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Universal deformation formulas and breaking symmetry*

Vincent Coll

Department of Mathematics, Ursinus College, Collegeville, PA 19426, USA

Murray Gerstenhaber**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA

Samuel D. Schack

Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14214-3093, USA

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Abstract

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We show that an algebra with a non-nilpotent Lie group of automorphisms or "symmetries" (e.g., smooth functions on a manifold with such a group of diffeomorphisms) may generally be deformed (in the function case, "quantized") in such a way that only a proper subgroup of the original group acts. This symmetry breaking is a consequence of the existence of certain "universal deformation formulas" which are elements, independent of the original algebra, in the tensor algebra of the enveloping algebra of the Lie algebra of the group.

Introduction

We show that if an algebra A admits a Lie group of automorphisms of dimension at least two then A may generally be deformed in such a way that the full group no longer acts, and we examine (but are unable to classify) the ways by which such "spontaneous symmetry breaking by deformation" may occur. A basic example of an algebra with a continuous group of symmetries is the algebra of smooth (or, in

Correspondence to: Professor S.D. Schack, Department of Mathematics, State University of New York at Buffalo, 106 Diefendorf Hall, Buffalo, NY 14214-3093, USA. Email: mthsds@ubvmsc.cc.buffalo.edu.

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** E-mail: mgersten@mail.sas.upenn.edu.

suitable cases, analytic or algebraic) functions on a manifold X which itself possesses a "large" Lie group G of automorphisms. This commutative function algebra is then generally deformed to a non-commutative one, a process frequently called "quantization". A special case, in turn, is that where the manifold is the group G itself, in which case its Hopf algebra, $\mathcal{O}(G)$, of polynomial functions is deformed into a "quantum group".

The phenomenon of general symmetry breaking just described results from the existence of certain "universal deformation formulas" (udfs), whose discussion will require a brief review of the algebraic deformation theory introduced in [5]. However, the oldest general formula [6], which serves a paradigm for the others, is both easy to understand and ubiquitous. Suppose that φ and ψ are commuting derivations of an algebra A of characteristic zero. We can then define a new formal multiplication (technically not on A itself but on the algebra $A[[t]]$ of formal power series on A—something we may suppress), by the *exponential deformation formula*

$$a *_t b = ab + t\varphi(a)\psi(b) + \frac{t^2}{2!}\varphi^2(a)\psi^2(b) + \frac{t^3}{3!}\varphi^3(a)\psi^3(b) + \cdots.$$

If A is associative then this gives a "one-parameter family" of associative multiplications on the "same" underlying vector space as that of A, a statement to be taken literally only when A is real or complex (or even p-adic!) and the series is well-defined for small values of t. The Lie group of symmetries of A which induced this deformation is that whose infinitesimal generators are φ and ψ. Note that A may be commutative but the deformed algebra generally need not be, for while φ and ψ commute there may be elements a and b in A with $\varphi(a)\psi(b) \neq \psi(a)\varphi(b)$, so $a *_t b \neq b *_t a$.

The formula displayed above remains meaningful for Lie algebras and may also create them. For if an associative algebra A deforms then the Lie algebra obtained by taking commutators also does so, and when A is commutative but deforms to a non-commutative algebra, then this commutator algebra becomes non-trivial. The infinitesimal of this associated Lie algebra deformation is then generally called a *Poisson bracket*.

A fundamental special case of this formula, antedating algebraic deformation theory by 15 years, first appears in Moyal's work [16] on statistical mechanics. Later, using algebraic deformation theory, Lichnerowitz [15] applied it to produce "star products", i.e., deformations of the Poisson–Lie algebra of a symplectic manifold, but needed the vanishing of certain deRham cohomology groups. Subsequently Dewilde and Lecomte removed the restrictive hypothesis, thereby showing that quantization was always possible (cf. [3]). The Berezin–Toeplitz–Wick calculus uses the exponential formula to realize a sizable sub-algebra of the algebra of Toeplitz operators on Fock space as a deformation of the algebra of functions having linear exponential growth (e.g., polynomials and linear exponentials in z and $\bar{z}$). More recently it has been used to obtain new quantum groups (cf. [8]). In this paper we will exhibit other udfs, and will show that, in general, an algebra A with a Lie group of automorphisms may be deformed so that only a nilpotent subgroup remains.

In what follows all rings will be assumed to contain unity. The *ground ring* k will be a commutative unital ring and "algebra" will always mean an associative unital k-algebra. All unsubscripted tensor and wedge products will be formed in the category of k-modules.[1]

Cohomology and deformation theory

Let M be a bimodule over the algebra A and $C^n(A, M) = \mathrm{Hom}_k(A^{\otimes n}, M)$ be the k-module of *Hochschild n-cochains of A with coefficients in M*. It is frequently convenient to view the n-cochains for $n \neq 0$ as k-multilinear maps

$$A \times \cdots \times A \ (n \text{ times}) \to M$$

and to identify $C^0(A, M) = \mathrm{Hom}_k(A^{\otimes 0}, M) = \mathrm{Hom}_k(k, M)$ with M. An element of $C^n(A, A)$ may be denoted F^n when it is useful to note its dimension. The *coboundary operator* $\delta^n: C^n(A, M) \to C^{n+1}(A, M)$ is defined by

$$\begin{aligned}(\delta^n F^n)&(a_1, \ldots, a_{n+1}) \\ &= a_1 F(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i F(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &\quad + (-1)^{n+1} F(a_1, \ldots, a_n) a_{n+1}\end{aligned}$$

and has square zero. We set $\ker \delta^n = Z^n(A, M)$, the module of *$n$-cocycles*; this contains $\mathrm{im}\, \delta^{n-1} = B^n(A, M)$, the module of *$n$-coboundaries*, and we obtain $H^n(A, M) = Z^n(A, M)/B^n(A, M)$, the nth *Hochschild cohomology group*. Also note that $Z^1(A, M) = \mathrm{Der}(A, M)$, the k-module of k-linear derivations of A into M, which is a Lie algebra when $M = A$. *Normalized n-cochains* satisfy the further condition $F(a_1, \ldots, a_n) = 0$ if any $a_i \in k$. These form a subcomplex $C_N^\bullet(A, M)$ of $C^\bullet(A, M)$, and the inclusion $C_N^\bullet(A, M) \hookrightarrow C^\bullet(A, M)$ induces an isomorphism, $H_N^\bullet(A, M) \cong H^\bullet(A, M)$ (cf. [9]). In the case $M = A$, which we need for deformation theory, the cohomology ring $H^\bullet(A, A)$ has the structure of a *G-algebra* (cf. [5]). That is, it is a graded k-module $H^\bullet = \{H^n\}$ together with two multiplications, $(\eta, \nu) \mapsto \eta\nu$ ("cup") and $(\eta, \nu) \mapsto [\eta, \nu]$ ("bracket") satisfying the following three properties:

(1) $(\eta, \nu) \mapsto \eta\nu$ is an associative graded commutative product; i.e., for $\eta^n \in H^n$ and $\nu^m \in H^m$ we have

$$\eta^m \nu^n = (-1)^{mn} \nu\eta \in H^{m+n}.$$

[1] Much of this paper was developed in the first author's 1990 Ph.D. Thesis "Universal deformation formulae" at the University of Pennsylvania.

(2) $(\eta, \nu) \mapsto [\eta, \nu]$ is a graded Lie product for which the grading is the degree reduced by 1, so $\eta^m \in H^m$ has *reduced degree* $m-1$ and $[\eta^m, \nu^n] \in H^{m+n-1}$. The bracket then satisfies

$$[\eta^m, \nu^n] = -(-1)^{(m-1)(n-1)}[\nu^n, \eta^m]$$

and

$$(-1)^{(m-1)(n-1)}[\eta^m, [\lambda^p, \nu^n]] + (-1)^{(p-1)(m-1)}[\lambda, [\nu, \eta]] + (-1)^{(n-1)(p-1)}[\nu, [\eta, \lambda]] = 0.$$

The associative and graded Lie products are connected by the following property:

(3) $[-, \lambda^p]$ is a graded derivation of degree $p-1$ of the associative algebra structure; that is, for all $\lambda^p \in H^p$,

$$[\eta^m \nu^n, \lambda^p] = [\eta, \lambda]\nu + (-1)^{m(p-1)}\eta[\nu, \lambda].$$

For $H^\bullet(A, A)$ we describe products at the cochain level, and observe that they descend to cohomology, where (3) also holds. The graded associative multiplication on $C^\bullet(A, A)$, denoted by $\smile$, is defined by

$$(F^m \smile G^n)(a_1, \ldots, a_m, b_1, \ldots, b_n) = F^m(a_1, \ldots, a_m)G^n(b_1, \ldots, b_n).$$

While the cup product uses the multiplication in A, the graded Lie bracket on $C^\bullet(A, A)$ does not. For this, first define $F^m \bar{\circ}_i G^n \in C^{m+n-1}(A, A)$ for $1 \leqslant i \leqslant m$ by

$$F^m \bar{\circ}_i G^n(a_1, \ldots, a_{m+n-1}) = F^m(a_1, \ldots, a_{i-1}, G(a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{m+n-1})$$

(G is "composed into" the ith slot of F). The *(total) composition product* is

$$F^m \bar{\circ} G^n = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} F^m \bar{\circ}_i G^n.$$

Note that the associativity of the multiplication α is equivalent to

$$\alpha \bar{\circ} \alpha = 0$$

or, equally well, $\alpha \bar{\circ}_1 \alpha = \alpha \bar{\circ}_2 \alpha$. The graded commutator

$$[F^m, G^n]_G = F^m \bar{\circ} G^n - (-1)^{(m-1)(n-1)} G^n \bar{\circ} F^m$$

is then a graded Lie product. Note that

$$\delta F^n = (-1)^{n+1}[\alpha, F^n]_G = -[F, \alpha]_G.$$

Also $C^1(A, A) = \mathrm{End}_k(A)$, is an associative algebra under composition, and $[-,-]_G$ coincides with the usual Lie product of endomorphisms since $F^1 \bar{\circ} G^1 = FG$. More generally, $F^1 \bar{\circ} G = F^1 \bar{\circ}_1 G = F \circ G$, for every cochain G. One has

$$\delta(F^m \smile G^n) = \delta F^m \smile G^n + (-1)^m F^m \smile \delta G^n,$$

so

$$Z^n \smile Z^m \subset Z^{n+m} \quad \text{and} \quad Z^n \smile B^m, B^n \smile Z^m \subset B^{n+m},$$

and one defines the cup product of cohomology classes by

$$[F^m] \smile [G^n] = [F^m \smile G^n] \quad \text{for } F^m \in Z^m, G^n \in Z^n.$$

At the cohomology level, $\smile$ is graded commutative,

$$[F^m] \smile [G^n] = (-1)^{mn}[G^n] \smile [F^m],$$

by virtue of the identity [5]

$$\delta(F^m \bar{\circ} G^n) - F^m \bar{\circ} \delta G^n - (-1)^{n-1} \delta F^m \bar{\circ} G^n$$
$$= (-1)^n (G^n \smile F^m - (-1)^{mn} F^m \smile G^n).$$

This also implies that the graded Lie product descends to cohomology. In particular, for derivations φ and ψ one has

$$\delta(\varphi\psi) = -(\psi \smile \varphi + \varphi \smile \psi),$$

which generalizes inductively to give

Lemma 1. *If $\varphi_1, \ldots, \varphi_n$ are derivations of A into itself, then*

$$\delta(\varphi_1 \cdots \varphi_n) = - \sum_{\substack{i_1 < \cdots < i_r \\ i_{r+1} < \cdots < i_n}} \varphi_{i_1} \varphi_{i_2} \cdots \varphi_{i_r} \smile \varphi_{i_{r+1}} \varphi_{i_{r+2}} \cdots \varphi_{i_n}.$$

The sum is over all partitions of $\{1, \ldots, n\}$ into two non-empty subsets. In particular, writing φ^r for the rth iterate of a single derivations φ, we have

$$\delta \varphi^n = - \sum_{i=1}^{n-1} \binom{n}{i} \varphi^i \smile \varphi^{n-i}. \qquad \square$$

That $[-,-]_G$ acts as graded derivations of $\smile$ on $H^\bullet(A, A)$ follows from another identity [5]:

$$[F^m \smile G^n, H^p]_G = [F^m, H^p]_G \smile G^n + (-1)^{m(p-1)} F^m \smile [G^n, H^p]_G + \delta E,$$

where

$$E = \sum_{i=1}^{p-1} \sum_{j=m+1}^{m+p-2} (-1)^{(m-1)i+(n-1)j} (H^p \bar{\circ}_i F^m) \bar{\circ}_j G^n.$$

We note a few other useful identities. Trivially,

$$(F^p \smile G) \bar{\circ}_i H = \begin{cases} (F \bar{\circ}_i H) \smile G & \text{for } i \leqslant p, \\ F \smile (G \bar{\circ}_{i-p} H) & \text{for } i > p, \end{cases}$$

so

$$(F^p \smile G) \bar{\circ} H^q = (F \bar{\circ} H) \smile G + (-1)^{p(q-1)} F \smile (G \bar{\circ} H).$$

An easy induction establishes

Lemma 2. *If $\varphi_1, \ldots, \varphi_p \in \mathrm{Der}(A, A)$ and $F, G \in C^\bullet(A, A)$ then*

$$(\varphi_1 \cdots \varphi_p) \bar{\circ} (F \smile G) = \sum_{\substack{i_1 < \cdots < i_r \\ i_{r+1} < \cdots < i_p}} (\varphi_{i_1} \cdots \varphi_{i_r} \bar{\circ} F) \smile (\varphi_{i_{r+1}} \cdots \varphi_{i_p} \bar{\circ} G).$$

Here the sum ranges over all partitions of $\{1, \ldots, r\}$ into two (possibly empty) subsets. (Interpret an empty composite as the identity map in $C^1(A, A)$.) In particular, we have the standard Leibniz rule,

$$\varphi^p \bar{\circ} (F \smile G) = \sum_{r=0}^{p} \binom{p}{r} (\varphi^r \bar{\circ} F) \smile (\varphi^{p-r} \bar{\circ} G). \qquad \square$$

The exterior algebra $\bigwedge^\bullet \mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ has a natural G-algebra structure in which the graded associative multiplication is just the exterior product, $\wedge$, and the bracket is the *Schouten bracket* $[-,-]_S$ described as follows:

(i) If $x_1, \ldots, x_p, y_1, \ldots, y_q \in \mathfrak{q}$, then

$$[x_1, y_1 \wedge \cdots \wedge y_q]_S = \sum (-1)^{j-1} y_1 \wedge \cdots \wedge [x_1, y_j] \wedge \cdots \wedge y_q.$$

(ii) Setting $x_1 \wedge \cdots \wedge x_p = \xi^p$ and $y_1 \wedge \cdots \wedge y_q = \eta^q$, then

$$[\xi^p, \eta^q]_S = \sum (-1)^{(i-1)} x_1 \wedge \cdots \wedge [x_i, \eta^q] \wedge \cdots \wedge x_p.$$

Combining these we obtain

(iii) If $x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p = \xi^{\hat{i},p}$ and $y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_q = \eta^{\hat{j},q}$, then

$$\begin{aligned}[\xi^p, \eta^q]_S &= [x_1 \wedge \cdots \wedge x_q, y_1 \wedge \cdots \wedge y_q]_S \\ &= (-1)^{(p-1)(q-1)} \sum (-1)^{i+j} [x_i, y_j] \wedge \xi^{\hat{i},p} \wedge \eta^{\hat{j},q},\end{aligned}$$

where $\hat{x}_i$, $\hat{y}_j$ denote the omission of x_i and y_j respectively.

The functor $\mathfrak{g} \mapsto \bigwedge^{\bullet}\mathfrak{g}$ is easily seen to be the left adjoint to the forgetful functor G-algebras $\to$ Lie algebras, $H^{\bullet} \mapsto H^1$. This means, in particular, that $[-,-]_S$ is the unique graded Lie bracket on $\bigwedge^{\bullet}\mathfrak{g}$ which restricts to the given bracket on $\mathfrak{g}$ *and*, together with $\wedge$, makes $\bigwedge^{\bullet}\mathfrak{g}$ a G-algebra.

A *deformation* of an (associative) algebra A is a new (associative) multiplication on the formal power series ring $A[[t]]$ of the form

$$\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$$

where $\alpha\,(=\alpha_0)$ is the original multiplication and each α_i is a k-bilinear map (2-cochain) $A \times A \to A$, tacitly extended to be $k[[t]]$ bilinear. We denote the deformed algebra by A_t. (Adaptation to other categories, e.g. Lie algebras, is clear.) A second deformation A'_t, with multiplication α'_t, is *equivalent* to A_t if there is a $k[[t]]$-algebra isomorphism $f_t\colon A'_t \to A_t$ of the form $f_t = \mathrm{id}_A + tf_1 + t^2 f_2 + \cdots$ where each f_i is a k-linear map (1-cochain) $A \to A$ (again extended to be $k[[t]]$-linear). The condition that f_t be an algebra map may be rewritten as

$$(\alpha_t \mathbin{\bar{\circ}}_2 f_t) \mathbin{\bar{\circ}}_1 f_t = f_t \mathbin{\bar{\circ}} \alpha'_t$$

or, equivalently, $\alpha'_t = f_t^{-1} \mathbin{\bar{\circ}} ((\alpha_t \mathbin{\bar{\circ}}_2 f_t) \mathbin{\bar{\circ}}_1 f_t)$. When this is the case, $\alpha'_1 = \alpha_1 + \delta f_1$. One may easily check that the associativity of α_t is equivalent to $\alpha_t \mathbin{\bar{\circ}} \alpha_t = 0$. Gathering the coefficients of t^n for each n gives the equivalent set of conditions

$$(*_n) \qquad \sum_{\substack{l+m=n \\ l,m>0}} \alpha_l \mathbin{\bar{\circ}} \alpha_m = \delta\alpha_n \quad \text{for all } n = 1, 2, \ldots,$$

from which one sees, in particular, that $\delta\alpha_1 = 0$. It follows that the *integrability* of an $\alpha_1 \in Z^2(A, A)$, i.e., the existence of a deformation with the given α_1 as its linear term, depends only on the cohomology class $[\alpha_1]$ of α_1. We may view $[\alpha_1] \in H^2(A, A)$ as the *infinitesimal* of the equivalence class of the deformation A_t. A *trivial* deformation is one equivalent to the $k[[t]]$-bilinear extension of the original multiplication. If one has only $\alpha + t\alpha_1 + \cdots + t^r\alpha_r$ satisfying $(*_n)$ for $n = 1, \ldots, r$ then

$$\alpha_1 \mathbin{\bar{\circ}} \alpha_r + \alpha_2 \mathbin{\bar{\circ}} \alpha_{r-1} + \cdots + \alpha_r \mathbin{\bar{\circ}} \alpha_1 \in Z^3(A, A),$$

and the class of this cocycle is the obstruction to finding an α_{r+1} such that $(*_{r+1})$ is satisfied. In particular, $\mathrm{Sq}[\alpha_1] = [\alpha_1 \bar{\circ} \alpha_1] \in H^3(A, A)$, is the *primary obstruction* to integrating $[\alpha_1]$.

Suppose now that φ and ψ are derivations of A into itself. Then $\varphi \smile \psi$ is a Hochschild 2-cocycle though not, in general, the infinitesimal of a deformation. However, if φ and ψ commute and k contains $\mathbb{Q}$, then

$$\alpha_t = \alpha + t\varphi \smile \psi + \frac{t^2}{2!}\varphi^2 \smile \psi^2 + \frac{t^3}{3!}\varphi^3 \smile \psi^3 + \cdots$$

defines a deformation, and in characteristic $p > 0$ continues to do so, provided $\varphi^p = \psi^p = 0$. (Here, as earlier, φ^n denotes the n-fold composite of φ with itself.) This deformation was denoted $\exp t(\varphi \smile \psi)$ in [7], but this notation is ambiguous; one should write $\alpha \exp t(\varphi \otimes \psi)$. For note that we may have $\varphi \smile \psi = \tilde{\varphi} \smile \tilde{\psi}$ for some other pair of commuting derivations while $\alpha \exp t(\varphi \otimes \psi) \neq \alpha \exp t(\tilde{\varphi} \otimes \tilde{\psi})$. With this notation, the last formula may be rewritten to give the *exponential deformation formula*:

$$\alpha_t = \alpha\left(1 \otimes 1 + t\varphi \otimes \psi + \frac{t^2}{2!}\varphi^2 \otimes \psi^2 + \frac{t^3}{3!}\varphi^3 \otimes \psi^3 + \cdots\right).$$

This may be "skew-symmetrized" (cf. [17]) as follows: Setting $\varphi \wedge \psi = \frac{1}{2}(\varphi \otimes \psi - \psi \otimes \varphi)$ one has $\varphi \wedge \psi = \varphi \otimes \psi - \delta(2\varphi\psi)$, so there is an equivalent deformation

$$\alpha_t' = \alpha\left(1 \otimes 1 + t\varphi \wedge \psi + \frac{t^2}{2!}(\varphi \wedge \psi)^2 + \cdots\right) = \alpha e^{t\varphi \wedge \psi}.$$

In particular when φ and ψ are commuting tangent vector fields on the smooth manifold X, this "integrates" the Poisson bracket $\varphi \wedge \psi$ to a deformation of $A = C^\infty(X)$.

Universal deformation formulas

The first known "universal deformation formula" (udf) based on a Lie algebra $\mathfrak{g}$ was the exponential deformation formula of the previous section. In this case $\mathfrak{g}$ is the unique abelian Lie algebra of dimension two, which we henceforth denote $\mathfrak{a}_2$. The construction of udfs based on $\mathfrak{a}_2$ and other Lie algebras parallels deformation of an algebra and their explicit construction is the subject of this section.

To begin with, we note that the structure of $C^\bullet(A, A)$ is richer still than described in the previous section — it is a "unital comp(osition) algebra": By definition, a (*right*) *comp algebra* $C^\bullet$ is a graded k-module $C^0, C^1, \ldots$ together with a distinguished element $\pi \in C^2$ and k-bilinear operations $\bar{\circ}_i : C^p \times C^q \to C^{p+q-1}$ for $i \geqslant 1$ such that $f^p \bar{\circ}_i g = 0$ if $i > p$,

$$(f^p \bar{\circ}_i g^q) \bar{\circ}_j h^r = \begin{cases} (f \bar{\circ}_j h) \bar{\circ}_{i+r-1} g & \text{if } j < i, \\ f \bar{\circ}_i (g \bar{\circ}_{j-i+1} h) & \text{if } i \leqslant j < q + i, \\ (f \bar{\circ}_{j-q+1} h) \bar{\circ}_i g & \text{if } j \geqslant q + i \end{cases}$$

and $\pi \bar{\circ}_1 \pi = \pi \bar{\circ}_2 \pi$. (The first and third parts of the equation displayed above are equivalent; we include them for symmetry and completeness.) We then define $\bar{\circ}$ and $[-, -]$ as for the case of $C^\bullet(A, A)$. Moreover, we can define a coboundary operator and a "cup product" by $\delta f = -[f, \pi]$ and $f \smile g = (\pi \bar{\circ}_2 g) \bar{\circ}_1 f$. (The condition on π insures that $\delta^2 = 0$.) Now all of our previous assertions for $C^\bullet(A, A)$ carry over to an arbitrary comp algebra $C^\bullet$. In particular, the homology $H(C^\bullet)$ is a G-algebra. (This is essentially what is proved in [5]; the claim in [7] that the same is true for the "composition complexes" defined therein is not quite correct because the relationship between $\bar{\circ}_i$ and $\smile$ is not precise enough there.) A comp algebra is *unital* if there are (necessarily unique) elements $1 \in C^0$ and $I \in C^1$ such that $\pi \bar{\circ}_1 1 = \pi \bar{\circ}_2 1 = I$ and $I \bar{\circ} f = f \bar{\circ}_i I = f$ for all p, all $f \in C^p$ and all $i \leqslant p$. The element 1 is then an identity for $\smile$ in both $C^\bullet$ and $H(C^\bullet)$. Also, $\pi = I \smile I = \delta I$.

As in the case of cochains, C^1 is an associative algebra under $\bar{\circ} = \bar{\circ}_1$ and, hence, a Lie algebra under the commutator $[-, -]$. The 1-cocycles $Z^1(C^\bullet)$ again comprise a Lie subalgebra. Now, with the evident maps, unital comp algebras form a category and, clearly, the assignment $C^\bullet \mapsto Z^1(C^\bullet)$ is a functor. This functor has a left adjoint. That is, for each Lie algebra $\mathfrak{g}$ there is a "universal" unital comp algebra, denoted $T^\bullet U\mathfrak{g}$, characterized by the fact that there is a natural bijection between Lie algebra maps $\mathfrak{g} \to Z^1(C^\bullet)$ and unital comp algebra maps $T^\bullet U\mathfrak{g} \to C^\bullet$. A special case occurs for each Lie algebra map $\mathfrak{g} \to \mathrm{Der}(A, A)$. Such morphisms arise, in particular, when a Lie group G acts on A and $\mathfrak{g}$ is the Lie algebra of G.

The universal unital comp algebra on $\mathfrak{g}$ is, as an algebra, just the tensor algebra $T^\bullet U\mathfrak{g}$ of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$. To define the comp algebra structure, first note that $U\mathfrak{g}$ is a bialgebra (in fact, a Hopf algebra) with comultiplication $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ defined by $\Delta g = 1 \otimes g + g \otimes 1$ for $g \in \mathfrak{g}$ (the *primitives* of $U\mathfrak{g}$) and $\Delta(uw) = \Delta(u)\Delta(w)$ for $u, w \in U\mathfrak{g}$. Then, following [7], we can make the tensor algebra TU on *any* bialgebra U (and so, in particular, for $U = U\mathfrak{g}$) into a comp algebra by setting

$$\begin{aligned}(u_1 \otimes \cdots \otimes u_p) \bar{\circ}_i (u'_1 \otimes \cdots \otimes u'_q) \\ = \sum u_1 \otimes \cdots \otimes u_{i-1} \otimes u_{i,(1)} u'_1 \otimes \cdots \otimes u_{i,(q)} u'_q \otimes u_{i+1} \otimes \cdots \otimes u_p,\end{aligned}$$

where, extending the Sweedler notation, we write $\Delta_q(u_i) = \sum u_{i,(1)} \otimes \cdots \otimes u_{i,(q)}$ for the iterated comultiplication $\Delta_q : U \to T^q U = U^{\otimes q}$. (Note that $\Delta_2 = \Delta$ while $\Delta_1 = \mathrm{Id}$; thus, on $T^1 U = U$, the operation $\bar{\circ} = \bar{\circ}_1$ is just the multiplication in U.) The elements

π, I and 1 are, respectively, $1 \otimes 1 \in U \otimes U$, $1 \in U$ and $1 \in k$. Elementary computations show that $\smile$ is just $\otimes$, that $\delta u = -[u, 1 \otimes 1] = u \otimes 1 + 1 \otimes u - \Delta u$ for $u \in U$ — so the 1-cocycles are just the primitives — and that $\delta a = 0$ for $a \in k = T^0 U$. In particular, $H^0(T^\bullet U\mathfrak{g}) = k$ and $H^1(T^\bullet U\mathfrak{g}) = Z^1(T^\bullet U\mathfrak{g}) = \mathfrak{g}$. It is now easily checked that $\mathfrak{g} \mapsto T^\bullet U\mathfrak{g}$ is, as claimed, left adjoint to $C^\bullet \mapsto Z^1(C^\bullet)$. Finally, we note that this complex has arisen in other considerations recently — notably in Drinfel'd's study of quasi-bialgebras [4] and Grabowski's work on star-products [13].

A *universal deformation formula* (udf) *based on* $\mathfrak{g}$ is a formal power series $\gamma_t = 1 \otimes 1 + t\gamma_1 + t^2\gamma_2 + \cdots$ in $(T^2 U\mathfrak{g})[[t]]$ such that $\gamma_t \bar{\circ} \gamma_t = 0$. The sense in which γ_t is "universal" is the following: if $\mathfrak{g} \to \mathrm{Der}(A, A) = Z^1(A, A)$ is a Lie algebra morphism then the induced comp algebra map $T^\bullet U\mathfrak{g} \to C^\bullet(A, A)$ will carry γ_t to a deformation. A udf γ'_t is *equivalent* to γ_t if there is a formal power series $a_t = 1 + ta_1 + t^2 a_2 + \cdots$ in $(T^1 U\mathfrak{g})[[t]]$ such that $(\gamma_t \bar{\circ}_2 a_t) \bar{\circ}_1 a_t = a_t \bar{\circ} \gamma'_t$. Note that, since $\bar{\circ}$ coincides with multiplication on $T^1 U\mathfrak{g} = U\mathfrak{g}$, every $a_t = \sum a_i t^i \in (T^1 U\mathfrak{g})[[t]]$ with $a_0 = 1$ is $\bar{\circ}$-invertible. The definition of equivalence can thus be rewritten as $\gamma'_t = a_t^{-1} \bar{\circ} ((\gamma_t \bar{\circ}_2 a_t) \bar{\circ}_1 a_t)$. Conversely, given any $a_t \in (T^1 U\mathfrak{g})[[t]]$ with $a_0 = 1$, it is easily checked that *conjugating* γ_t by a_t, as in the latter formula, produces a (necessarily equivalent) udf γ'_t with $\gamma'_1 = \gamma_1 + \delta a_1$. The *infinitesimal* of a udf is thus well-defined as a cohomology class — namely that of γ_1 — in $H^2(T^\bullet U\mathfrak{g})$. Clearly, equivalent udfs always induce equivalent deformations.

Note that any udf based on $\mathfrak{g}$ induces, in particular, a deformation of the tensor algebra $TU\mathfrak{g}$. However, since $H^n(TV, TV) = 0$ for $n \geqslant 2$ and every vector space V, these deformations — and all others — are necessarily trivial. Of course, for other algebras the induced deformations are generally nontrivial.

If $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ then any udf based on $\mathfrak{g}_0$ may be constructed as a udf based on $\mathfrak{g}$. A formula is *strictly based on* $\mathfrak{g}$ if it does not arise from a proper subalgebra. For example, if $\mathfrak{g} = \mathfrak{a}_2$ then the exponential deformation formula is a udf strictly based on $\mathfrak{a}_2$.

We now compute the cohomology of $T^\bullet U\mathfrak{g}$. First note that, by the earlier stated universal property of $\bigwedge^\bullet \mathfrak{g}$, the identity map $\mathfrak{g} \to \mathfrak{g} = Z^1(T^\bullet U\mathfrak{g}) = H^1(T^\bullet U\mathfrak{g})$ induces a G-algebra map $\bigwedge^\bullet \mathfrak{g} \to H(T^\bullet U\mathfrak{g})$.

Theorem 3. *The canonical G-algebra map* $\bigwedge^\bullet \mathfrak{g} \to H(T^\bullet U\mathfrak{g})$ *is an isomorphism.*

Proof. The Poincaré–Birkoff–Witt theorem (PBW) asserts that if $x_1, x_2, \ldots$ is an ordered linear basis for the Lie algebra $\mathfrak{g}$ the "standard" monomials of the form $x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$ comprise a basis for $U\mathfrak{g}$. We will call $i_1 + i_2 + \cdots + i_r$ the *degree* of such a monomial. Since nothing depends on the actual Lie bracket of $\mathfrak{g}$, we may, without any loss of generality, assume that $\mathfrak{g}$ is an abelian Lie algebra with basis $x_1, x_2, \ldots$. Then $U\mathfrak{g}$ is just the polynomial ring $k[x_1, x_2, \ldots]$.

Now let $y_1, y_2 \ldots$ be new variables and set $S = k[y_1, y_2, \ldots]$. If $I = (i_1, \ldots, i_n)$ is a multi-index with each $i_r \geqslant 0$ then set $y^I = y_1^{i_1} \cdots y_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$. Also, if $I' = (i'_1, \ldots, i'_m)$ then, extending one or the other by zeros if necessary, set

$I + I' = I' + I = (i_1 + i'_1, i_2 + i'_2, \ldots)$. Write ∂_i for the derivation $\partial/\partial y_i$ of S into itself and ∂^I for $\partial_1^{i_1}\partial_2^{i_2}\cdots\partial_n^{i_n}$. Then $\partial^I y^I = 1$ but $\partial^I y^J = 0$ for all $J \neq I$ with $|J| \leqslant |I|$.

Now, each $C^n_{\mathrm{N}}(S, S)$ (*normalized* n-cochains) has a natural topology in which a basis $U_1 \supset U_2 \supset \cdots$ of neighborhoods of 0 is defined by letting U_r consist of those F with $F(x^{I_1}, \ldots, x^{I_n}) = 0$ whenever $|I_1 + \cdots + I_n| \leqslant r$. Each U_r/U_{r+1} is spanned, over S, by the images of those $\partial^{I_1} \smile \cdots \smile \partial^{I_n}$ with $|I_1 + \cdots + I_n| = r$. We may therefore write every $F \in C^n_{\mathrm{N}}(S, S)$ uniquely as an infinite k-linear combination of the form $\sum c_{J, I_1, \ldots, I_n} y^J(\partial^{I_1} \smile \cdots \smile \partial^{I_n})$ with each $|I_r| > 0$. (However, $|J| = 0$ is permitted.) As in the complex $T^\bullet U\mathfrak{g}$, we assign $y^J(\partial^{I_1} \smile \cdots \smile \partial^{I_n}) \in C^\bullet_{\mathrm{N}}(S, S)$ degree $|I_1 + \cdots + I_n|$. Then δ preserves degrees. Moreover, for each fixed multi-index J, the space of (infinite) linear combinations, as above, of the terms $y^J(\partial^{I_1} \smile \cdots \smile \partial^{I_n})$ forms a subcomplex of $C^\bullet_{\mathrm{N}}(S, S)$. The map $\mathfrak{g} \to \mathrm{Der}(S, S)$, $x_i \mapsto \partial_i$ induces the cochain map

$$T^\bullet U\mathfrak{g} \to C^\bullet(S, S), \qquad x^{I_1} \otimes \cdots \otimes x^{I_n} \mapsto \partial^{I_1} \smile \cdots \smile \partial^{I_n},$$

which is obviously a monomorphism and restricts to a map of normalized complexes $T^\bullet_{\mathrm{N}} U\mathfrak{g} \to C^\bullet_{\mathrm{N}}(S, S)$. Here $T^\bullet_{\mathrm{N}} U\mathfrak{g}$ is the span of those $x^{I_1} \otimes \cdots \otimes x^{I_n}$ in which each $|I_r| > 0$ and it is easily checked that the inclusion $T^\bullet_{\mathrm{N}} U\mathfrak{g} \hookrightarrow T^\bullet U\mathfrak{g}$ induces a cohomology isomorphism. Note that the image of the map $T^\bullet_{\mathrm{N}} U\mathfrak{g} \to C^\bullet_{\mathrm{N}}(S, S)$ lies in the subcomplex with $J = 0$.

The Hochschild–Kostant–Rosenberg theorem ([14]; see also [17]), asserts, in particular, that since S is a polynomial ring over $k \supseteq \mathbb{Q}$, the natural S-module map

$$\textstyle\bigwedge^\bullet_S \mathrm{Der}(S, S) \to H^\bullet(S, S), \qquad y^J(\partial^{I_1} \wedge \cdots \wedge \partial^{I_n}) \mapsto y^J(\partial^{I_1} \smile \cdots \smile \partial^{I_n})$$

is an isomorphism. As δ preserves degrees, this means that the summands of degree n in an n-cocycle themselves comprise a cocycle having the same cohomology class while the other summands constitute a coboundary. This is true in particular for the subcomplex with $J = 0$ and, so, for the image of the map $T^\bullet_{\mathrm{N}} U\mathfrak{g} \to C^\bullet_{\mathrm{N}}(S, S)$. It now follows that this map induces a cohomology monomorphism with image $\bigwedge^\bullet \mathfrak{g}$. In particular, $H(T^\bullet_{\mathrm{N}} U\mathfrak{g}) \cong \bigwedge^\bullet \mathfrak{g}$ and it is easily checked that this isomorphism is inverse to the canonical map $\bigwedge^\bullet \mathfrak{g} \to H(T^\bullet_{\mathrm{N}} U\mathfrak{g}) \cong H(T^\bullet U\mathfrak{g})$. □

As in the case of deformations, the equation $\gamma_t \bar{\circ} \gamma_t = 0$ for a udf $\gamma_t = 1 \otimes 1 + t\gamma_1 + t^2\gamma_2 + \cdots$ (where $\gamma_i \in U\mathfrak{g} \otimes U\mathfrak{g}$) implies that γ_1 is a 2-cocycle, i.e., is in $Z^2(T^\bullet U\mathfrak{g})$. Since up to equivalence of formulas only the cohomology class of γ_1 matters, we may view the class $[\gamma_1]$ in $H^2(T^\bullet U\mathfrak{g}) \cong \bigwedge^2 \mathfrak{g}$ as the infinitesimal of γ_t. The γ_i satisfy the equations $(*_r)$ as before. If we have $\gamma_1, \ldots, \gamma_n$ satisfying $(*_r)$ for $r = 1, \ldots, n$ the primary obstruction to extending the "approximate formula" $\Gamma_n = 1 \otimes 1 + t\gamma_1 + \cdots + t^n\gamma_n$ one more term is the class

$$\mathrm{Obs}(\Gamma_n) = [\gamma_1 \bar{\circ} \gamma_n + \gamma_2 \bar{\circ} \gamma_{n-1} + \cdots + \gamma_n \bar{\circ} \gamma_1] \in \textstyle\bigwedge^3 \mathfrak{g}.$$

When $\dim_k \mathfrak{g} = 2$ we have $\bigwedge^3 \mathfrak{g} = 0$, and so, by Theorem 3, the obstruction is trivial and every infinitesimal can be integrated.

For $\dim_k \mathfrak{g} \geqslant 3$ the obstruction map $\mathrm{Sq}: \bigwedge^2 \mathfrak{g} \to \bigwedge^3 \mathfrak{g}$ sends γ_1 to $\mathrm{Sq}(\gamma_1) = [\gamma_1 \bar{\circ} \gamma_1] = \frac{1}{2}[\gamma_1, \gamma_1]_S$. Suppose now that $\dim_k \mathfrak{g} < \infty$. Those γ_1 with $\mathrm{Sq}(\gamma_1) = 0$ form an algebraic subset, $V_1(\mathfrak{g}) = V_1$ of $\bigwedge^2 \mathfrak{g}$ which is stable under all automorphisms of $\mathfrak{g}$ and therefore, in particular, under the action of the adjoint Lie group G as well as under $\mathrm{ad}(g)$ for $g \in \mathfrak{g}$. Those γ_1 which can be prolonged to an approximate formula $1 \otimes 1 + t\gamma_1 + t^2\gamma_2 + t^3\gamma_3$ form an algebraic subset V_2 of V_1, and so forth. We thus obtain a descending sequence $V_1 \supseteq V_2 \supseteq \cdots$ of (quadratic) algebraic subvarieties of $V_0 = \bigwedge^2 \mathfrak{g} \subset U\mathfrak{g} \otimes U\mathfrak{g}$. In a finite number of steps we arrive at V_∞ *the locus of integrability* consisting of all $\gamma_1 \in \bigwedge^2 \mathfrak{g}$ for which there is a full universal deformation formula beginning $1 \otimes 1 + t\gamma_1 + \cdots$. Denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x.

Theorem 4. *If* $\mathfrak{g}$ *is a nilpotent Lie algebra with index of nilpotence* N, *then* $V_\infty = V_{\lfloor (3N+1)/2 \rfloor}$.

Proof. If $1 \otimes 1 + t\gamma_1 + t^2\gamma_2 + \cdots + t^n\gamma_n$ is an approximate deformation formula then each of the tensor factors of the obstruction 3-cocycle is a word in $U\mathfrak{g}$ and the total length of these words is $2 + 2n$. After reordering, using PBW, some terms will have total degree 3 and according to Theorem 3, these determine the cohomology class of the obstruction. Each term is reordered by the introduction of a sequence of commutators in each tensor factor. Since there are only three tensor factors, each term will have one (or more) tensor factors requiring $(2 + 2n - 3)/3$ iterated commutators. Noting that $\mathfrak{g}$ has index of nilpotence $N < (2n - 1)/3$ for $n > \lfloor (3N + 1)/2 \rfloor$ finishes the proof. □

Determining the locus of integrability is generally difficult but we conjecture the following:

Conjecture. If $\mathfrak{g}$ is semisimple then $V_1 = V_\infty$.

For $\dim_k \mathfrak{g} = 2$, the cohomology theory has shown that every infinitesimal can be integrated, so udfs based on $\mathfrak{g}$ surely exist (as noted earlier). As also mentioned, the exponential deformation formula is a udf based on $\mathfrak{a}_2$. Although not previously described as such the *quasi-exponential formula* of [2] is a udf (strictly) based on the unique non-abelian Lie algebra $\mathfrak{b}_2$ of dimension two. We repeat it without proof.

Theorem 5. *Let* φ, ψ *be generators of* $\mathfrak{b}_2$ *with* $[\varphi, \psi] = \varphi$ *and set* $[\psi]_n = \psi(\psi - 1)\cdots(\psi - n + 1)$. *If* $k \supset \mathbb{Q}$ *then*

$$e(t; \varphi, \psi) = 1 \otimes 1 + t\varphi \otimes \psi + \frac{t^2}{2!}\varphi^2 \otimes [\psi]_2 + \frac{t^3}{3!}\varphi^3 \otimes [\psi]_3 + \cdots$$

is a udf strictly based on $\mathfrak{b}_2$. □

The absolute values of the coefficients of the poloynomial $[x]_n = x(x-1)\cdots(x-n+1)$ are the *Stirling numbers of the first kind*; the coefficient of x^i is precisely the number of elements of the symmetric group on n letters, expressible as a product of i disjoint cycles. If A is a finite-dimensional algebra over $\mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{b}_2 \to \mathrm{Der}(A, A)$ is a Lie algebra map then the deformation of A induced by $e(t; \varphi, \psi)$ has a nonzero radius of convergence which is generally finite (the reciprocal of the absolute value of the largest eigenvalue of φ). The skew symmetrized version of the quasi-exponential is not exponential. However, Giaquinto has observed that the quasi-exponential may also be written as

$$e(t; \varphi, \psi) = \exp\left(t\varphi \otimes \psi - \frac{t^2}{2}\varphi^2 \otimes \psi + \frac{t^3}{3}\varphi^3 \otimes \psi - \cdots\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n}(-1)^{n+1}\varphi^n \otimes \psi\right)$$

$$= \exp(\ln(1 + t\varphi) \otimes \psi).$$

Denote by $\mathfrak{sl}_2$ the three-dimensional simple Lie algebra with basis h, x, y and commutator relations

$$[h, x] = x, \quad [h, y] = -y, \quad [x, y] = 2h.$$

Theorem 6. $V_1 = V_\infty = \{\gamma = a(h \wedge x) + b(h \wedge y) + c(x \wedge y) \in \bigwedge^2 \mathfrak{sl}_2 \mid ab - c^2 = 0\}$. *Moreover, every non-zero $\gamma \in V_\infty$ is integrable to a udf based strictly on $\mathfrak{sl}_2$.*

Proof. First note that $\bigwedge^3 \mathfrak{sl}_2$ is generated (over k) by $h \wedge x \wedge y$ and that for γ and $\gamma' = a'(h \wedge x) + b'(h \wedge y) + c'(x \wedge y) \in \bigwedge^2 \mathfrak{sl}_2$ we have

$$[\gamma, \gamma']_S = (2ab' + 2a'b - 4cc')h \wedge x \wedge y.$$

The primary obstruction to γ, namely $\frac{1}{2}[\gamma, \gamma]_S$, thus vanishes if and only if $ab - c^2 = 0$, and this equation then defines V_1. Now suppose that $\Gamma_n = 1 \otimes 1 + t\gamma_1 + \cdots + t^n\gamma_n$ is a udf modulo t^{n+1}, that $\gamma_1 \neq 0$, and that, for some d, $d(h \wedge x \wedge y)$ is the obstruction to extending Γ_n to be a udf modulo t^{n+2}. Replacing γ_n by $\gamma_n + \gamma'$ with $\gamma' \in \bigwedge^2 \mathfrak{sl}_2$ changes the obstruction to $d(h \wedge x \wedge y) + [\gamma, \gamma']_S$. Clearly then we can always alter the last term of Γ_n by a cocycle to make the approximate series integrable. If $d = 0$ we may use $\gamma' = 0$. Thus every $\gamma_1 \in V_1$ can be extended first to $1 \otimes 1 + t\gamma_1 + t^2\gamma_2$ and then, by adjusting γ_2, to a udf with infinitesimal γ_1. We have proved the first assertion.

For the second assertion, note that it is trivial if $abc \neq 0$. Otherwise, any coefficient being zero implies that γ_1 is $a(h \wedge x)$ or $b(h \wedge y)$, so we may assume without loss of generality that $\gamma_1 = -h \wedge x$. Now, as h and x generate a subalgebra of $\mathfrak{sl}_2$ isomorphic to $\mathfrak{b}_2$, the quasi-exponential $e(t; x, h)$ is a udf with infinitesimal $x \wedge h = -h \wedge x = \gamma_1$. In particular, $1 \otimes 1 + t(x \wedge h) + \frac{t^2}{2}(x^2 \otimes h(h-1))$ is unobstructed and, by the foregoing, remains so if we adjust the quadratic term by any cocycle γ' with $[x \wedge h, \gamma']_S = 0$. The latter condition requires only that $b' = 0$, so we may choose, for example, $\gamma' = x \wedge y$. Any integral of the adjusted series then has quadratic term $\frac{1}{2}(x^2 \otimes h(h-1)) + x \wedge y$ and, so, is strictly based on $\mathfrak{sl}_2$. $\square$

We now consider udfs based on the Lie algebra $\mathfrak{sl}_n$ for $n \geqslant 3$. Recalling that $\mathfrak{sl}_n$ may be viewed (in its natural representation) as the set of $n \times n$ matrices with trace 0, we denote by e_{ij} the matrix with a 1 in the ijth location and zeros elsewhere. Set $\beta = 2\sum_{1<j<n} e_{1j} \otimes e_{jn}$ and write $H_+^{[m]} = H_+(H_+ + 1)\cdots(H_+ + m - 1)$, where $H_+ = e_{11} - e_{nn}$. Then, with

$$\mu_m = H_+^{[m]} \otimes e_{1n}^m + \binom{m}{1}(H_+^{[m-1]} \otimes e_{1n}^{m-1})\beta + \binom{m}{2}(H_+^{[m-2]} \otimes e_{1n}^{m-2})\beta^2 + \cdots + \beta^m,$$

a straightforward induction argument gives

Theorem 7. *The series $\mu_t = 1 \otimes 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \cdots$ is a udf based on $\mathfrak{sl}_n$.* $\square$

Note that the latter formula is based strictly on a subset of a one-dimensional extension of a Heisenberg algebra generated by $\{e_{11} - e_{nn}, e_{1i}, e_{nj} \mid i = 2, \ldots, n$ and $j = 1, \ldots, n-1\}$.

Breaking symmetry

It is natural to ask, when a non-abelian Lie group G operates as automorphisms of an algebra A, whether there will generally be deformations of A—whose form depends only on the structure of the Lie group—such that the deformed algebra no longer admits the operation of G. That is, does the very existence of symmetries provide paths to breaking them? While we cannot as yet answer this question in such generality, we can address the analogous issue for udfs.

Let G be a Lie group operating on an algebra A by automorphisms and $\mathfrak{g}$ be the Lie algebra of G. We may view $\varphi \in \mathfrak{g}$ as the *infinitesimal* of the formal automorphism $\exp t\varphi = e^{t\varphi} = \mathrm{Id} + t\varphi + \frac{t^2}{2!}\varphi^2 + \cdots$. Abusing language we refer also to $\varphi \in \mathfrak{g}$ as a *symmetry* of A.

The Hochschild cohomology groups of an algebra A are upper semicontinuous functions of the algebra in the following sense: if A_t is a deformation of A then its cohomology groups are subquotients of those of A [1, 7, 10]. In particular, if k is a field then the dimensions of the cohomology groups of A cannot increase under deformation. Indeed, if the characteristic of k is 0 and A undergoes a "jump" deformation, i.e., one that remains constant for generic $t \neq 0$, then $\dim_k H^2(A, A)$ must strictly decrease (assuming that it was finite at the start) [10].

To compare $H^n(A, A)$ and $H^n(A_t, A_t)$, first consider $C^n(A_t, A_t) = C^n(A[[t]], A[[t]]) \supset C^n(A, A)$. The coboundary operator δ_t in $C^\bullet(A_t, A_t)$ is (as usual) given by $\delta_t F_t = -[F_t, \alpha_t]_G$, so $\delta_t = \delta + t\delta_1 + t^2\delta_2 + \cdots$ where $\delta = \delta_0 = -[-, \alpha]_G$ is the coboundary operator in $C^\bullet(A, A)$ and each δ_i is a k-linear map $C^n(A, A) \to C^{n+1}(A, A)$ extended to be $k[[t]]$-linear. Note that if $\delta_t F_t = 0$ then, in particular, F_0 is a cocycle. (The upper semicontinuity of cohomology follows from this.) We shall say that a derivation $\varphi \in \mathrm{Der}(A, A)$ *lifts* under a deformation (A_t, α_t) if there is a derivation $\varphi_t \in \mathrm{Der}(A_t, A_t)$ having the form $\varphi_t = \varphi + t\varphi_1 + \cdots$; equivalently, there are cochains $\varphi_i \in C^1(A, A)$ for which

$$\sum_n \left(\sum_{i=0}^{n} [\varphi_{n-i}, \alpha_i]_G \right) t^n = [\varphi_t, \alpha_t]_G = 0.$$

(Here $\alpha_0 = \alpha$ and $\varphi_0 = \varphi$.) Otherwise, φ is said to *break*. Since cohomology is upper semicontinuous, no new derivations may appear at $t \neq 0$. For this reason, once a symmetry is broken, there is no way to recover it *via* deformation. We note trivially that if φ is central in $\mathrm{Der}(A, A)$ then it necessarily lifts, so the exponential deformation formula breaks no symmetries; it may however trivialize them in the sense that they become coboundaries.

In analogy with the case of algebras, given an $x \in \mathfrak{g}$ and a udf $\gamma_t = 1 \otimes 1 + \sum t^i \gamma_i$ we say that x *lifts* if there are elements $x_i \in U\mathfrak{g}$ for which $x_t = x + \sum t^i x_i$ satisfies $[x_t, \gamma_t]_G = 0$. Equivalently,

$$(**_n) \qquad \sum_{i=0}^{n} [x_{n-i}, \gamma_i]_G = 0$$

for all n. (Here $\gamma_0 = 1 \otimes 1$ and $x_0 = x$.) Otherwise, γ_t *breaks* x. Any map $f\colon \mathfrak{g} \to \mathrm{Der}[A, A)$ carries liftable elements of $\mathfrak{g}$ to liftable derivations (with respect to the induced deformation $f(\gamma_t)$). However, the image of an element which breaks may nonetheless be a liftable derivation. For example, when $A = TV$, the tensor algebra on a k-module V, *every* derivation is liftable under *every* deformation simply because TV is rigid, i.e., all its deformations are trivial. The general problem is that the containment of the image of $T^1 U\mathfrak{g}$ in $C^1(A)$ may be proper. Consequently, a failure to solve $(**_n)$ in $T^1 U\mathfrak{g}$ does not preclude the possibility of finding solutions in $C^1(A, A)$. Nonetheless, in some generic sense it must be true that if γ_t breaks x then there is some map $f\colon \mathfrak{g} \to \mathrm{Der}(A, A)$ such that $f(\gamma_t)$ breaks $f(x)$. We also make the

Conjecture. Let G be a non-nilpotent Lie group acting as automorphisms on a finite-dimensional algebra A through an inclusion $\mathfrak{g} \hookrightarrow \mathrm{Der}(A, A)$ having the property that $\mathfrak{g} \to H^1(A, A)$ is also a monomorphism. Then every non-nilpotent $\varphi \in \mathfrak{g}$ breaks under some deformation of A. More generally, there is a sequence of deformations of A which breaks all symmetries outside of the subalgebra generated by the nilpotent elements of $\mathfrak{g}$ and one other element.

We return to this conjecture after examining the easier question of liftability of elements of a Lie algebra.

Theorem 8. *Let h and x be a basis for $\mathfrak{b}_2$ with $[h, x] = x$. Then $\gamma_t = e(t; h, x)$ breaks h but x is liftable to $x_t = x + x\ln(1 + tx)$.*

Proof. If h were to lift, to say $h_t = h + th_1 + \cdots$, we would have from $(**_1)$, $[h, x \otimes h]_G + [h_1, 1 \otimes 1]_G = 0$. That is, $[x \otimes h, h]_G = [h_1, 1 \otimes 1]_G = \delta h_1$ and, so, $-h \bar{\circ} (x \otimes h) + (x \otimes h) \bar{\circ} h = \delta h_1$. But the left-hand side of the latter equation is

$$-hx \otimes h - x \otimes h^2 + xh \otimes h + x \otimes h^2 = [x, h] \otimes h = x \otimes h,$$

which is not a coboundary in $T^2 U\mathfrak{b}_2$. So h breaks. The reader may easily check the lifting of x. $\square$

Remarkably, no perturbation of the formula $e(t; x, h)$ will break x. More specifically, we have the following:

Theorem 9. *Let γ_t be a formula strictly based on $\mathfrak{sl}_2$ with infinitesimal $x \wedge h$. Then x lifts while γ_t breaks both h and y.*

Proof. The breaking of h and y is established as in the previous theorem. That x lifts is more subtle and follows from Lemma 10.

We first note that the complex $T^\bullet U\mathfrak{sl}_2$ is the direct sum of two subcomplexes:

$$T^\bullet U\mathfrak{sl}_2 = T^\bullet_x U\mathfrak{sl}_2 \oplus T^\bullet_{\sim x} U\mathfrak{sl}_2,$$

where $T^\bullet_x U\mathfrak{sl}_2$ is the ideal generated by $\{h^a x^b y^c \mid b \neq 0\}$, i.e., the linear combinations of terms $h^{a_1} x^{b_1} y^{c_1} \otimes h^{a_2} x^{b_2} y^{c_2} \otimes \cdots$ in which at least one $b_i \neq 0$. (Here we have ordered the standard PBW basis as h, x, y.) Elements of $T^\bullet_{\sim x} U\mathfrak{sl}_2$ are linear combinations of the basis elements $h^{a_1} y^{c_1} \otimes h^{a_2} y^{c_2} \otimes \cdots$. We call the elements of $T^\bullet_x U\mathfrak{sl}_2$ *x-terms*. It is trivial that these are indeed subcomplexes. Our proof of the theorem requires

Lemma 10. *If $\alpha, \beta \in U\mathfrak{sl}_2 \otimes U\mathfrak{sl}_2$ and α is an x-term then the composition products $\alpha \bar{\circ} \beta$ and $\beta \bar{\circ} \alpha$ are also x-terms. Also, if $\gamma_t = 1 \otimes 1 + \sum t^i \gamma_i$ is a formula strictly based on $\mathfrak{sl}_2$ with infinitesimal $h \wedge x$ then γ_t is equivalent to a udf $\hat{\gamma}_t$ in which each $\hat{\gamma}_i$ is an x-term.*

Proof of Theorem 9 (continued). Assuming the lemma for a moment, suppose that $x_t = x + tx_1 + t^2 x_2 + \cdots + t^{n-1} x_{n-1}$ is a lifting of x modulo t^n. In as much as liftability is invariant under equivalence, we may assume that every γ_n is an x-term. The obstruction to extending x_t to a lifting modulo t^{n+1} is $[x_{n-1}, \gamma_1]_G + \cdots + [x_1, \gamma_{n-1}]_G + [x, \gamma_n]_G$, for this is precisely $-\delta x_n$ if such a lifting exists. The lemma insures that this is an x-term and, so, its cohomology class in $\bigwedge^2 \mathfrak{sl}_2$ is

$a(h \wedge x) + b(x \wedge y)$ for some $a, b \in k$. Now writing $\bar{\sigma}$ (rather than $[\sigma]$) for the cohomology class of σ, we have, by assumption, $\bar{\gamma}_1 = \overline{h \otimes x} = h \wedge x$ and so $[y, \bar{\gamma}_1]_S = -x \wedge y$ and $[h, \bar{\gamma}_1]_S = h \wedge x$. Therefore, if we replace x_{n-1} by $x_{n-1} + by - ah$, then the cohomology class of the obstruction to extending the new $(n-1)$st order lifting vanishes, so the obstruction is a coboundary and x_n can be found. Since it is trivial that x lifts to first order, this inductive procedure shows that x lifts. □

Proof of Lemma 10. For the first assertion, it suffices to consider the case in which $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta_2$ with each α_i and β_i being in the standard basis given by PBW. We show only that $\alpha \bar{\circ} \beta$ is an x-term, the argument for $\beta \bar{\circ} \alpha$ being similar. Since $\alpha \bar{\circ} \beta = (\alpha_1 \bar{\circ} \beta) \otimes \alpha_2 + \alpha_1 \otimes (\alpha_2 \bar{\circ} \beta)$ we may assume without loss of generality that $\alpha_1 = h^p x^q y^r$ is an x-term $(q \neq 0)$ and must show that $\alpha_1 \bar{\circ} \beta$ is an x-term. Lemma 2 implies that $\alpha_1 \bar{\circ} \beta$ is the sum of terms $h^{p_1} x^{q_1} y^{r_1} \beta_1 \otimes h^{p_2} x^{q_2} y^{r_2} \beta_2$ where $p_1 + p_2 = p$, $q_1 + q_2 = q$ and $r_1 + r_2 = r$. In particular, either $q_1 \neq 0$ or $q_2 \neq 0$. It thus suffices to show that $h^p x^q y^r h^a x^b y^c$ is an x-term if $q \neq 0$. Since $y^r h^a x^b y^c$ is a linear combination of the PBW basis elements $h^l x^m y^n$, it suffices to show that $h^p x^q h^l x^m y^n$ is an x-term whenever $q > 0$. For this, note that the relation $[h, x] = x$ easily implies that $x^q h^l = (h - q)^l x^q$. So $h^p x^q h^l x^m y^n = h^p (h - q)^l x^{q+m} y^n$, which is clearly an x-term, as required.

For the second assertion note that $x \otimes h$ is an x-term and assume inductively that each γ_i is an x-term for $1 \leqslant i < n$. Clearly, γ_n may be written as $\gamma_n = \gamma'_n + \gamma''_n$ where $\gamma'_n \in T^2_x U\mathfrak{sl}_2$ while $\gamma''_n \in T^2_{\sim x} U\mathfrak{sl}_2$. Since $\delta\gamma_n = \sum_{1=1}^{i=n-1} \gamma_i \bar{\circ} \gamma_{n-i}$, the first assertion implies that it is an x-term. As $T^\bullet_x U\mathfrak{sl}_2$ and $T^\bullet_{\sim x} U\mathfrak{sl}_2$ are subcomplexes, we have that $\delta\gamma'_n = \delta\gamma_n$ and $\delta\gamma''_n = 0$, so $\gamma''_n = a(h \wedge y) + \delta\theta_n$ where $\theta_n \in T^1_{\sim x} U\mathfrak{sl}_2$. Conjugating γ_t by $1 - \theta_n t^n$ produces an equivalent udf $\tilde{\gamma}_t$ in which $\tilde{\gamma}_i = \gamma_i$ for $i < n$ and $\tilde{\gamma}_n = v_n - \delta\theta_n = \gamma'_n + a(h \wedge y)$. We have

$$\delta\tilde{\gamma}_{n+1} = \sum_{n > i \geqslant 2} \gamma_i \bar{\circ} \gamma_{n+1-i} + [\gamma_1, \gamma'_n]_G + [\gamma_1, a(h \wedge y)]_G.$$

Since this is a 3-cocycle, the sum of its degree-three terms is also a cocycle and has the same cohomology class, namely 0. Now, the proof of the first assertion shows that, as γ'_n and γ_i for $i < n$ are all x-terms, every summand of $[\gamma_1, \gamma'_n]_G$ and each $\gamma_i \bar{\circ} \gamma_{n+1-i}$ has an x in *two* tensor factors. Hence each such summand with degree three reduces to $x \wedge x \wedge z \in \bigwedge^3 \mathfrak{sl}_2 \cong H^3(T^\bullet U\mathfrak{sl}_2)$ for some $z \in \mathfrak{sl}_2$, which means that it contributes 0 to the class of $\delta\tilde{\gamma}_{n+1}$. It follows that, with $\bar{\sigma}$ once again denoting the cohomology class of σ,

$$\begin{aligned} 0 = \overline{\delta\tilde{\gamma}_n} &= \overline{[\gamma_1, a(h \wedge y)]_G} = [\bar{\gamma}_1, a(h \wedge y)]_S \\ &= a[x \wedge h, h \wedge y]_S = -2a(h \wedge x \wedge y), \end{aligned}$$

and so $a = 0$ and $\tilde{\gamma}_n = \gamma'_n$, which is an x-term. This is just what we need to proceed by induction, replacing γ_t with $\tilde{\gamma}_t$. The equivalent udf $\hat{\gamma}_t$ is then the result of conjugating γ_t by $\prod(1 - \theta_n t^n)$. □

We remark that there exist udfs based strictly on $\mathfrak{sl}_2$ which break x, namely those with infinitesimal $y \wedge h$.

Our conjecture concerning the breaking of symmetry would follow from Theorem 8 if we knew that whenever A is finite-dimensional and $\mathrm{Der}(A)$ contains a copy of $\mathfrak{b}_2$, some deformation induced by a universal formula breaks the approximate symmetry. This requires the following easy lemma:

Lemma 11. *Every non-nilpotent Lie algebra* $\mathfrak{g}$ *contains a copy of* $\mathfrak{b}_2$.

Proof. If $\mathfrak{g}$ is not nilpotent then Engel's theorem provides an $x \in \mathfrak{g}$ for which $\mathrm{ad}(x)$ is not nilpotent. Consequently, $\mathrm{ad}(x)$ must have a non-zero eigenvalue, λ. If $y \in \mathfrak{g}$ is corresponding non-zero eigenvector then x/λ and y together generate a copy of $\mathfrak{b}_2$. □

We conclude the paper with an example of a udf based on a non-abelian Lie algebra which, like the exponential deformation formula breaks no symmetries. Let $\mathfrak{h}$ denote the *Heisenberg algebra*, which has basis x, y, z and commutator relations $[x, y] = z$ and $[x, z] = [y, z] = 0$. In its natural representation $\mathfrak{h}$ may be described as the set of all strictly upper triangular 3×3 matrices with the commutator product. The primary obstruction to a 2-cocycle $ax \wedge z + by \wedge z + cx \wedge y$ is $-2c^2 x \wedge y \wedge z$, so $V_1(\mathfrak{h}) = \{(ax + by) \wedge z\}$. Since the factors of $(ax + by) \wedge z$ generate a copy of $\mathfrak{a}_2$, every element of V_1 is integrable (e.g., by the exponential formula) and $V_1 = V_\infty$. To produce a formula based strictly on $\mathfrak{h}$ (rather than the subalgebra generated by $ax + by$ and z), it will clearly suffice to modify the exponential formula for the case $x \wedge y$ of the foregoing by introducing a summand of $y \otimes z$ in, say, the quadratic term. We may achieve this as follows: Any Lie algebra automorphism $\mathfrak{g} \to \mathfrak{g}$ induces a map $T^\bullet U\mathfrak{g} \to T^\bullet U\mathfrak{g}$. In characteristic 0 such an automorphism can be obtained as the exponential $e^{t\sigma}$ of a derivation σ of $\mathfrak{g}$. Now if γ_t is a udf based on $\mathfrak{g}$ then so too will be $\gamma_t' = e^{t\sigma}(\gamma_t)$.

For the case of $\mathfrak{g} = \mathfrak{h}$, we let $\sigma = y\partial_x$. Then upon applying $e^{t(y\partial_x)}$ to the exponential deformation formula $e^{t(x \otimes z)} = \sum \frac{1}{k!} x^k \otimes z^k = \sum \alpha_k$ we obtain

$$\gamma_t' = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \frac{1}{m!} (y\partial_x)^m \alpha_{n-m}$$

and so γ_t' is strictly based on $\mathfrak{h}$. If A is given then the deformation induced by the latter udf is equivalent to the exponential deformation of A and therefore breaks no symmetries.

In summary. *If* $\mathfrak{g}$ *is non-abelian and contains* $\mathfrak{b}_2$ *then there is a udf based on* $\mathfrak{g}$ *which generally breaks some symmetry. Otherwise* $\mathfrak{g}$ *must be nilpotent and, as in the example of the Heisenberg algebra, may break no symmetry.*

We do not know if there are any nilpotent Lie algebras which generically break. It is curious that the very existence of a non-nilpotent symmetry group generally provides a path to the breaking of symmetry!

Finally, we note that although our focus has been on associative algebras much of what we do is applicable to other categories. If γ_t is a udf based on $\mathfrak{g}$ and A is any structure whose "deformation cochain complex" is a comp algebra, $C^\bullet$, and $\mathfrak{g} \to Z^1(C^\cdot)$ is a Lie algebra morphism then the induced map $T^\bullet U\mathfrak{g} \to C^\bullet$ carries γ_t to a deformation of A. This applies, in particular, to coalgebras but not to Lie algebras (cf. [10, 12]).

Note added in proof. Giaquinto and Zhang have observed the following udf $\gamma_t = \sum t^n \gamma_n$ based on $\mathfrak{sl}_n$ with infinitesimal $\gamma_1 = h \otimes e_{1k} + \beta$, where $h = \frac{1}{2}(e_{11} - e_{kk})$ and $\beta = \sum_{j=2}^{k-1} e_{1j} \otimes e_{jk}$: set $h_{n,i} = (h+n-1)(h+n-2)\cdots(h+i)$, then $\gamma_n = \frac{1}{n!}\sum_{i=0}^{n} \beta^i (h_{n,i} \otimes e_{1k}^{n-i})$.

References

[1] J. Coffee, Filtered and associated graded rings, Bull. Amer. Math. Soc. 78 (1972) 584–587.
[2] V. Coll, M. Gerstenhaber and A. Giaquinto, An explicit deformation formula with non-commuting xderivations, Ring Theory, Vol. 1, Israel Mathematical Conference Proceedings (Weizmann Science Press, New York, 1989) 396–403.
[3] M. DeWilde and P. Lecompte, Formal deformations of the Poisson Lie algebra of a symplectic manifold and star-products. Existence, equivalence, derivations, in: M. Hazewinkel and M. Gerstenhaber, eds., Deformation Theory of Algebras and Structures and Applications, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, Vol. 247 (Kluwer, Dordrecht, 1988) 897–958.
[4] V.G. Drinfel'd, Quasi-Hopf algebras, Leningrad J. Math. 1 (1990) 1419–1457.
[5] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.
[6] M. Gerstenhaber, On the deformation of rings and algebras, Ann. of Math. 79 (1964) 59–103.
[7] M. Gerstenhaber, On the deformation of rings and algebras III, Ann. of Math. 88 (1968) 1–34.
[8] M. Gerstenhaber, A. Giaquinto and S.D. Schack, Quantum Symmetry, in: P.P. Kulish, ed., Quantum Groups, Lecture Notes in Mathematics, Vol. 1510 (Springer, Berlin, 1992) 9–46.
[9] M. Gerstenhaber and S.D. Schack, Relative Hochschild cohomology, rigid algebras, and the bockstein, J. Pure Appl. Algebra 43 (1986) 53–74.
[10] M. Gerstenhaber and S.D. Shack, Algebraic cohomology and deformation theory in: M. Hazewinkel and M. Gerstenhaber, eds., Deformation Theory of Algebras and Structures and Applications, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, Vol. 247 (Kluwer, Dordrecht, 1988) 11–264.
[11] M. Gerstenhaber and S.D. Schack, Bialgebra cohomology, deformations and quantum groups, Proc. Nat. Acad. Sci. 87 (1990) 478–481.
[12] M. Gerstenhaber and S.D. Schack, Algebras, bialgebras, quantum groups, and algebraic deformations, in: M. Gerstenhaber and J. Stasheff, eds., Deformation Theory and Quantum Groups with Applications to Mathematical Physics, Contemporary Mathematics, Vol. 134 (American Mathematical Society, Providence, RI, 1992) 51–92.
[13] J. Grabowski, Abstract Jacobi and Poisson structures. Quantization and star-products, J. Geom. Phys. 9 (1992) 45–73.
[14] G. Hochschild, B. Kostant and A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962) 383–408.
[15] A. Lichnerowitz, Quantum mechanics and deformations of geometrical dynamics, in: A.O. Barut, ed., Quantum Theory, Groups, Fields and Particles (Reidel, Dordrecht, 1982) 3–82.
[16] J.E. Moyal, Quantum mechanics as a statistical theory, Proc. Cambridge Philos. Soc. 45 (1949) 99–124.
[17] J. Vey, Deformation du Crochét de Poisson sur une varieté symplectique, Comment. Math. Helv. 50 (1975) 421–454.