha)^M$.

Let $G \subset C$ be a generic set of conditions over M. Define $\bar{G}: \kappa \times \omega \to V(\kappa)^M$ by $\bar{G}(\alpha, n) = x$ if and only if $(\exists f \in G)(f(\alpha, n) = x)$. For $x \in M$ let $E(G, x) = \{k: (\exists f \in G)((k, f) \in x)\}$. Here E stands for "evaluation."

For limit ordinals $\lambda < \kappa$ we define $T(G, \lambda) \in \mathscr{P}(\omega)^\omega$ by $T(G, \lambda)(m) = E(G, \bar{G}(\lambda, m))$.

For $f, g \in C$ we define $g \mid f$ by $(g \mid f)(\alpha, m) = f(\alpha, m)$ if $f(\alpha, m)$ and $g(\alpha, m)$ are defined; $g(\alpha, m)$ if $g(\alpha, m)$ is defined and $f(\alpha, m)$ is undefined; undefined if $g(\alpha, m)$ is undefined. For $f \in C$, we define $G \mid f = \{g \mid f: g \in G\}$. Note that $G \mid f$ is also generic.

LEMMA 5.1.1. *Let $G \subset C$ be generic over M, $\lambda < \kappa$ be a limit ordinal, and $f \in C$. Then $T(G, \lambda) \approx T(G \mid f, \lambda)$.*

Proof. By symmetry, it is enough to prove that $\operatorname{rng}(T(G, \lambda)) \subset \operatorname{rng}(T(G \mid f, \lambda))$. Let $m \in \omega$, and consider $T(G, \lambda)(m) = E(G, \bar{G}(\lambda, m))$. Let $x = \{(k, g \mid f): g \in C \cap \bar{G}(\lambda, m)\}$. Then $x \in V(\lambda)^M$ and $E(G \mid f, x) = E(G, \bar{G}(\lambda, m))$. Obviously by genericity, there are infinitely many $r \in \omega$ such that $\bar{G}(\lambda, r) = x$, and hence there is an r such that $\overline{G \mid f}(\lambda, r) = x$. Therefore $T(G, \lambda)(m) = T(\overline{G \mid f}, \lambda)(r)$.

LEMMA 5.1.2. *Let $\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$ be limit ordinals, and let $f \in C$. Then for all k, $f \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$ if and only if $f \restriction ((\lambda + 1) \times \omega) \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$.*

Proof. By way of contradiction, assume $f \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$, and $g \Vdash k \notin F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$, where $f \restriction ((\lambda + 1) \times \omega) \leqslant g$. Let $G \subset C$ be generic over M, where $g \in G$. Obviously $T(G, \lambda) = T(G \mid f, \lambda)$. By Lemma 5.1.1, each $T(G, \lambda_i) \approx T(G \mid f, \lambda_i)$. Hence by the symmetry condition on F, $F(T(G,\lambda), T(G,\lambda_1),..., T(G,\lambda_n)) = F(T(G\mid f,\lambda), T(G\mid f,\lambda_1),..., T(G\mid f,\lambda_n))$. This is a contradiction since $g \in G$, $f \in G\mid f$.

We are now prepared to apply the combinatorics in [14]. The following definitions are taken from there.

DEFINITION 1. A partition C of the set X is a collection of pairwise disjoint sets, the union of which is X. Two elements in the same set of C are called C-equivalent. The set of subsets of X of cardinality n is denoted by $[X]^n$. If C is a partition of $[X]^n$, then $Y \subset X$ is C-homogeneous if and only if every two elements of $[Y]^n$ are C-equivalent.

DEFINITION 2. (i) C is an *f-partition system* of $[\alpha]^n$ if and only if f is a cardinal-valued function with domain including α such that for each $\nu < \alpha$, C_ν is a partition of $[\alpha]^n$ and $\mathrm{card}(C_\nu) \leqslant f(\nu)$.

(ii) C is a *partition system* of $[\alpha]^n$ if and only if for some $f\colon \alpha \to \alpha$, C is an f-partition system of $[\alpha]^n$.

(iii) If C is an f-partition system of $[\alpha]^n$ then $X \subset \alpha$ is *C-homogeneous* if and only if for each $\nu \in X$, the set $X - (\nu + 1)$ is C_ν-homogeneous.

(iv) $P(k,\alpha)$ is the class of all cardinals μ such that for any partition system C of $[\mu]^k$ there is a C-homogeneous set of length α.

The following is from Theorem 3.1 of [14].

LEMMA 5.1.3. *The following is provable in ZFC. For all k, if μ is a k-Mahlo cardinal, then for all $\alpha < \mu$, $\mu \in P(k+1,\alpha)$.*

It is convenient to give a slightly altered form of the above partition relation. Let $P'(k,\mu,\omega)$ assert the following: $k \geqslant 1$, μ is a cardinal, and for every function $H\colon \mu^{k+1} \to V(\mu)$ such that each $H[\{\alpha\} \times \mu^k] \in V(\mu)$, $\alpha < \mu$, there is an infinite strictly increasing sequence of limit ordinals $\{\lambda_m\}$ such that for any $s < t_1 < \cdots < t_k$, $s < r_1 < \cdots < r_k$, $H(\lambda_s, \lambda_{t_1},..., \lambda_{t_k}) = H(\lambda_s, \lambda_{r_1},..., \lambda_{r_k})$.

LEMMA 5.1.4. *The following is provable in ZFC. If $\mu > \omega$ is a strongly inaccessible cardinal, and $1 \leqslant k$, then $\mu \in P(k,\omega)$ if and only if $P'(k,\mu,\omega)$.*

Proof. Left to the reader.

LEMMA 5.1.5. *In M, $P'(n,\kappa,\omega)$.*

Proof. By Lemmas 5.1.3 and 5.1.4.

We now define a specific $H\colon \kappa^{n+1} \to V(\kappa)^M$ as follows. For limit ordinals

$\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$, we let $H(\lambda, \lambda_1, \ldots, \lambda_n) = \{(k, f): f \in C_{\lambda+1} \,\&\, f \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))\}$. Define H to be ϕ at other inputs.

LEMMA 5.1.6. *Let $G \subset C$ be generic over M, and let $\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$ be limit ordinals. Then $F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n)) = \{k: (\exists f \in G)((k, f) \in H(\lambda, \lambda_1, \ldots, \lambda_n))\} = E(G, H(\lambda, \lambda_1, \ldots, \lambda_n))$.*

Proof. Let $k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$. Let $g \in G$ be such that $g \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$. By Lemma 5.1.2, $g \restriction ((\lambda + 1) \times \omega) \Vdash k \in F(T(G, \lambda), T(G, \lambda_1), \ldots, T(G, \lambda_n))$, and so $(k, g \restriction ((\lambda + 1) \times \omega)) \in H(\lambda, \lambda_1, \ldots, \lambda_n)$. The converse is evident.

Now we apply Lemma 5.1.5 to H to obtain a strictly increasing sequence $\{\lambda_m\}$ of length ω of limit ordinals below κ such that for all $s < t_1 < \cdots < t_n$, $s < r_1 < \cdots < r_n$, $H(\lambda_s, \lambda_{t_1}, \ldots, \lambda_{t_n}) = H(\lambda_s, \lambda_{r_1}, \ldots, \lambda_{r_n})$. Fix $G \subset C$ to be generic over M.

For each $m < \omega$ let f_m be the condition with domain $\{(\lambda_{m+1}, 0)\}$ and value $H(\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n})$. Recursively define $G_0 = G \mid f_0$, $G_{m+1} = G_m \mid f_{m+1}$. Note that each G_m is generic.

LEMMA 5.1.7. *For each $s < t_1 < \cdots < t_n < t$, $F(T(G_t, \lambda_s), T(G_t, \lambda_{t_1}), \ldots, T(G_t, \lambda_{t_n})) = T(G_t, \lambda_{s+1})(0)$.*

Proof. $F(T(G_t, \lambda_s), T(G_t, \lambda_{t_1}), \ldots, T(G_t, \lambda_{t_n})) = E(G_t, H(\lambda_s, \lambda_{s+1}, \ldots, \lambda_{s+n}))$ by Lemma 5.1.6 and the indiscernibility of $\{\lambda_n\}$. On the other hand, $T(G_t, \lambda_{s+1})(0) = E(G_t, f_s(\lambda_{s+1}, 0)) = E(G_t, H(\lambda_s, \lambda_{s+1}, \ldots, \lambda_{s+n}))$.

LEMMA 5.1.8. *For each $s < t_1 < \cdots < t_n$, $F(T(G_s, \lambda_s), T(G_{t_1}, \lambda_{t_n}), \ldots, T(G_{t_n}, \lambda_{t_n})) = T(G_{s+1}, \lambda_{s+1})(0)$.*

Proof. Observe that $T(G_t, \lambda_i) = T(G_i, \lambda_i)$ for $i < t$.

This completes the proof of Proposition P with $\sim$ replaced by $\approx$, since we may take $\{T(G_m, \lambda_m)\}$ as our sequence from $\mathscr{P}(\omega)^\omega$.

We now only assume that $y \sim z \to F(x, y) = F(x, z)$, and elaborate on the above argument to obtain the same conclusion.

Let $G \subset C$ be generic over M. Define the all important map G^*: $\{\lambda < \kappa: \lambda$ is a limit ordinal$\} \times \omega \to V(\kappa)^M$ as follows.

We first make the convention that for ordered pairs (a, b), $(a, b)(1) = a$ and $(a, b)(2) = b$.

If $\bar{G}(\lambda, 0) \in C_\lambda \times V(\lambda)^M$, let $G^*(\lambda, 0) = \bar{G}(\lambda, 0)$; otherwise let $G^*(\lambda, 0) = (\phi, \phi)$. If $\bar{G}(\lambda, m + 1) \in C_\lambda \times V(\lambda)^M$ and the domain of $\bar{G}(\lambda, m + 1)(1)$ includes the domain of $G^*(\lambda, m)(1)$, let $G^*(\lambda, m + 1) = \bar{G}(\lambda, m + 1)$; otherwise let $G^*(\lambda, m + 1) = (p, \phi)$, where p is constantly ϕ and has the same domain as $G^*(\lambda, m)(1)$.

Observe that each $\operatorname{dom}(G^*(\lambda, m)(1)) \subset \operatorname{dom}(G^*(\lambda, m + 1)(1))$.

Similarly, for conditions $f \in C$ we define the partial function f^*: $\{\lambda < \kappa$: λ is a limit ordinal$\} \times \omega \to V(\kappa)^M$ as follows. If $f(\lambda, 0) \in C_\lambda \times V(\lambda)^M$, let $f^*(\lambda, 0) = f(\lambda, 0)$; otherwise let $f^*(\lambda, 0) = (\phi, \phi)$. If $f(\lambda, m+1) \in C_\lambda \times V(\lambda)^M$ and the domain of $f(\lambda, m+1)(1)$ includes the domain of $f^*(\lambda, m)(1)$, let $f^*(\lambda, m+1) = f(\lambda, m+1)$; otherwise let $f^*(\lambda, m+1) = (p, \phi)$, where p is constantly ϕ and has the same domain as $f^*(\lambda, m)(1)$. We intend that $\mathrm{dom}(f^*) = \{(\lambda, i)$: $\lambda < \kappa$ is a limit ordinal and $\{\lambda\} \cdot [0, i] \subset \mathrm{dom}(f)\}$.

Let $\lambda < \kappa$ be a limit ordinal. Instead of using $T(G, \lambda)$, we use the more sophisticated $J(G, \lambda) \in \mathscr{P}(\omega)^\omega$ defined by $J(G, \lambda)(m) = E(G \mid G^*(\lambda, m)(1), G^*(\lambda, m)(2))$.

Until further notice, we fix limit ordinals $\lambda_1 < \cdots < \lambda_n < \kappa$, and $G \subset C$ generic over M. We essentially prove that $(J(G, \lambda_1), \ldots, J(G, \lambda_n)) \sim (J(G \mid f, \lambda_1), \ldots, J(G \mid f, \lambda_n))$ for any $f \in C$ (this follows from Lemma 5.1.13).

We say that $f \in C$ is (m, r)-saturated if (a) $r \in [0, n]$, $\mathrm{dom}(f) = x \cdot [0, m) \cup \{\lambda_1, \ldots, \lambda_r\} \cdot [0, 2m) \cup \{\lambda_{r+1}, \ldots, \lambda_n\} \cdot [0, m)$, for some $x \subset \kappa$ disjoint from $\{\lambda_1, \ldots, \lambda_n\}$, (b) if $f(\lambda_i, j)$ is defined then $\mathrm{dom}(f^*(\lambda_i, j)(1)) \subset \mathrm{dom}(f)$, and (c) if $i \in [1, r]$, $j \in [m, 2m)$, then $\mathrm{dom}(f^*(\lambda_i, j)(1)) = \mathrm{dom}(f) \cap (\lambda_i \times \omega)$.

Let $f, g \in C$ be (m, r)-saturated. We say that f, g are *dual* if $\mathrm{dom}(f) = \mathrm{dom}(g)$, and for all $1 \leqslant i \leqslant r$, $0 \leqslant j < m$, $f \mid f^*(\lambda_i, j)(1)$, $g \mid g^*(\lambda_i, m+j)(1)$ agree on $\lambda_i \times \omega$, $f \mid f^*(\lambda_i, m+j)(1)$, $g \mid g^*(\lambda_i, j)(1)$ agree on $\lambda_i \times \omega$, $f^*(\lambda_i, j)(2) = g^*(\lambda_i, m+j)(2)$, and $f^*(\lambda_i, m+j)(2) = f^*(\lambda_i, j)(2)$.

LEMMA 5.1.9. *Let $f, g \in C$, $\alpha < \lambda_1$, and f, g agree on $\alpha \times \omega$. Then there is an m and conditions f' and g' such that f', g' agree on $\alpha \times \omega$, $f \leqslant f'$, $g \leqslant g'$, $\mathrm{dom}(f') = \mathrm{dom}(g')$, and f', g' are both $(m, 0)$-saturated.*

Proof. Left to the reader.

LEMMA 5.1.10. *Let $f', g' \in C$ be $(m, 0)$-saturated, with $\mathrm{dom}(f') = \mathrm{dom}(g')$. Then there are (m, n)-saturated dual $\hat{f}$, $\hat{g}$ such that $f' \leqslant \hat{f}$, $g' \leqslant \hat{g}$, and f', $\hat{f}$ agree off of $\{\lambda_1, \ldots, \lambda_n\}$, g', $\hat{g}$ agree off of $\{\lambda_1, \ldots, \lambda_n\}$.*

Proof. We construct (m, r)-saturated dual conditions f_r, g_r by induction. Take $f_0 = f'$, $g_0 = g'$. We extend f_r to f_{r+1} by defining $f_{r+1}(\lambda_r, m+j) = (g_r \upharpoonright (\lambda_r \times \omega) \mid g_r^*(\lambda_r, j)(1), g_r^*(\lambda_r, j)(2))$, and extend g_r to g_{r+1} by defining $g_{r+1}(\lambda_r, m+j) = (f_r \upharpoonright (\lambda_r \times \omega) \mid f_r^*(\lambda_r, j)(1), f_r^*(\lambda_r, j)(2))$, $j \in [0, m)$.

LEMMA 5.1.11. *Let f, g be (m, n)-saturated, and dual. Let $G \subset C$ be generic over M. Then for all $1 \leqslant i \leqslant n$, $0 \leqslant j < m$,* (a) *$(G \mid f) \mid (G \mid f)^*(\lambda_i, j)(1)$ and $(G \mid g) \mid (G \mid g)^*(\lambda_i, m+j)(1)$ agree on C_{λ_i},* (b) *$(G \mid g) \mid (G \mid g)^*(\lambda_i, j)(1)$ and $(G \mid f) \mid (G \mid f)^*(\lambda_i, m+j)(1)$ agree on C_{λ_i},* (c) *$(G \mid f)^*(\lambda_i, j)(2) = (G \mid g)^*(\lambda_i, m+j)(2)$, and* (d) *$(G \mid g)^*(\lambda_i, j)(2) = (G \mid f)^*(\lambda_i, m+j)(2)$. For all $1 \leqslant i \leqslant n, j \geqslant 2m$,* (e) *$(G \mid f) \mid (G \mid f)^*(\lambda_i, j)(1)$*

and $(G \mid g) \mid (G \mid g)^*(\lambda_i, j)(1)$ *agree on* C_{λ_i}, *and* (f) $(G \mid f)^*(\lambda_i, j)(2) = (G \mid g)^*(\lambda_i, j)(2)$.

Proof. First observe that for all $k \in [0, 2m)$, $(G \mid f)^*(\lambda_i, k) = f^*(\lambda_i, k)$, $(G \mid g)^*(\lambda_i, k) = g^*(\lambda_i, k)$.

Let $1 \leqslant i \leqslant n$, $0 \leqslant j < m$. By duality, $f^*(\lambda_i, j)(2) = g^*(\lambda_i, m+j)(2)$, $g^*(\lambda_i, j)(2) = f^*(\lambda_i, m+j)(2)$. Hence we have shown (c), (d).

Observe that $(G \mid f) \mid f^*(\lambda_i, k)(1) = G \mid (f \mid f^*(\lambda_i, k)(1))$, $(G \mid g) \mid g^*(\lambda_i, k)(1) = G \mid (g \mid g^*(\lambda_i, k)(1))$. For (a) it suffices to prove that $G \mid (f \mid f^*(\lambda_i, j)(1))$ and $G \mid (g \mid g^*(\lambda_i, m+j)(1))$ agree on C_{λ_i}. Since $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f \mid f^*(\lambda_i, j)(1)$, $g \mid g^*(\lambda_i, m+j)(1)$ agree on $\lambda_i \times \omega$, we are done.

For (e), (f), let $1 \leqslant i \leqslant n$, $j \geqslant 2m$. Then obviously $(G \mid f)^*(\lambda_i, j) = (G \mid g)^*(\lambda_i, j)$, since $\mathrm{dom}(f^*(\lambda_i, j)(2m-1)) = \mathrm{dom}(g^*(\lambda_i, j)(2m-1))$. Hence we have (f). Since the domains of $(G \mid f)^*(\lambda_i, j)(1) = (G \mid g)^*(\lambda_i, j)(1)$ include $\mathrm{dom}(f) \cap (\lambda_i \times \omega)$, (e) follows.

LEMMA 5.1.12. *Let* $\hat{f}$, $\hat{g}$ *be* (m, n)*-saturated, and dual. Let* $G \subset C$ *be generic over* M. *Then for all* $1 \leqslant i \leqslant n$, $0 \leqslant j < m$, $J(G \mid \hat{f}, \lambda_i)(j) = J(G \mid \hat{g}, \lambda_i)(m+j)$, $J(G \mid \hat{g}, \lambda_i)(j) = J(G \mid \hat{f}, \lambda_i)(m+j)$. *For all* $1 \leqslant i \leqslant n$, $j \geqslant 2m$, $J(G \mid \hat{f}, \lambda_i)(j) = J(G \mid \hat{g}, \lambda_i)(j)$.

Proof. This follows formally from Lemma 5.1.11.

LEMMA 5.1.13. *Let* $f, g \in C$, $\alpha < \lambda_1$, *and* f, g *agree on* $\alpha \times \omega$. *Then there are* $f \leqslant \hat{f}$, $g \leqslant \hat{g}$ *such that* $\hat{f}$, $\hat{g}$ *agree on* $\alpha \times \omega$, *and* $(J(G \mid \hat{f}, \lambda_1),\ldots, J(G \mid \hat{f}, \lambda_n)) \sim (J(G \mid \hat{g}, \lambda_1),\ldots, J(G \mid \hat{g}, \lambda_n))$.

Proof. By Lemmas 5.1.9, 5.1.10, and 5.1.12.

LEMMA 5.1.14. *Let* $\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$ *be limit ordinals, and let* $f \in C$. *Then for all* k, $f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1),\ldots, J(G, \lambda_n))$ *if and only if* $f \upharpoonright ((\lambda+1) \times \omega) \Vdash k \in F(J(G, \lambda), J(G, \lambda_1),\ldots, J(G, \lambda_n))$.

Proof. By way of contradiction, assume $f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1),\ldots, J(G, \lambda_n))$, and $g \Vdash k \notin F(J(G, \lambda), J(G, \lambda_1),\ldots, J(G, \lambda_n))$, where $f \upharpoonright ((\lambda+1) \times \omega) \leqslant g$. By extending f, we can assume that f, g agree on $(\lambda+1) \times \omega$. So by Lemma 5.1.13 there are $f \leqslant \hat{f}$, $g \leqslant \hat{g}$ such that $\hat{f}$, $\hat{g}$ agree on $(\lambda+1) \times \omega$, and $(J(G \mid \hat{f}, \lambda_1),\ldots, J(G \mid \hat{f}, \lambda_n)) \sim (J(G \mid \hat{g}, \lambda_1),\ldots, J(G \mid \hat{f}, \lambda_n))$. Since $J(G \mid \hat{f}, \lambda) = J(G \mid \hat{g}, \lambda)$, we see that $F(J(G \mid \hat{f}, \lambda), J(G \mid \hat{f}, \lambda_1),\ldots, J(G \mid \hat{f}, \lambda_n)) = F(J(G \mid \hat{g}, \lambda), J(G \mid \hat{g}, \lambda_1),\ldots, J(G \mid g, \lambda_n))$. This is a contradiction.

As before, we now define a specific $W: \kappa^{n+1} \to V(\kappa)^M$ as follows. For limit ordinals $\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$, we let $W(\lambda, \lambda_1,\ldots, \lambda_n) = \{(k, f): f \in C_{\lambda+1}$ & $f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1),\ldots, J(G, \lambda_n))\}$. Define W to be ϕ at other inputs.

LEMMA 5.1.15. *Let $G \subset C$ be generic over M, and let $\lambda < \lambda_1 < \cdots < \lambda_n < \kappa$ be limit ordinals. Then* $F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_n)) = \{k : (\exists f \in G)((k, f) \in W(\lambda, \lambda_1, \ldots, \lambda_n))\} = E(G, W(\lambda, \lambda_1, \ldots, \lambda_n))$.

Proof. See the proof of Lemma 5.1.6.

As before, we apply Lemma 5.1.5 to W obtain a strictly increasing sequence $\{\lambda_m\}$ of length ω of limit ordinals below κ such that for all $s < t_1 < \cdots < t_n$, $s < r_1 < \cdots < r_n$, $W(\lambda_s, \lambda_{t_1}, \ldots, \lambda_{t_n}) = W(\lambda_s, \lambda_{r_1}, \ldots, \lambda_{r_n})$. Fix $G \subset C$ to be generic over M.

As before, for each $m < \omega$ let f_m be the condition with domain $\{(\lambda_{m+1}, 0)\}$ and value $(\phi, W(\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}))$. Recursively define $G_0 = G \mid f_0$, $G_{m+1} = G_m \mid f_{m+1}$.

LEMMA 5.1.16. *For each* $s < t_1 < \cdots < t_n$, $F(J(G_s, \lambda_s), J(G_{t_1}, \lambda_{t_1}), \ldots, J(G_{t_n}, \lambda_{t_n})) = J(G_{s+1}, \lambda_{s+1})(0)$.

Proof. See the proof of Lemma 5.1.8.

This completes the proof of Proposition P, by taking $\{J(G_m, \lambda_m)\}$ as our infinite sequence.

We now begin the proof of the second part of Theorem 5.1.

We define the Σ_k formulas of set theory as those prenex formulas with $k-1$ alterations of like quantifiers, starting with existential quantifiers. Thus Σ_0 formulas have no quantifiers, and Σ_1 formulas have only existential quantifiers. It is understood that the matrix of a prenex formula is allowed to have bounded quantifiers, which are not counted.

Let $n, k \geqslant 1$. An *(n, k)-special sequence of ordinals of length m* is a strictly increasing sequence of infinite ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ such that for all $s < t_1 < \cdots < t_n < m$, $s < r_1 < \cdots < r_n < m$, $\beta \leqslant \alpha_s$, and Σ_k formulas $\varphi(x_1, \ldots, x_{n+2})$ of set theory with only the free variables shown, we have $L(\alpha_m) \models \varphi(\beta, \alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_n}) \leftrightarrow \varphi(\beta, \alpha_s, \alpha_{r_1}, \ldots, \alpha_{r_n})$.

We now fix $n \geqslant 2$, $k \geqslant 6$, $m \geqslant n+4$, and assume that there is an (n, k)-special sequence of ordinals of length m. We fix $\alpha_1 < \cdots < \alpha_m$ to be the (n, k)-special sequence of ordinals of length m such that the m-tuple $(\alpha_m, \ldots, \alpha_1)$ is lexicographically least. We wish to establish that $L(\alpha_m) \models ZF_{k-6}$, and if $n \geqslant 3$ then $L(\alpha_m) \models \alpha_i$ is an inaccessible cardinal, for all $i < m$. Here ZF_l is ZF based on Σ_l formulas in the replacement scheme, and Δ_0-separation.

LEMMA 5.1.17. *Let* $1 \leqslant p \leqslant n$, $1 < t_1 < \cdots < t_p < m$, $1 < r_1 < \cdots < r_p < m$, *and let φ be Σ_k. Then for all* $\beta \leqslant \alpha_1$, $L(\alpha_m) \models \varphi(\beta, \alpha_1, \alpha_{t_1}, \ldots, \alpha_{t_p}) \leftrightarrow \varphi(\beta, \alpha_1, \alpha_{r_1}, \ldots, \alpha_{r_p})$.

Proof. The case $p = n$ is immediate. We argue by backwards induction. Suppose this is true for p, $1 < p \leqslant n$. By using α_{m-1} as a dummy variable, if

$1 < t_1 < \cdots < t_{p-1} < m-1$, $1 < r_1 < \cdots < r_{p-1} < m-1$, $\beta < \alpha_1$, then $L(\alpha_m) \models \varphi(\beta, \alpha_1, \alpha_{t_1}, ..., \alpha_{t_{p-1}}) \leftrightarrow \varphi(\beta, \alpha_1, \alpha_{r_1}, ..., \alpha_{r_{p-1}})$. Now assume $t_{p-1} = m-1$. Since there are at least p numbers in $(1, m-1)$, there is a number $j \in (1, m-1)$ and distinct numbers $r_1, ..., r_p$ in $(1, m-1)$ such that $(t_1, ..., t_{p-1}, j)$ and $(r_1, ..., r_p)$ have the same order pattern. Thus we can apply the induction hypothesis with dummy variables α_j, α_{r_p}, to obtain for all $\beta < \alpha_1$, $L(\alpha_m) \models \varphi(\beta, \alpha_1, \alpha_{t_1}, ..., \alpha_{t_{p-1}}) \leftrightarrow \varphi(\beta, \alpha_1, \alpha_{r_1}, ..., \alpha_{r_{p-1}})$. Thus we have reduced the case $t_{p-1} = \alpha_{m-1}$ to $t_{p-1} < \alpha_{m-1}$, and we are done.

LEMMA 5.1.18. *Let* $1 \leqslant p \leqslant n$, $2 < t_1 < \cdots < t_p < m$, $2 < r_1 < \cdots < r_p < m$, *and let* φ *be* Σ_k. *Then for all* $\beta \leqslant \alpha_2$, $L(\alpha_m) \models \varphi(\beta, \alpha_2, \alpha_{t_1}, ..., \alpha_{t_p}) \leftrightarrow \varphi(\beta, \alpha_2, \alpha_{r_1}, ..., \alpha_{r_p})$.

Proof. Use the same argument as that for Lemma 5.1.17, except that $(1, m-1)$ is replaced by $(2, m-1)$.

LEMMA 5.1.19. *Each* α_i, $1 < i \leqslant m$, *is a limit ordinal.*

Proof. If α_m is a successor ordinal then let $\alpha_m = \lambda + a$, where λ is a limit ordinal and $a < \omega$. If $\alpha_{m-1} \geqslant \lambda$ then α_{m-1} is the unique solution to a π_1 predicate over $L(\alpha_m)$, which is a contradiction. Hence $\alpha_{m-1} < \lambda$. Since $L(\lambda)$ is the unique solution to a π_1 predicate over $L(\alpha_m)$, we see that $\alpha_1, ..., \alpha_{m-1}, \lambda$ is (n, k)-special. This is a contradiction.

If some α_i, $1 < i < m$, is a successor ordinal then they all are. Also, if $\alpha_2 - 1 \leqslant \alpha_1$ then $\alpha_3 - 1 \leqslant \alpha_1$, which is impossible. Hence in this case we can see that $\alpha_1, \alpha_2 - 1, ..., \alpha_{m-1} - 1$, α_m is (n, k)-special. This is also a contradiction.

LEMMA 5.1.20. $L(\alpha_m) \models \alpha_i$ *is a cardinal, for all* $1 < i < m$.

Proof. Let $|\alpha_2|, ..., |\alpha_{m-1}|$ be the respective cardinals of $\alpha_2, ..., \alpha_{m-1}$ from the point of view of $L(\alpha_m)$ (i.e., the smallest ordinals in one-one correspondence with them in $L(\alpha_m)$). If any $|\alpha_i| = \alpha_i$, $1 < i < m$, then all $|\alpha_i| = \alpha_i$ and we are done. So we may assume that each $|\alpha_i| < \alpha_i$, $1 < i < m$.

In the first case suppose that the $|\alpha_i|$, $1 < i < m$, are not identical. Then they strictly increase.

If $|\alpha_2| \leqslant \alpha_1$ then $|\alpha_2| = \beta \leftrightarrow |\alpha_3| = \beta$ holds, where $\beta = |\alpha_2|$. This is a contradiction. Hence $|\alpha_2| > \alpha_1$. Observe that α_1, $|\alpha_2|, ..., |\alpha_{m-1}|$, α_m is (n, k)-special since $|\alpha_2|, ..., |\alpha_{m-1}|$ are uniformly defined from $\alpha_2, ..., \alpha_{m-1}$ by a Σ_2 formula. This is a contradiction.

In the second case suppose that the $|\alpha_i|$, $1 < i < m$, are identical. Let $|\alpha_2| = \beta < \alpha_2$. For each $1 < i < m$ let $f_i\colon \alpha_i \to \beta$ be the first constructed such one-one surjective map in $L(\alpha_m)$. Let $\gamma < \beta$ be $f_5(\alpha_3)$. Now $f_5(\alpha_3) = \gamma \leftrightarrow f_5(\alpha_4) = \gamma$. This is a contradiction.

LEMMA 5.1.21. *$L(\alpha_m)$ satisfies the power set axiom.*

Proof. If not, then $L(\alpha_m)$ satisfies that there is a largest cardinal, β. If $\alpha_{m-1} < \beta$ then $\alpha_1,\dots,\alpha_{m-1}, \beta$ is (n,k)-special (since β is the unique solution to a π_2 predicate in $L(\alpha_m)$). Hence $\beta \leqslant \alpha_{m-1}$. Therefore by Lemma 5.1.20, $\beta = \alpha_{m-1}$. But "α_{m-1} is the largest cardinal" $\leftrightarrow$ "α_{m-2} is the largest cardinal." This is a contradiction.

LEMMA 5.1.22. *Suppose that for all Σ_{j+4} formulas $\varphi(x, y)$ with only the free variables shown, and ordinals β which are the unique solution to a Σ_{j+4} formula in $L(\alpha_m)$, we have $L(\alpha_m) \models (\forall\gamma < \beta)(\exists\delta)(\varphi(\gamma, \delta)) \rightarrow (\exists\mu)(\forall\gamma < \beta)(\exists\delta < \mu)(\varphi(\gamma, \delta))$. Then $L(\alpha_m) \models ZF_j$.*

Proof. Left to the reader. Use least counterexamples to kill parameters.

LEMMA 5.1.23. *$L(\alpha_m)$ satisfies ZF_{k-6}.*

Proof. By Lemma 5.1.22, it suffices to assume the following, and obtain a contradiction: $L(\alpha_m) \models (\forall\gamma < \beta)(\exists\delta)(\varphi(\gamma, \delta))$, $L(\alpha_m) \models \neg(\exists\mu)(\forall\gamma < \beta)(\exists\delta < \mu)\, \varphi(\gamma, \delta)$, where φ is a Σ_{k-2} formula with all free variables shown, and β is the unique solution to a Σ_{k-2} predicate in $L(\alpha_m)$. If $\beta > \alpha_2$ then $\beta > \alpha_{m-1}$, and so $\alpha_1,\dots,\alpha_{m-1}, \beta$ is (n,k)-special. Hence $\beta \leqslant \alpha_2$. For each $\gamma < \beta$ let $A_\gamma = (\mu\delta)(\varphi(\gamma, \delta))$.

We claim that for all $\gamma < \beta$, $A_\gamma < \alpha_3$ or $A_\gamma > \alpha_{m-1}$. To see this, suppose $A_\gamma \geqslant \alpha_3$. Then $A_\gamma \geqslant \alpha_3 \leftrightarrow A_\gamma \geqslant \alpha_{m-1}$. Hence $A_\gamma \geqslant \alpha_{m-1}$. Also $A_\gamma = \alpha_{m-1} \leftrightarrow A_\gamma = \alpha_3$. Hence $A_\gamma \neq \alpha_{m-1}$.

Fix γ to be the least ordinal such that $A_\gamma > \alpha_{m-1}$. Then for each $3 \leqslant i < m$, γ is the least ordinal such that $A_\gamma > \alpha_i$. Thus there is a Σ_k formula $\psi(x, y)$ with only the free variables shown, such that for all $3 \leqslant i < m$, γ is the unique solution to $\psi(\alpha_i, y)$ over $L(\alpha_m)$.

We now claim that $\alpha_1,\dots,\alpha_{m-1}, A_\gamma$ is (n,k)-special. To see this, let $s < t_1 < \cdots < t_n < m$, $s < r_1 < \cdots < r_n < m$, $\beta \leqslant \alpha_s$. We must verify that $L(A_\gamma) \models \rho(\beta, \alpha_s, \alpha_{t_1},\dots,\alpha_{t_n}) \leftrightarrow \rho(\beta, \alpha_s, \alpha_{r_1},\dots,\alpha_{r_n})$, where ρ is Σ_k. Now $L(A_\gamma) \models \rho(\beta, \alpha_s, \alpha_{t_1},\dots,\alpha_{t_n})$ if and only if $L(\alpha_m) \models (\exists\gamma)(\psi(\alpha_{t_n}, \gamma)\ \&\ L(A_\gamma) \models \rho(\beta, \alpha_s, \alpha_{t_1},\dots,\alpha_{t_n}))$, and $L(A_\gamma) \models \rho(\beta, \alpha_s, \alpha_{r_1},\dots,\alpha_{r_n})$ if and only if $L(\alpha_m) \models (\exists\gamma)(\psi(\alpha_{r_n}, \gamma)\ \&\ L(A_\gamma) \models \rho(\beta, \alpha_s, \alpha_{r_1},\dots,\alpha_{r_n}))$. This completes the proof that $\alpha_1,\dots,\alpha_{m-1}, A_\gamma$ is (n,k)-special, which is the desired contradiction.

Before assuming that $n \geqslant 3$ in order to obtain stronger conclusions, we state the following.

LEMMA 5.1.24. *The following is provable in $ZF - \mathcal{P}$. If there is a $(2, k+6)$-special sequence of ordinals of length 6, then there is an $L(\lambda) \models ZF_k$.*

Now assume that $n \geqslant 3$.

LEMMA 5.1.25. *$L(\alpha_m) \models \alpha_i$ is an inaccessible cardinal, for $1 < i < m$.*

Proof. It suffices to assume that none of $\alpha_2, \ldots, \alpha_{m-1}$ are inaccessible in $L(\alpha_m)$. Suppose that at least one of them is a successor cardinal. Then all of them are. If $\alpha_2 = \alpha_1^+$ then $\alpha_3 = \alpha_1^+$, which is a contradiction. Hence $\alpha_1, \alpha_2^-, \ldots, \alpha_{m-1}^-, \alpha_m$ is (n, k)-special, where α_i^- is the cardinal preceding α_i.

Thus the α_i, $1 < i < m$, are limit cardinals. We assume that they are not regular. In the first case, suppose that their cofinalities are not identical. Then $\mathrm{cf}(\alpha_2), \ldots, \mathrm{cf}(\alpha_{m-1})$ is either strictly increasing or strictly decreasing. If $\mathrm{cf}(\alpha_2) \leqslant \alpha_1$ then $\mathrm{cf}(\alpha_2) = \mathrm{cf}(\alpha_3)$, which is a contradiction. Hence if $\mathrm{cf}(\alpha_2), \ldots, \mathrm{cf}(\alpha_{m-1})$ are strictly increasing, then α_1, $\mathrm{cf}(\alpha_2), \ldots, \mathrm{cf}(\alpha_{m-1})$, α_m is (n, k)-special, which is a contradiction. Finally, if $\mathrm{cf}(\alpha_2), \ldots, \mathrm{cf}(\alpha_{m-1})$ is strictly decreasing then $\mathrm{cf}(\alpha_3) < \alpha_2$, and hence $\mathrm{cf}(\alpha_3) = \mathrm{cf}(\alpha_4)$, which is again a contradiction.

Thus let $\mathrm{cf}(\alpha_i) = \beta$, for all $1 < i < m$. Then $\beta < \alpha_2$. For each $1 < i < m$, let $h_i: \beta \to (\alpha_1, \alpha_i)$ be the first constructed such strictly increasing cofinal map in $L(\alpha_m)$. For each $1 < i < m - 1$ let γ_i be the least ordinal such that $h_{i+1}(\gamma_i) > \alpha_i$. Observe that since each $\gamma_i < \beta < \alpha_2$, we see that all the γ_i, $2 < i < m - 1$, are identical. Since $\gamma_2 = \gamma_3 \leftrightarrow \gamma_3 = \gamma_4$ (using $n \geqslant 3$), we see that all the γ_i, $1 < i < m - 1$, are identically some $\gamma < \beta < \alpha_2$.

It is immediate that for all $1 < i < m - 1$, $\alpha_1 < h_i(\gamma) < \alpha_i$, and $h_{i+1}(\gamma) > \alpha_i$. We now claim that α_1, $h_2(\gamma), \ldots, h_{m-1}(\gamma)$, α_m is (n, k)-special, which will complete the proof by contradiction.

Let $2 < j < i < m$. Since $\gamma < \beta < \alpha_2$, we see that "$\gamma$ is the least ordinal such that $h_i(\gamma) > \alpha_{i-1}$" $\leftrightarrow$ "γ is the least ordinal such that $h_i(\gamma) > \alpha_j$." Since the left hand side is true, the right hand side is true.

Let $s < t_1 < \cdots < t_n < m$, $s < r_1 < \cdots < r_n < m$, $\beta \leqslant h_s(\gamma)$, where for convenience we define h_1 to be constantly α_1. We must show $L(\alpha_m) \models \varphi(\beta, h_s(\gamma), h_{t_1}(\gamma), \ldots, h_{t_n}(\gamma)) \leftrightarrow \varphi(\beta, h_s(\gamma), h_{r_1}(\gamma), \ldots, h_{r_n}(\gamma))$, where φ is Σ_k. The left hand side is equivalent to $L(\alpha_m) \models (\exists\gamma)(\gamma$ is the least ordinal such that $h_{t_n}(\gamma) > \alpha_{t_{n-1}}$ & $\varphi(\beta, h_s(\gamma), h_{t_1}(\gamma), \ldots, h_{t_n}(\gamma))$, and the right hand side is equivalent to the corresponding statement involving $s, r_1, \ldots, r_n$. This completes the proof of the lemma.

In order to relate the case $n \geqslant 3$ to Mahlo cardinals, we use the work in [14].

The following definition is from [14].

DEFINITION 3. (i) A function $f: \kappa \to \kappa$ is *m-normal* if and only if f is a continuous, strictly increasing function such that whenever $\nu < \kappa$, then $f(\nu)$ is a strong limit carinal which is not m-Mahlo.

(ii) $S(m, n, r)$ if and only if for every inaccessible κ and every m-normal function $f: \kappa \to \kappa$, there is an f-partition system C of $[\kappa]^n$ such that each C-homogeneous set has length $<r$.

The following lemma is from [14, p. 289].

LEMMA 5.1.26. *The following is provable in ZFC*: $S(n-1, n+2, n+5)$.

LEMMA 5.1.27. *The following is provable in* $ZFC + GCH$. *Let* $n \geqslant 1$, *and* κ *be an inaccessible cardinal. Let* $f\colon \kappa \to \kappa$ *be the function* $f(\alpha) = \omega_{\omega \cdot \alpha}$. *If every f-partition system C of* $[\kappa]^{n+2}$ *has a C-homogeneous set of length* $n+5$ *then there is an* $(n-1)$*-Mahlo cardinal below* κ.

Proof. Immediate from Lemma 5.1.26.

We will be concerned with the following partition relation $P^*(\kappa, n, b)$. Let $H\colon \kappa^{n+1} \to \mathscr{P}(\kappa)$. Then there are inaccessible cardinals $\alpha_1 < \cdots < \alpha_b < \kappa$ such that for all $s < t_1 < \cdots < t_n \leqslant b$, $s < r_1 < \cdots < r_n \leqslant b$, we have $H(\alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_n}) \cap \alpha_s = H(\alpha_s, \alpha_{r_1}, \ldots, \alpha_{r_n}) \cap \alpha_s$.

LEMMA 5.1.28. *The following is provable in* $ZFC + GCH$. *Let* $n \geqslant 1$, *and let* κ *be an inaccessible cardinal. If* $P^*(\kappa, n+2, n+5)$ *then there is an* $(n-1)$*-Mahlo cardinal below* κ.

Proof. By Lemma 5.1.27, it suffices to prove that every f-partition system C of $[\kappa]^{n+2}$ has a C-homogeneous set of length $n+5$, where $f(\alpha) = \omega_{\omega \cdot \alpha}$. Observe that for each $\gamma < \kappa$, $C_\gamma\colon [\kappa]^{n+2} \to \omega_{\omega \cdot \gamma}$. Set $H(\beta, \alpha_1, \ldots, \alpha_{n+2}) = C_\beta(\{\alpha_1, \ldots, \alpha_{n+2}\})$, for $\beta < \alpha_1 < \cdots < \alpha_{n+2}$; 0 elsewhere. If β is inaccessible then $H(\beta, \alpha_1, \ldots, \alpha_{n+2}) \subset \beta$. Hence any H-homogeneous sequence of inaccessible cardinals is C-homogeneous.

LEMMA 5.1.29. *Let* $n \geqslant 3$, *and assume that for each k there is an* (n, k)*-special sequence of length* $n+4$. *Then for each k there is an ordinal* λ *such that* $L(\lambda) \models ZFC_k + GCH + (\exists\kappa)(\kappa$ *is an inaccessible cardinal such that* $P^*(\kappa, n-1, n+2))$.

Proof. Let $\alpha_1, \ldots, \alpha_{n+4}$ be the (n, k)-special sequence of length $n+4$ such that $(\alpha_{n+4}, \ldots, \alpha_1)$ is lexicographically least, $k \geqslant 7$. Set $\lambda = \alpha_{n+4}$. Then by Lemmas 5.1.23 and 5.1.25, $L(\lambda) \models ZF_{k-6} +$ "α_i is an inaccessible cardinal," for $1 < i < n+4$. Working within $L(\lambda)$, we want to prove that $P^*(\alpha_{n+3}, n-1, n+2)$. Suppose that $P^*(\alpha_{n+3}, n-1, n+2)$ fails in $L(\lambda)$. In $L(\lambda)$, let $H\colon \alpha_{n+3}^n \to \mathscr{P}(\alpha_{n+3})$ be the first constructed counterexample. For $s < t_1 < \cdots < t_{n-1} < n+3$, $s < r_1 < \cdots < r_{n-1} < n+3$, and $\beta \leqslant \alpha_s$, we have $\beta \in H(\alpha_s, \alpha_{t_1}, \ldots, \alpha_{t_{n-1}}) \leftrightarrow \beta \in H(\alpha_s, \alpha_{r_1}, \ldots, \alpha_{r_{n-1}})$, because $\alpha_1, \ldots, \alpha_{n+4}$ is (n, k)-special and H is suitably defined from α_{n+3}. This is a contradiction.

LEMMA 5.1.30. *The following is provable in* $ZF - \mathscr{P}$. *Let* $n \geqslant 4$, *and assume that for all k there is an* (n, k)*-special sequence of length* $n+4$. *Then* *for all k there is an ordinal λ such that $L(\lambda) \models ZF_k +$ "there is an $(n-4)$-Mahlo cardinal."*

Proof. From Lemmas 5.1.28 and 5.1.29.

We will need a refinement of Lemmas 5.1.24 and 5.1.30 involving possibly non-standard models as follows.

Let T be the formal system of set theory with the following axioms: (i) extensionality, (ii) pairing, (iii) union, (iv) transitive closures, (v) Δ_0-separation, (vi) there is no largest ordinal, (vii) for every ordinal α, $L(\alpha)$ exists, (viii) $(\forall x)(\exists\alpha)(x \in L(\alpha))$, and (ix) transfinite induction on $\in$ for all formulas. An ω-model of T is a model of T whose natural numbers are isomorphic to ω.

LEMMA 5.1.31. *The following is provable in $ZF - \mathscr{P}$. (1) Suppose that for all k there is an ω-model $(\omega, R) \models T$ and five "ordinals" $\omega \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_5$ in (ω, R) such that for all $s < t_1 < t_2 < 6$, $s < r_1 < r_2 < 6$, $\beta \leqslant \alpha_s$, and Σ_k formulas φ, $(\omega, R) \models \varphi(\beta, \alpha_s, \alpha_{t_1}, \alpha_{t_2}) \leftrightarrow \varphi(\beta, \alpha_s, \alpha_{r_1}, \alpha_{r_2})$. Then for all k there is an ω-model of ZF_k. (2) Let $n \geqslant 4$ and suppose that for all k there is an ω-model $(\omega, R) \models T$ and $n+3$ "ordinals" $\omega \leqslant \alpha_1 < \cdots < \alpha_{n+3}$ such that for all $s < t_1 < \cdots < t_n < n+4$, $s < r_1 < \cdots < r_n < n+4$, $\beta \leqslant \alpha_s$, and Σ_k formulas φ, $(\omega, R) \models \varphi(\beta, \alpha_s, \alpha_{t_1}, \dots, \alpha_{t_n}) \leftrightarrow \varphi(\beta, \alpha_s, \alpha_{r_1}, \dots, \alpha_{r_n})$. Then for all k there is an ω-model of $ZFC_k +$ "there is an $(n-4)$-Mahlo cardinal."*

Proof. Because of axiom (ix), we can push down to least special sequences as in the proofs of Lemmas 5.1.24 and 5.1.30.

We now show how to construct models (ω, R) as in Lemma 5.1.31 from what we call *critical sequences* of subsets of ω, which are closer to Proposition P. We then produce these critical sequences directly from Proposition P.

For $x \subset \omega$, let $|x| = \{\{m: 2^n 3^m \in x\}: n \in \omega\}$.

Let $\mathscr{L}$ be the language of second order arithmetic augmented with variables α_n for subsets of $\mathscr{P}(\omega)$, in addition to variables x_n for subsets of ω and variables a_n for elements of ω. We do not have quantifiers ranging over subsets of $\mathscr{P}(\omega)$. We view any $A \subset \mathscr{P}(\omega)$ as an interpretation of $\mathscr{L}$, where third order variables may be assigned any subset of $\mathscr{P}(\omega)$, and second order variables may be assigned any element of $\mathscr{P}(\omega)$. The $\in$ symbol has two roles: $t \in x_n$, and $x_n \in \alpha_m$, for numerical terms t. The second order set quantifiers range over A.

A formula φ in $\mathscr{L}$ is Σ_k^1, $k \geqslant 0$, if it is in prenex form with $k-1$ alterations of quantifiers, beginning with an existential quantifier, followed by only bounded numerical quantifiers. Thus we make no distinction here between numerical and set quantifiers. (Σ_0 formulas have only bounded numerical quantifiers, and we allow primitive recursive function symbols.)

We say that $x_1,\dots,x_m \in \mathscr{P}(\omega)$ is an *(n,k)-critical sequence of sets of length m* if and only if for all $0 \leqslant q \leqslant n$, the following hold: (i) for all $s < t_1 < \cdots < t_q < t \leqslant m$ and Σ_k^1 formulas φ, we have $\{a \in \omega: |x_t| \models \varphi(a, x_s, |x_{t_1}|,\dots,|x_{t_q}|)\} \in |x_{s+1}|$, and (ii) for all $s < t_1 < \cdots < t_q < t \leqslant m$, $s < r_1 < \cdots < r_q < r \leqslant m$, and Σ_k^1 formulas φ, we have $|x_t| \models \varphi(x_s, |x_{t_1}|,\dots,|x_{t_q}|)$ if and only if $|x_r| \models \varphi(x_s, |x_{r_1}|,\dots,|x_{r_q}|)$. In particular, note that each $|x_i| \subset |x_{i+1}|$, and each $x_i \in |x_{i+1}|$.

We now fix an $(n, 2k)$-critical sequence of sets $x_1,\dots,x_m$, where $n \geqslant 2$. At various points in the argument, we need to assume that m, k are sufficiently large. We will not pay much attention here as to how large m and k must be.

LEMMA 5.1.32. *Let $1 \leqslant p < m$. Then every set of integers hyperarithmetic in x_p is in $|x_{p+1}|$.*

Proof. Let $1 \leqslant p < m$, and assume by way of contradiction that there is an index e of an x_p-recursive well ordering such that not every set recursive in $H_e(x_p)$ exists in $|x_{p+1}|$. Choose e with this property such that the length of $\{e\}(x_p)$ is minimized. Using x_p as a parameter, we can in $|x_{p+1}|$, put together the $H_k(x_p)$ for initial segments k in $\{e\}(x_p)$, so as to be able to define $H_e(x_p)$ over $|x_{p+1}|$. In fact, we can define any set recursive in $H_e(x_p)$ over $|x_{p+1}|$ with x_p as a parameter. From clause (i), we see hat any set recursive in $H_e(x_p)$ is present in $|x_{p+1}|$.

Let $(x,R) \models T$. An *initial segment* of (x,R) is any $y \subset x$ such that for $n \in y$, $m \in x$, if $(x,R) \models (\forall a)(n \in L(a) \rightarrow m \in L(a))$, then $m \in y$. A *regular segment* is any $y \subset x$ of the form $\{n: (x,R) \models n \in L(a)\}$, where a is a limit ordinal in (x,R), or x itself. A *proper regular segment* is a regular segment which is not x.

For each $1 \leqslant p < m$, let K_p be the set of all $(x,R) \models T$ which are coded in $|x_p|$ such that (x,R) is satisfied to be well founded in $|x_{p+1}|$.

LEMMA 5.1.33. *For $1 \leqslant p < m$, every element of K_p is well founded with respect to all sets Σ_k^0 in any finite number of elements of $|x_m|$.*

Proof. Fix $j \geqslant 1$. We prove this for any j elements of $|x_m|$. By indiscernibility, it is enough to prove this for $p \leqslant 2$. Let $(x,R) \in K_p$. Then by indiscernibility, (x,R) is satisfied to be well founded in $|x_m|$. Since all sets arithmetic in j elements of $|x_3|$ are in $|x_m|$, we have $|x_3| \models$ "(x,R) is well founded with respect to all sets Σ_k^0 in j sets." Hence this is also satisfied in $|x_m|$.

LEMMA 5.1.34. *For $1 \leqslant p < m$, any two isomorphisms in $|x_m|$ from an initial segment of one element of K_p onto an initial segment of another, cohere. Furthermore, their domains and ranges are either total or given by an $L(a)$. For (x,R), $(y,S) \in K_p$, either* (a) *there is an isomorphism from*

(x, R) onto (y, S) in $|x_{p+1}|$, (b) *there is an isomorphism from (x, R) onto a proper regular segment of (y, S) in $|x_{p+1}|$, or* (c) *there is an isomorphism from a proper regular segment of (x, R) onto (y, S) in $|x_{p+1}|$. (Such an isomorphism is called a comparison map.)*

Proof. The first two sentences follow straightforwardly from Lemma 5.1.33. Now let $p \leqslant 2$. Let (x, R), $(y, S) \in K_p$, and consider all isomorphisms from regular segments of (x, R) onto regular segments of (y, S), which are present in $|x_{p+1}|$. These isomorphisms cohere into a single such isomorphism, h. Observe that $h \in |x_{p+1}|$.

Now assume that h is neither total nor surjective, and let $\mathrm{dom}(h) = L(a)$, $\mathrm{rng}(h) = L(b)$. Then clearly there is a longer isomorphism than h from a regular segment of (x, R) onto a regular segment of (y, S), which is hyperarithmetic in x_{p+1}. By Lemma 5.1.32, this longer isomorphism is in $|x_{p+2}|$.

We now consider all isomorphisms from regular segments of (x, R) onto regular segments of (y, S) which are present in $|x_{p+2}|$. These again cohere into a single such isomorphism $h^* \in |x_{p+2}|$. By the above, h^* is longer than h. But for every j, $j \in \mathrm{dom}(h^*) \leftrightarrow j \in \mathrm{dom}(h)$, by indiscernibility. This is a contradiction.

We have established that (a), (b), or (c) holds for $p \leqslant 2$. It then holds for all $1 \leqslant p < m$ by indiscernibility. This completes the proof.

LEMMA 5.1.35. *For $1 \leqslant p < m-1$, there is a proper regular initial segment of an element of K_{p+1} which of longer (in the sense of $|x_m|$) than any element of K_p.*

Proof. By indiscernibility, it suffices to prove this for $p \leqslant 2$. All of the elements of K_p cohere in the sense of $|x_{p+1}|$ by Lemma 5.1.34. They can be put together using the comparison maps in $|x_{p+1}|$ to form a limit structure which is in $|x_{p+1}|$. Now we replace it by the next largest structure obtained by adding ω new levels of the constructible hierarchy on top. The resulting structure is hyperarithmetic in x_{p+1}, and so is in $|x_{p+2}|$. And this structure is obviously longer than all elements of K_p. Furthermore, it is clear by its construction that it is satisfied to be well founded in $|x_m|$ (if not, then there would be a non-well foundedness in an element of K_p which is Σ_k^0 in finitely many elements of $|x_m|$). Hence $|x_m| \models$ "there is a proper regular initial segment of an element of K_{p+2} which is longer than all elements of K_p." Therefore $|x_m| \models$ "there is a proper regular initial segment of an element of K_{p+1} which is longer than all elements of K_p." This completes the proof.

We now make K_{m-2} into a relational structure $\mathfrak{A}$ as follows. The domain of $\mathfrak{A}$ consists of all pairs $((x, R), i)$, where $(x, R) \in K_{m-2}$ and $i \in x$. The equality relation of $\mathfrak{A}$ is given by $((x, R), i) \equiv ((y, S), j)$ if and only if (x, R) and (y, S) are isomorphic in $|x_m|$, and the isomorphism sends i to j. The $\in$-

relation of $\mathfrak{A}$ is given by $((x,R),i)\,E\,((y,S),j)$ if and only if for the comparison map h from (x,R) to (y,S) in $|x_m|$, we have $S(h(i),j)$.

Observe that the whole structure $\mathfrak{A}$ can be coded up into one subset of ω in $|x_{m-1}|$, in the sense that it codes an enumeration of all elements of $\mathfrak{A}$, the $\equiv$, E on $\mathfrak{A}$, and all of the relevant comparison mappings. The complete diagram of $\mathfrak{A}$, as well as all sets hyperarithmetic in this coding up of $\mathfrak{A}$, is therefore present in $|x_m|$. From this it follows that $\mathfrak{A}$ satisfies axiom (ix) of T, and hence $\mathfrak{A} \models T$ (with $=$ interpreted as $\equiv$).

Let $1 \leqslant p < m-3$. Let Q_p be the set of all $((x,R),i) \in K_{m-2}$ such that $(x,R) \models$ "i is a limit ordinal," and the regular initial segment determined by i in (x,R) is of the least length greater than or equal to the lengths of elements of K_p.

The following is left to the reader.

LEMMA 5.1.36. *For $1 \leqslant p < m-3$, Q_p is nonempty. All elements of Q_p are $\equiv$. For each $1 < p < m-4$, $u \in Q_p$, $v \in Q_{p+1}$, we have $\mathfrak{A} \models$ "u, v are infinite ordinals and $u < v$." Q_p, as a 3-ary relation on $|x_m|$, is definable by a Σ^1_k formula $\psi(x, R, i, |x_p|, |x_{m-2}|)$ over $|x_m|$, which is independent of p.*

Recall that we allowed bounded set quantifiers in the matrix of Σ_k formulas of set theory. If we do not allow bounded quantifiers, then we speak of Σ^*_k formulas. In the theory T we can produce appropriate universal Σ_k formulas in order to show the following: for every b there is a k such that every Σ_b formula is provably equivalent, in T, to a Σ^*_k formula.

LEMMA 5.1.37. *For each $1 \leqslant p < m-3$, let $a_p \in Q_p$. Assume $n = 3$. Then for all $1 \leqslant s < t_1 < t_2 < m-3$, $1 \leqslant s < r_1 < r_2 < m-3$, uEa_s, and Σ^*_k formulas φ, we have $\mathfrak{A} \models \varphi(u, a_s, a_{t_1}, a_{t_2}) \leftrightarrow \varphi(u, a_s, a_{r_1}, a_{r_2})$.*

Proof. Since uEa_s, u is equivalent, in $\mathfrak{A}$, to an $((x,R),i) \in |\mathfrak{A}|$, where $(x,R) \in |x_s|$. Then the left side of this equivalence can be viewed as a statement in $|x_m|$ about x_s, $|x_{t_1}|$, $|x_{t_2}|$, and $|x_{m-2}|$. The right side can be viewed as the corresponding statement in $|x_m|$ about x_s, $|x_{r_1}|$, $|x_{r_2}|$, and $|x_{m-2}|$.

LEMMA 5.1.38. *The following is provable in $ZF - \mathscr{P}$. If for every k, m there is a $(3, k)$-critical sequence of sets of length m, then for all k there is an ω-model of ZF_k.*

Proof. By Lemmas 5.1.31 and 5.1.37.

The same argument also shows the following.

LEMMA 5.1.39. *The following is provable in $ZF - \mathscr{P}$. Let $n \geqslant 4$. If for* *every k, m there is a $(n+1, k)$-critical sequence of sets of length m, then for all k there is an ω-model of ZFC_k + "there is an $(n-4)$-Mahlo cardinal."*

LEMMA 5.1.40. *The following is provable in $ZF-\mathscr{P}$. If Proposition* P *holds for $n=4$, $m<\omega$, and for all finitely Borel functions F, then for every k, m there is a $(3, k)$-critical sequence of sets of length m. If Proposition* P *holds for all finitely Borel functions F, even for only $m<\omega$, then for every n, k, there is an (n, k)-critical sequence of sets of length m.*

Proof. Let k, m be given. We define a finitely Borel function $F\colon \mathscr{P}(\omega)^\omega \times (\mathscr{P}(\omega)^\omega)^4 \to \mathscr{P}(\omega)$ as follows. Let $x_1,\dots,x_5 \in \mathscr{P}(\omega)^\omega$. For $x \in \mathscr{P}(\omega)^\omega$, let $\bar{x} = \{2^n 3^m\colon m \in x(n)\}$. We define $F(x_1,\dots,x_5)$ by cases as follows.

Case 1. It is not the case that $\mathrm{Rng}(x_1) \subset \mathrm{Rng}(x_2)$. Then set $F(x_1,\dots,x_5)$ to be the first term of x_1 not in $\mathrm{Rng}(x_2)$.

Case 2. Case 1 does not apply, and there is a Σ_k^1 formula φ such that $\{a \in \omega\colon \mathrm{Rng}(x_5) \models \varphi(a, \bar{x}_1, \mathrm{Rng}(x_2), \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\} \notin \mathrm{Rng}(x_2)$. Then let $\bar{\varphi}$ be the Σ_k^1 formula with least Gödel number with this property, and set $F(x_1,\dots,x_5) = \{a \in \omega\colon \mathrm{Rng}(x_5) \models \bar{\varphi}(a, \bar{x}, \mathrm{Rng}(x_2), \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\}$.

Case 3. Cases 1, 2 do not apply. Set $F(x_1,\dots,x_5) = \{\#(\varphi)\colon \varphi$ is Σ_k^1 and $\mathrm{Rng}(x_5) \models \varphi(\bar{x}_1, \mathrm{Rng}(x_2), \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\}$.

Now apply Proposition P to produce an appropriate sequence $\{x_j\}$ from $\mathscr{P}(\omega)^\omega$ of length $m+8$. Let $s \leqslant m+4$. Then obviously Case 1 cannot apply in the definition of $F(x_s, x_{s+1}, x_{s+2}, x_{s+3}, x_{s+4})$ since it is $x_{s+1}(0)$. This establishes that $\mathrm{Rng}(x_s) \subset \mathrm{Rng}(x_{s+1})$ for $s \leqslant m+4$.

Now let $s < t_1 < t_2 < t_3 < t \leqslant m+4$, $s < r_1 < r_2 < r_3 < r \leqslant m+4$. Then Case 1 does not apply to $F(x_s, x_{t_1}, x_{t_2}, x_{t_3}, x_t)$, $F(x_s, x_{r_1}, x_{r_2}, x_{r_3}, x_r)$. Now since $\mathrm{Rng}(x_{s+1}) \subset \mathrm{Rng}(x_{t_1})$, $\mathrm{Rng}(x_{s+1}) \subset \mathrm{Rng}(x_{r_1})$, we see that since these two values of F are both $x_{s+1}(0)$, Case 2 never applies. So Case 3 applies, and hence for all Σ_k^1 formulas φ, $\mathrm{Rng}(x_t) \models \varphi(\bar{x}_s, \mathrm{Rng}(x_{t_1}), \mathrm{Rng}(x_{t_2}), \mathrm{Rng}(x_{t_3}))$ if and only if $\mathrm{Rng}(x_r) \models \varphi(\bar{x}_s, \mathrm{Rng}(x_{r_1}), \mathrm{Rng}(x_{r_2}), \mathrm{Rng}(x_{r_3}))$. Since Case 2 does not apply, for all Σ_k^1 formulas φ, $\{a \in \omega\colon \mathrm{Rng}(x_t) \models \varphi(a, \bar{x}_s, \mathrm{Rng}(x_{t_1}), \mathrm{Rng}(x_{t_2}), \mathrm{Rng}(x_{t_3}))\} = \{a \in \omega\colon \mathrm{Rng}(x_t) \models \varphi(a, \bar{x}_s, \mathrm{Rng}(x_{s+1}), \mathrm{Rng}(x_{s+2}), \mathrm{Rng}(x_{s+3}))\} \in \mathrm{Rng}(x_{s+1})$. Through the use of dummy variables, this is enough to show that $\{\bar{x}_j\}_{j<m}$ is a $(3, k)$-critical sequence of sets of length m.

The remainder of the lemma is proved analogously.

LEMMA 5.1.41. *The following is provable in $ZF-\mathscr{P}$. If Proposition* P *holds for $n=4$ and $m<\omega$, even for just finitely Borel functions F, then ZF is consistent. If Proposition* P *holds for all n, $m<\omega$, even for just finitely Borel functions F, then each $ZFC + (\exists\kappa)(\kappa$ is $\bar{n}$-Mahlo$)$ has an ω-model.*

Proof. By Lemmas 5.1.38–5.1.40.

The proof of Theorem 5.1 is now complete by the second incompleteness theorem. Actually, we have proved the following sharper version of Theorem 5.1.

THEOREM 5.2. *$ZFC+(\forall n)(\exists k)(\kappa$ is n-Mahlo) proves Proposition* P. *However, for no n does $ZFC+(\exists\kappa)(\kappa$ is n-Mahlo$)+V=L$ prove Proposition* P *even if we restrict to $m<\omega$ and to finitely Borel functions F. $ZFC+V=L$ does not prove Proposition* P *for $n=4$, $m<\omega$, and finitely Borel functions F.*

We also have the following, using the discussion in Section 1.

COROLLARY 5.3. *It is necessary and sufficient to use Mahlo cardinals of arbitrarily high finite order in order to prove Proposition* P, *even for $m<\omega$ and finitely Borel F. It is necessary to go beyond ZFC in order to prove Proposition* P *for $n=4$, $m<\omega$, and finitely Borel functions F.*

In Theorem 5.2., we have attempted to isolate the weakest form of Proposition P which is independent of *ZFC*, as precisely as possible. Unfortunately we have been only partially successful. In particular, we do not know if Proposition P can be proved in *ZFC* for $n=3$ (even if we restrict to $m<\omega$ and finitely Borel functions).

We now wish to give a proof of Proposition P for $n=3$, $m<\omega$ within *MKC*.

As in the proof of Proposition P, we fix a Borel function $F\colon Q\times Q^3\to I$ with the invariance condition. We let A be a countable admissible set which contains a Borel code for F. Thus in the appropriate sense, F is given by a formula in $\mathscr{L}_A$. In *MKC* we can fix a countable transitive set M satisfying *VBC* together with the replacement scheme for all formulas in $\mathscr{L}_A$, and such that A is countable in M.

We define the same notion of forcing over M as before, except that we view ourselves as adding a system $\{f_\alpha\}$ of mutually generic enumerations $f_\alpha\colon\omega\to V(\alpha)^M$ for *every* ordinal α of M. Thus the forcing conditions consist of all finite functions in M whose domain is included in $On\times\omega$.

Let $(*)$ be the following proposition in class theory. Suppose H is a function of four variables on On defined by a formula φ of set theory in $\mathscr{L}_A$ in the following way: $H(\lambda,\lambda_1,\lambda_2,\lambda_3)=\{x\in V(\lambda_3)\colon V(\lambda_3)\models\varphi(x,\lambda,\lambda_1,\lambda_2)\}$ for limit ordinals $\lambda<\lambda_1<\lambda_2<\lambda_3$; $\varnothing$ otherwise. Then there are arbitrarily long finite increasing sequences of limit ordinals $\{\lambda_k\}_{k<m}$ such that for any $s<t_1<t_2<t_3\leqslant m$, $s<r_1<r_2<r_3\leqslant m$, we have $H(\lambda_s,\lambda_{t_1},\lambda_{t_2},\lambda_{t_3})\cap V(\lambda_s+\omega)=H(\lambda_s,\lambda_{r_1},\lambda_{r_2},\lambda_{r_3})\cap V(\alpha_s+\omega)$.

LEMMA 5.4.1. *If* $(*)$ *holds in* M *then Proposition* P *holds for* $n = 4$ *and* $m < \omega$.

Proof. It is clear from the proof of Proposition P that it suffices to verify that the crucial function $H(\lambda, \lambda_1, \lambda_2, \lambda_3) = \{(k, f): f \in C_{\lambda+1} \,\&\, f \Vdash k \in F(J(G,\lambda), J(G,\lambda_1), J(G,\lambda_2), J(G,\lambda_3))\}$ is of the form $\{x \in V(\lambda_3): V(\lambda_3) \models \varphi(x, \lambda, \lambda_1, \lambda_2)\}$, for φ in $\mathscr{L}_A$. It is standard that in this context, we can replace $\Vdash$ with $\Vdash_{\lambda_3+1}$, which is the restriction of $\Vdash$ to only conditions in C_{λ_3+1}. The forcing term $F(J(G,\lambda), J(G,\lambda_1), J(G,\lambda_2), J(G,\lambda_3))$ can be viewed as an infinitary forcing term with constants for every element of $V(\lambda_3) \cup \{\lambda_3\}$ (in the ground model M), which as a set is in $L_\mu[V(\lambda_3)]$, where $\mu \in A$, is of ordinal rank in A, and is definable over $L_\mu[V(\lambda_3)]$ by a formula in $\mathscr{L}_A$ from $\lambda, \lambda_1, \lambda_2, \lambda_3$ which does not depend on $\lambda, \lambda_1, \lambda_2, \lambda_3$. The construction of a ψ in $\mathscr{L}_A$ such that $H(\lambda, \lambda_1, \lambda_2, \lambda_3) = \{x \in V(\lambda_3): L_\mu[V(\lambda_3)] \models \psi(x, \lambda, \lambda_1, \lambda_2)\}$ is done by recursion on the infinitary forcing term using $\Vdash_{\lambda_3+1}$. Since $H(\lambda, \lambda_1, \lambda_2, \lambda_3) \subset V(\lambda + \omega)$, we can obtain the desired φ by interpreting $L_\mu[V(\lambda_3)]$ within $V(\lambda_3)$.

Now let $(**)$ be the following proposition in class theory. Let H be a function of three variables on On, and assume that for each ordinal α, $\{H(\alpha, \alpha_1, \alpha_2): \alpha < \alpha_1 < \alpha_2\}$ is a set. Then there are arbitrarily long finite increasing sequences of ordinals $\{\alpha_k\}_{k \leqslant m}$ such that for any $s < t_1 < t_2 \leqslant m$, $s < r_1 < r_2 \leqslant m$, we have $H(\alpha_s, \alpha_{t_1}, \alpha_{t_2}) = H(\alpha_s, \alpha_{r_1}, \alpha_{r_2})$.

LEMMA 5.4.2. *If* $(**)$ *holds in* M *then* $(*)$ *holds in* M.

Proof. In M, let $H(\alpha, \alpha_1, \alpha_2, \alpha_3) = \{x \in V(\alpha_3): V(\alpha_3) \models \varphi(\alpha, \alpha_1, \alpha_2)\}$, where $\varphi \in \mathscr{L}_A$. Let $\{\mu_\alpha\}$ be an increasing transfinite sequence of infinite cardinals of length On, in M, such that each $V(\mu_\alpha)$ is an elementary substructure of each later $V(\mu_\beta)$ for all subformulas of φ. Now let $H'(\alpha, \alpha_1, \alpha_2) = H(\mu_\alpha, \mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_2+1}) \cap V(\mu_\alpha + \omega)$. Let $m \geqslant 1$. Apply $(**)$ to H' to obtain an appropriate $\{\alpha_k\}_{k \leqslant m}$. Since the value of $H(\gamma, \gamma_1, \gamma_2, \gamma_3) \cap V(\gamma + \omega)$ depends only on γ, γ_1, γ_2 for γ, γ_1, γ_2, $\gamma_3 \in \{\mu_\alpha\}$, we see that $H(\mu_{\alpha_s}, \mu_{\alpha_{t_1}}, \mu_{\alpha_{t_2}}, \mu_{\alpha_{t_3}}) \cap V(\mu_{\alpha_s} + \omega) = H'(\alpha_s, \alpha_{t_1}, \alpha_{t_2}) \cap V(\mu_{\alpha_s} + \omega) = H'(\alpha_s, \alpha_{r_1}, \alpha_{r_2}) \cap V(\mu_{\alpha_s} + \omega) = H(\mu_{\alpha_s}, \mu_{\alpha_{r_1}}, \mu_{\alpha_{r_2}}, \mu_{\alpha_{r_3}}) \cap V(\mu_{\alpha_s} + \omega)$. Hence $\{\mu_{\alpha_k}\}_{k \leqslant m}$ satisfies the conclusion of $(*)$ in M.

The following is from Theorem 3.2 of [14].

LEMMA 5.4.3. *Let* κ *be an inaccessible cardinal. Let* $H: \kappa^3 \to V(\kappa)$, *and assume that for each ordinal* α, $\{H(\alpha, \alpha_1, \alpha_2): \alpha < \alpha_1 < \alpha_2 < \kappa\}$ *is an element of* $V(\kappa)$. *Then there are arbitrarily long finite increasing sequences of ordinals* $\{\alpha_k\}_{k \leqslant m}$ *such that for any* $s < t_1 < t_2 \leqslant m$, $s < r_1 < r_2 \leqslant m$, *we have* $H(\alpha_s, \alpha_{t_1}, \alpha_{t_2}) = H(\alpha_s, \alpha_{r_1}, \alpha_{r_2})$.

LEMMA 5.4.4. *VBC proves* $(**)$.

Proof. Imitate the proof of Lemma 5.4.3 by replacing the inaccessible cardinal κ by On.

LEMMA 5.4.5. *Proposition* P *holds for* F *and any* $m < \omega$ *in* M.

Proof. By Lemmas 5.3.1–5.3.4.

THEOREM 5.4. *MKC proves Proposition* P *for* $n = 3$, $m < \omega$.

Proof. The choice of F was arbitrary, and so this follows from Lemma 5.4.5. Replacement for formulas in $\mathscr{L}_A$ is needed for the proof of Lemma 5.4.2.

Here is a slight variant of Proposition P.

PROPOSITION Q. *Let* $F\colon Q \times Q^n \to I$ *be a Borel function such that* $x \in Q$, $y, z \in Q^n$, $y \sim z$ *implies* $F(x, y) = F(x, z)$. *Then there is a sequence* $\{x_k\}$ *of length* $m \leqslant \omega$ *from* Q *such that the value of* F *at subsequences of length* $n+1$ *depends only on the first term of the subsequence, and is always a coordinate of the second term of the subsequence.*

It is easily verified that Theorems 5.1–5.4 hold for Proposition Q.

We now wish to consider a more substantial variant of Propositions P, Q where we can pinpoint an instance which is provable in *MKC* but not in $ZFC + V = L$.

First consider the following invariant of the most elementary Borel diagonalization theorem. Let H be the group of all permutations σ of N which are the identity map except at finitely many places. Let $F\colon Q \to Q$ be a Borel function such that for all $\sigma \in H$, $F(x \circ \sigma) = F(x) \circ \sigma$. Then for some x, $F(x)$ is a subsequence of x. This can be proved in the same way as basic Borel diagonalization, and has precisely the same metamathematical properties. Now we combine it with Ramsey's theorem.

PROPOSITION R. *Let* $F\colon Q \times Q^n \to Q$ *be a Borel function such that for all* $\sigma_1, \ldots, \sigma_n \in H$, $F(x, y_1 \circ \sigma_1, \ldots, y_n \circ \sigma_n) = F(x, y_1, \ldots, y_n) \circ \sigma_1$. *Then there is a sequence* $\{x_k\}$ *of length* $m \leqslant \omega$ *such that the value of* F *at subsequences of* $\{x_k\}$ *of length* $n+1$ *depends only on the first two terms of the subsequence, and is always an infinite subsequence of the second term of the subsequence of* $\{x_k\}$.

THEOREM 5.5. *Proposition* R *is provable in* $ZFC + (\forall n)(\exists \kappa)(\kappa$ *is* n-*Mahlo*), *but not in any* $ZFC + (\exists \kappa)(\kappa$ *is* $\bar{n}$-*Mahlo*) $+ V = L$, *for any* n, *even for* $m < \omega$ *and finitely Borel functions. In particular, Proposition* R *for* $n = 4$ *and finite* m *is provable in MKC but not in* $ZFC + V = L$.

We sketch a proof of Proposition R for $n+1$, $n \geqslant 1$. Let $F\colon Q \times Q^{n+1} \to Q$

be given as in the hypothesis of Proposition R, and let M, κ be as in the proof of Proposition P. Define forcing, $\bar{G}$, G^*, f^*, $g \mid f$, $G \mid f$, E, and J as in the proof of Proposition P.

LEMMA 5.5.1. *Let $p \geqslant 0$ and let $\lambda < \lambda_1 < \cdots < \lambda_{n+1} < \kappa$ be limit ordinals. Let $f \in C$ be such that f is defined on $\{\lambda_1\} \times [0, p]$. Let α be the rank in the cumulative hierarchy of the range of f on $\{\lambda_1\} \times [0, p]$. Then for all $j \in [0, p]$, $k \in \omega$, $f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j)$ if and only if $f \upharpoonright (((\lambda + 1) \times \omega) \cup (\alpha \times \omega) \cup \{\lambda_1\} \times [0, p]) \Vdash k \in F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j)$.*

Proof. Similar to the proof of Lemma 5.1.14, as follows. By way of contradiction, let $f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j)$, and $g \Vdash k \notin F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j)$, where $f \upharpoonright (((\lambda + 1) \times \omega) \cup (\alpha \times \omega) \cup \{\lambda_1\} \times [0, p]) \leqslant g$. By extending f, we can assume that f, g agree on $(\lambda + 1) \times \omega$, $\alpha \times \omega$, and $\{\lambda_1\} \times [0, p]$, and are both defined on $\{\lambda_1\} \times [0, p]$. We can extend f, g to $\hat{f}$, $\hat{g}$ so that they still agree on $(\lambda + 1) \times \omega$, $\alpha \times \omega$, and $\{\lambda_1\} \times [0, p]$, and such that the following holds: Let $G \subset C$ be generic over M. Then for $1 < i \leqslant n + 1$, $J(G \mid \hat{f}, \lambda_i) \sim J(G \mid \hat{g}, \lambda_i)$, and there is a $\sigma \in H$ which is the identity on $[0, p]$ such that $J(G \mid \hat{g}, \lambda_1) = J(G \mid \hat{f}, \lambda_1) \circ \sigma$, and also $J(G \mid \hat{f}, \lambda) = J(G \mid \hat{g}, \lambda)$. We then have for all $j \in [0, p]$, $F(J(G \mid \hat{f}, \lambda), J(G \mid \hat{f}, \lambda_1), \ldots, J(G \mid \hat{f}, \lambda_{n+1}))(j) = F(J(G \mid \hat{g}, \lambda), J(G \mid \hat{g}, \lambda_1), \ldots, J(G \mid \hat{g}, \lambda_{n+1}))(j)$. This is the desired contradiction since $f \in G \mid \hat{f}$, $g \in G \mid \hat{g}$.

As in the proof of Proposition P, we define a specific function $Y: \kappa^{n+2} \to V(\kappa)^M$ as follows. For limit ordinals $\lambda < \lambda_1 < \cdots < \lambda_{n+1} < \kappa$, we let $Y(\lambda, \lambda_1, \ldots, \lambda_{n+1}) = \{(k, j, f): f \in C_{\lambda_1 + 1}\ \&\ f \Vdash k \in F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j)\}$. Define Y to be $\varnothing$ at other inputs.

The following corresponds to Lemma 5.1.15.

LEMMA 5.5.2. *Let $G \subset C$ be generic over M, and let $\lambda < \lambda_1 < \cdots < \lambda_{n+1} < \kappa$ be limit ordinals. Then $F(J(G, \lambda), J(G, \lambda_1), \ldots, J(G, \lambda_{n+1}))(j) = \{k: (\exists f \in G)((k, j, f) \in Y(\lambda, \lambda_1, \ldots, \lambda_{n+1}))\} = E(G, \{(k, f): f \in C_\alpha\ \&\ (k, j, f^*) \in Y(\lambda, \lambda_1, \ldots, \lambda_{n+1})\})$, where α is the maximum of $\lambda + 1$ and the rank of the range of $\bar{G}$ on $\{\lambda_1\} \times [0, j]$, and f^* is the union of f with the finite function $\bar{G} \upharpoonright \{\lambda_1\} \times [0, j]$.*

As before, we apply Lemma 5.1.5 to Y to obtain a strictly increasing sequence $\{\lambda_m\}$ of length ω of limit ordinals below κ such that the value of Y at subsequences of length $n + 2$ depends on only the first *two* terms. Then evidently $\{J(G, \lambda_m)\}$ obeys the conclusion of Proposition R, for any generic $G \subset C$.

The proof that Proposition R is provable in MKC for $n = 4$, $m < \omega$, is entirely analogous to the proof in MKC of Proposition P for $n = 3$, $m < \omega$.

We now prove the remainder of Theorem 5.5.

We say that $x_1, \ldots, x_m$ is an *(n, k)-fundamental sequence of sets of length m* if $x_1, \ldots, x_m$ is an (n, k)-critical sequence of sets and (iii) for all Σ_k^1 formulas φ, $1 \leqslant s < t \leqslant m$, and $x \in |x_t|$, $\{a \in \omega : |x_t| \models \varphi(a, x_s, x)\} \in |x_t|$.

We fix $x_1, \ldots, x_m$ to be a $(2, 3k)$-fundamental sequence of sets of length m, where k, m are sufficiently large.

LEMMA 5.5.3. *Let $s < t \leqslant m$, $x \in |x_t|$, and assume that $y_1, \ldots, y_b$ are Σ_k^1 definable over $|x_t|$ from x_s, x. Then every set hyperarithmetic in (x_s, x, y) is in $|x_t|$.*

Proof. Let e be an index of a well ordering of least length, recursive in (x_s, x, y) such that $H_e(x_s, x, y)$ does not exist in $|x_t|$. Then by (iii), $H_e(x_s, x, y)$ does exist in $|x_t|$. Thus every appropriate $H_e(x_s, x, y)$ exists in $|x_t|$. The lemma follows by another application of (iii).

We now let W be the set of all $(x, R) \models T$ such that (a) for every $y \in |x_m|$, $(x, R, y) \in |x_m|$, and (b) (x, R) is well founded with respect to all sets Σ_k^0 in any finite number of elements of $|x_m|$.

LEMMA 5.5.4. *There is a unique comparison map in $|x_m|$ between any two elements of W. If $\alpha_1, \ldots, \alpha_a \in W$, $\beta_1, \ldots, \beta_b$ are the comparison maps in $|x_m|$ between various of the α_i, and $x \in |x_m|$, then every set hyperarithmetic in $(\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b, x)$ is in $|x_m|$.*

Proof. For the first part, use (a) to join the two elements of W. Use (b) to show that partial comparison maps cohere. Use (iii) to obtain a largest partial comparison map. And use Lemma 5.5.3 to extend this largest map if it is not a comparison map. For the second part, from (a) we see that $(\alpha_1, \ldots, \alpha_a, x) \in |x_m|$. Now $\beta_1, \ldots, \beta_b$ are suitably definable from $(\alpha_1, \ldots, \alpha_a)$. Apply Lemma 5.5.3.

LEMMA 5.5.5. $K_{m-2} \subset W$.

Proof. Let $(x, R) \in K_{m-2}$. By Lemmas 5.1.33 and (iii), $(x, R) \in W$.

We now make W into a relational structure $\mathscr{B}$ as follows. The domain of $\mathscr{B}$ consists of all pairs $((x, R), n)$, where $(x, R) \in W$ and $n \in x$. The equality relation of $\mathscr{B}$ is given by $((x, R), n) \equiv ((y, S), m)$ if and only if the comparison map from (x, R) to (y, S) sends n to m. The $\in$-relation of $\mathscr{B}$ is given by $((x, R), n) E((y, S), m)$ if and only if for the comparison map h from (x, R) to (y, S) in $|x_m|$, we have $S(h(n), m)$.

LEMMA 5.5.6. *$\mathscr{B} \models T$ with axiom* (ix) *for all Σ_k^* formulas.*

Proof. Lemma 5.5.4 provides us with enough flexibility to argue about $\mathscr{B}$. Clearly $\mathscr{B} \models T$ except for (ix). To verify (ix) for Σ_k^* formulas, choose representatives for the parameters in the Σ_k^* formula φ, and assume that φ has a solution but no $\in$-least solution in $\mathscr{B}$. Choose a representative for a solution. Then we get a non-well foundedness in the solution in $|x_m|$, which is a contradiction.

Recall the definition of Q_p, $1 \leqslant p < m-3$. As before, we have the following.

LEMMA 5.5.7. *All elements of each Q_p, $1 \leqslant p < m-3$, are $\equiv$. For each $1 < p < m-4$, $u \in Q_p$, $v \in Q_{p+1}$, we have $\mathscr{B} \models$ "u, v are infinite ordinals and $u < v$." Q_p, as a 3-ary relation on $|x_m|$, is definable by a Σ_k^1 formula $\psi(x, R, n, |x_p|)$ over $|x_m|$, which is independent of p.*

LEMMA 5.5.8. *For each $1 \leqslant p < m-3$, let $a_p \in Q_p$. Then for all $1 \leqslant s < t_1 < t_2 < m-3$, $1 \leqslant s < r_1 < r_2 < m-3$, uEa_s, and Σ_k^* formulas φ, we have $\mathscr{B} \models \varphi(u, a_s, a_{t_1}, a_{t_2}) \leftrightarrow \varphi(u, a_s, a_{r_1}, a_{r_2})$.*

Proof. The left side of this equivalence can be viewed as a statement in $|x_m|$ about x_s, $|x_{t_1}|$, $|x_{t_2}|$. The right side can be viewed as the corresponding statement about x_s, $|x_{r_1}|$, $|x_{r_2}|$. Note that we do not need $|x_{m-2}|$ to define $\mathscr{B}$, as we did to define $\mathscr{A}$.

LEMMA 5.5.9. *The following is provable in $ZF - \mathscr{P}$. If for every k, m there is a $(2, k)$-fundamental sequence of sets of length m, then for all k there is an ω-model of ZF_k.*

Proof. By Lemmas 5.5.8 and 5.1.31. We need to remark that in Lemma 5.1.31, we need axiom (ix) of T for all Σ_k^* formulas only.

LEMMA 5.5.10. *The following is provable in $ZF - \mathscr{P}$. If Proposition* R *holds for $n = 4$, $m < \omega$, for all finitely Borel functions F, then for every k, m there is a $(2, k)$-fundamental sequence of sets of length m.*

Proof. Let k, m be given. We define a finitely Borel function F: $\mathscr{P}(\omega)^\omega \times (\mathscr{P}(\omega)^\omega)^4 \to \mathscr{P}(\omega)^\omega$ as follows. Let $x_1, \dots, x_5 \in \mathscr{P}(\omega)^\omega$. We define $F(x_1, \dots, x_5)$ by cases as follows.

Case 1. It is not the case that $\mathrm{Rng}(x_1) \subset \mathrm{Rng}(x_2)$. Then set $F(x_1, \dots, x_5)$ to be constantly the first term of x_1 not in $\mathrm{Rng}(x_2)$.

Case 2. Case 1 does not apply, and there is a Σ_k^1 formula φ such that for some j, $\{a \in \omega: \mathrm{Rng}(x_5) \models \varphi(a, \bar{x}_1, x_2(j), \mathrm{Rng}(x_2), \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\} \notin \mathrm{Rng}(x_2)$. Then let $\bar{\varphi}$ be the Σ_k^1 formula with least Gödel number with this property, and set $F(x_1, \dots, x_5)(j) = \{a \in \omega: \mathrm{Rng}(x_5) \models \bar{\varphi}(a, \bar{x}_1, x_2(j), \mathrm{Rng}(x_2), \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\}$, for all j.

Case 3. Cases 1, 2 do not apply. Set $F(x_1, \dots, x_5)$ to be constantly $\{\#(\varphi)$: φ is Σ_k^1 and $\mathrm{Rng}(x_5) \models \varphi(\bar{x}_1, \mathrm{Rng}(x_3), \mathrm{Rng}(x_4))\}$.

Now apply Proposition R to produce an appropriate sequence $\{x_j\}$ from

$\mathscr{P}(\omega)^\omega$ of length $2m+8$. Let $s \leqslant 2m+4$. Then obviously Case 1 cannot apply in the definition of $F(x_s, x_{s+1}, x_{s+2}, x_{s+3}, x_{s+4})$ since it is a subsequence of x_{s+1}. This establishes that $\mathrm{Rng}(x_s) \subset \mathrm{Rng}(x_{s+1})$ for $s \leqslant 2m+4$.

Let $s < t_1 < t_2 < t \leqslant 2m+4$, $s < r_1 < r_2 < r \leqslant 2m+4$, where all of these numbers are even. Then Cases 1 and 2 do not apply to $F(x_s, x_{s+1}, x_{t_1}, x_{t_2}, x_t)$ since this is a subsequence of x_{s+1}. Hence for all Σ_k^1 formulas φ, $\{a \in \omega: \mathrm{Rng}(x_t) \models \varphi(a, \bar{x}_s, \mathrm{Rng}(x_{t_1}), \mathrm{Rng}(x_{t_2}))\} \in \mathrm{Rng}(x_{s+1}) \subset \mathrm{Rng}(x_{s+2})$. Now by Case 3, for all Σ_k^1 formulas φ, $\mathrm{Rng}(x_r) \models \varphi(\bar{x}_s, \mathrm{Rng}(x_{r_1}), \mathrm{Rng}(x_{r_2}))$ if and only if $\mathrm{Rng}(x_t) \models \varphi(\bar{x}_s; \mathrm{Rng}(x_{t_1}), \mathrm{Rng}(x_{t_2}))$. Hence we have shown that $\{\bar{x}_{2j}\}$, $1 \leqslant j \leqslant m$, is $(2, k)$-critical. Since Case 2 never applies, we also see that $\{\bar{x}_{2j}\}$, $1 \leqslant j \leqslant m$, is $(2, k)$-fundamental.

The same argument shows the following.

LEMMA 5.5.11. *The following is provable in $ZF - \mathscr{P}$. If Proposition* R *holds for $n, m < \omega$, for all finitely Borel functions F, then for every k, n, m there is an (n, k)-fundamental sequence of sets of length m.*

This completes the proof of Theorem 5.5.

The following is obtained using the discussion in Section 1.

COROLLARY 5.6. *It is necessary and sufficient to use Mahlo cardinals of arbitrarily high finite order in order to prove Proposition* R, *even for $m < \omega$ and finitely Borel functions F. It is necessary to go beyond ZFC and sufficient to use MKC in order to prove Proposition* R *for $n = 4$, $m < \omega$, and finitely Borel functions F.*

We now mention an important metamathematical point. It is already clear that if we relativize Propositions P–R to the universe of constructible sets, then the resulting propositions also have the metamathematical properties cited in this section. (In fact, the relativizations are equivalent, because the statements are π_2^1.)

However, we can say much more. Let S be the axioms of set theory consisting of extensionality, pairing, union, infinity, Δ_0-separation, Σ_1-replacement, and "every well ordering of ω is isomorphic to an ordinal." Let A be any transitive model of S (perhaps a proper class) which is given by a description within ZFC. By this we mean that A is defined as $\{x: \varphi(x)\}$ for some formula φ of set theory with only 'x' free, and such that $ZFC \vdash$ "A is a transitive model of S." Typical cases are the minimum model of ZF, and the minimum model of $ZF - \mathscr{P}$.

THEOREM 5.7. *The relativizations of Propositions* P–R *to A (using Borel codes) share the same metamathematical properties given earlier in this section for Propositions* P–R.

Proof. Firstly, A must be an admissible class such that $x \in A \cap \mathscr{P}(\omega) \rightarrow x^+ \in A$, where x^+ is the least admissible set with x present. This tells us that the relativized forms follow immediately from the original forms, in ZFC. Secondly, in obtaining all of our reversals in this section for Propositions P–R, we used only arithmetically coded Borel functions F. But Propositions P–R for such F follow immediately from their relativizations to A (even just for such F).

The following metamathematical equivalents can be obtained from examination of the proofs given in this section.

THEOREM 5.8. *The following are provably equivalent in* $ZF - \mathscr{P}$: (a) *any of Propositions* P–R *formulated with Borel codes, with or without restriction to finitely Borel functions, with or without restriction to* $m < \omega$, (b) *for every* $x \subset \omega$, $n < \omega$, *there is an* ω*-model of* $ZFC + (\exists\kappa)$ (κ *is* $\bar{n}$*-Mahlo) in which* x *is present. The following are provably equivalent in* $ZF - \mathscr{P}$: (c) *Proposition* R *formulated with Borel codes, for* $n = 4$, $m < \omega$, (d) *for every* $x \subset \omega$, $n < \omega$, *there is an* ω*-model of* ZFC_n *in which* x *is present.*

Finally, we remark that if we strengthen the invariance condition on F in Propositions P, Q to assert that if $y_1 \approx z_1, \ldots, y_n \approx z_n$ then $F(x, y_1, \ldots, y_n) = F(x, z_1, \ldots, z_n)$, and if we strengthen the invariance condition on F in Proposition R to assert that if $y_2 \approx z_2, \ldots, y_n \approx z_n$, $\sigma \in H$ then $F(x, y_1 \circ \sigma, z_2, \ldots, z_n) = F(x, y_1, y_2, \ldots, y_n) \circ \sigma$, then we retain all of the metamathematical properties cited here. (In fact, we have provable equivalences within $ZFC - \mathscr{P}$.)

APPENDIX

Here we give a Baire category proof of the basic Borel diagonalization theorem, Proposition C, as referred to in the Introduction.

The proof is essentially equivalent to the forcing proof given in Section 3. The unusual feature here is that, instead of applying the Baire category theorem to a separable space such as the reals under the infinite product topology $\mathbb{R}^N$, where $\mathbb{R}$ is given its usual separable topology, we must use the infinite product topology $\underset{\sim}{\mathbb{R}}^N$, where $\underset{\sim}{\mathbb{R}}$ is the reals under the *discrete topology.*

The necessity of using the discrete topology on $\mathbb{R}$, or at least some non-separable topology, is indicated by the following result: Theorem 3.1 cannot be proved in a suitable formalization of "separable" or "essentially countable" set theory, given below.

We have already shown that Proposition C cannot be proved in the usual formalization of countable set theory—namely, $ZFC - \mathscr{P}$. In this theory, no uncountable sets can be proved to exist. In fact, $ZFC - \mathscr{P}$ + "all sets are countable" is not only consistent, but also does not suffice to prove Proposition C.

Thus in $ZFC - \mathscr{P}$, we cannot construct $\mathbb{R}$ as a set. In essentially countable set theory, we use the axiom "every countable set has a power set" to construct all the usual separable spaces. We will give the exact formalization later.

We begin with a lemma from general topology. Let T be a topological space. The *Borel sets* in T form the least σ-algebra of subsets of T containing the open sets. A *nowhere dense set* is a set whose closure contains no open set. A *meager set* is a countable union of nowhere dense sets.

LEMMA 1. *Every Borel set in T differs from some open set by a meager set; i.e., every Borel set has the Baire property.*

Proof. It is obvious that the Borel sets form the least class of sets containing all open sets, closed under complements, and closed under countable unions. Suppose $A \Delta B$ is meager, B open. Then $-A \Delta -B$ is meager. Now $-B \Delta \operatorname{Int}(-B)$ is meager. Hence $-A \Delta \operatorname{Int}(-B)$ is meager. Finally, let $\{A_n\}$ be given, $\{B_n\}$ open, and each $A_n \Delta B_n$ meager. Then $\bigcup_n A_n \Delta \bigcup_n B_n \subset \bigcup_n (A_n \Delta B_n)$, which is meager.

Let $\underset{\sim}{\mathbb{R}}$ be $\mathbb{R}$ with the discrete topology. Let $\underset{\sim}{\mathbb{R}}^N$ be $(\underset{\sim}{\mathbb{R}})^N$ with the infinite product topology.

LEMMA 2. *No nonempty open set in $\underset{\sim}{\mathbb{R}}^N$ is meager in $\underset{\sim}{\mathbb{R}}^N$.*

Proof. A set in $\underset{\sim}{\mathbb{R}}^N$ is said to be *dense* if its closure is everything. Alternatively, a set V in $\underset{\sim}{\mathbb{R}}^N$ is dense if it meets every nonempty open set in $\underset{\sim}{\mathbb{R}}^N$. It suffices to prove that the intersection of any sequence of dense open sets is dense.

For finite sequences s of real numbers, we let G_s be the set of all elements of $\mathbb{R}^N$ which extend s. The G_s form a basis for the topology of $\underset{\sim}{\mathbb{R}}^N$. Let $\{V_n\}$ be a sequence of dense open sets, and let V be any open set. Let $\{s_n\}$ be a strictly increasing sequence of finite sequences of real numers defined as follows. Choose s_1 so that $G_{s_1} \subset V_1 \cap V$. Choose s_{n+1} so that $G_{s_{n+1}} \subset G_{s_n} \cap V_{n+1}$. Observe that $\{s_n\}$ determines an element of $\mathbb{R}^N$ which is in $\bigcap_n V_n \cap V$. This completes the proof of the lemma.

We now assume that $F: \mathbb{R}^N \to \mathbb{R}$ is a Borel function such that if $\mathbf{y}$ is obtained from $\mathbf{x}$ by permuting finitely many coordinates, then $F(\mathbf{x}) = F(\mathbf{y})$.

For each rational numer q, let $A_q = \{\mathbf{x}: F(\mathbf{x}) < q\}$.

LEMMA 3. *For each q, either A_q or its complement is meager in $\underset{\sim}{\mathbb{R}}^N$.*

Proof. Any permutation τ of finitely many coordinates induces a homeomorphism of $\mathbb{R}^N$. Let $A_q \Delta V$ be meager, where V is open. Then each $\tau[A_q] \Delta \tau[V] = A_q \Delta \tau[V]$ is meager. Therefore $A_q \Delta \bigcup_\tau \tau[V]$ is meager. It is easy to see that if $V \neq \emptyset$ then $\bigcup_\tau \tau[V]$ is dense. In this case, since $\bigcup_\tau \tau[V] - A_q$ is meager and $-\bigcup_\tau \tau[V]$ is meager, we see that $-A_q$ is meager. If $V = \emptyset$ then clearly A_q is meager.

LEMMA 4. *There is an* $\mathbf{x}$ *such that* $F(\mathbf{x}) \in \mathrm{Rng}(\mathbf{x})$.

Proof. Consider the set S of rationals q such that A_q is meager. It is obvious that S is closed under $<$. Now S cannot be all rationals, since a countable union of meager sets is meager. Similarly (using Lemma 3) $S \neq \emptyset$. Let z be the least upper bound of S. Then $\{\boldsymbol{x}: F(\boldsymbol{x}) < z\}$ is meager. Since for each rational $q > z$, $\{\boldsymbol{x}: F(\boldsymbol{x}) \geqslant q\}$ is meager (by Lemma 3), we see that $\{\boldsymbol{x}: F(\boldsymbol{x}) = z\}$ is the complement of a meager set. Now, the complement of any meager set must be dense. Hence $\{\boldsymbol{x}: F(\boldsymbol{x}) = z\}$ has an element which begins with z, and we are done.

Lemma 4 concludes this proof of Proposition C.

We now present the axiomatic system of essentially countable set theory. The idea here is that even though uncountable sets can be constructed in *ECST*, only limited use can be made of their uncountability. We give the axioms of *ECST* informally.

1. Extensionality. Two sets are equal if and only if they have the same elements.
2. Pairing. Each $\{a, b\}$ exists.
3. Union. Each $\{x: (\exists y \in a)(x \in y)\}$ exists.
4. Infinity. The least set ω such that $\phi \in \omega \;\&\; (\forall x)(x \in \omega \to x \cup \{x\} \in \omega)$ exists.
5. Limited separation. $\{x \in a: \varphi(x)\}$ exists, where $\varphi(x)$ is any formula (possibly with parameters) in which all quantifiers are bounded to sets (so called Δ_0-formulas).
6. Limited power set. The set of all subsets of any countable set exists.
7. Limited replacement. If $(\forall x \in a)(\exists! y)(\varphi(x, y))$ then $(\exists f)(\forall x \in a)(\varphi(x, f(x)))$, where $\varphi(x, y)$ is any formula (possibly with parameters) in which all quantifiers are bounded to sets (so-called Δ_0-formulas).
8. Choice. Every set of nonempty sets has a choice function.

It is possible to add certain stronger principles such as well orderings and transfinite recursion, but we do not go into these here.

The reader can convince himself that all of the usual separable mathematics can be formalized in *ECST*. One of the trickier matters is to construct the set of all Borel sets in a space. One cannot simply take the intersection of all σ-algebras containing the open sets for two reasons. Firstly, we do not know that there are any such σ-algebras since, e.g., we cannot construct the power set of the space (the space has power c). Secondly, taking the intersection would normally be done via separation—however, the formula would have unbounded set quantifiers.

Instead, one first develops the theory of well founded trees of finite sequences of natural numbers, and then uses them to describe the recipes for constructing the Borel sets. Topmost nodes are labelled with open sets. Other nodes are labelled with the instructions "complement," "union," or "intersection." For any such labelled tree and any point in the space, one proves that there is a unique "answering" function which assigns to each node the answer "yes" or "no" as to whether the point belongs to the Borel set given by the tree at that node. A point gets into the Borel set given by the labelled tree if the answer is "yes" at the vertex of the tree. We can then use limited separation and limited replacement to obtain the set of all Borel sets.

However, the proof given here of Proposition C cannot be given in *ECST*. The principal difficulty is with Lemma 1 applied to a non-separable space such as $\mathbb{R}^N$. Here separation is needed with quantifiers ranging over *subsets of* $\mathbb{R}^N$ just to construct, e.g., the interior of any subset of $\mathbb{R}^N$.

The following will be proved elsewhere, and establishes that Proposition C is unprovable in *ECST*.

THEOREM. *ECST is a conservative extension of second order arithmetic* (Z_2) *for* π_3^1 *sentences.*

REFERENCES

1. A. BLASS, A partition theorem for perfect sets, *Proc. Amer. Math. Soc.* **82**, No. 2 (June 1981), 271–277.
2. P. J. COHEN, The independence of the continuum hypothesis, *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 1143–1148; **51** (1964), 105–110.
3. H. FRIEDMAN, Higher set theory and mathematical practice, *Ann. Math. Logic* **2** (1971), 326–357.
4. H. FRIEDMAN, Borel structures in mathematics, I, Ohio State University, August 1979, unpublished notes.
5. H. FRIEDMAN, Some applications of Borel determinacy to analytic relations, in preparation.
6. L. HARRINGTON, A powerless proof of a theorem of Silver, University of California, at Berkeley, November 1976, unpublished notes.
7. L. HARRINGTON, Analytic determinacy and $0^{\#}$, *J. Symbolic Logic* **43** (1978), 685–693.
8. E. HEWITT AND L. J. SAVAGE, Symmetric measures on cartesian products, *Trans. Amer. Math. Soc.* **80** (1955), 470–501.
9. D. A. MARTIN, Measurable cardinals and analytic games, *Fund. Math.* **66** (1970), 287–291.

10. D. A. MARTIN, Borel determinacy, *Ann. of Math.* **102** (1975), 363–371.
11. D. A. MARTIN, Analysis and Σ_4^0 games, Rockefeller University, March 1974, unpublished notes.
12. J. PARIS AND L. HARRINGTON, A mathematical incompleteness in Peano arithmetic, *in* "Handbook of Mathematical Logic" (Jon Barwise, Ed.), pp. 1133–1142, North-Holland, Amsterdam, 1977.
13. F. P. RAMSEY, On a problem in formal logic, *Proc. London Math. Soc. Ser. 2* **30** (1930), 264–286.
14. J. H. SCHMERL, A partition property characterizing cardinals hyperinaccessible of finite type, *Trans. Amer. Math. Soc.* **188** (1974), 281–291.