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Tame kernels and further 4-rank densities

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Abstract

There has been recent progress on computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$ for F a quadratic number field. For certain quadratic number fields, this progress has led to "density results" concerning the 4-rank of tame kernels. These results were first mentioned in Conner and Hurrelbrink (J. Number Theory 88 (2001) 263) and proven in Osburn (Acta Arith. 102 (2002) 45). In this paper, we consider some additional quadratic number fields and obtain further density results of 4-ranks of tame kernels. Additionally, we give tables which might indicate densities in some generality.

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1. Introduction

We are interested in the structure of the 2-Sylow subgroup of $K_2(\mathcal{O}_F)$ for F a quadratic number field. As $K_2(\mathcal{O}_F)$ is a finite abelian group, it is a product of cyclic groups of prime power order. We say the 2^j-rank, $j \geqslant 1$, of $K_2(\mathcal{O}_F)$ is the number of cyclic factors of $K_2(\mathcal{O}_F)$ of order divisible by 2^j. For any number field, the 2-rank of the tame kernel is given by Tate's 2-rank formula (see [12]). In the case where F is a quadratic number field, Browkin and Schinzel [3] simplified the 2-rank formula. In their formula, we can determine the 2-rank by counting the number of elements in $\{\pm 1, \pm 2\}$ which are norms from the given quadratic field and the number of odd

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primes which are ramified in the given quadratic field. Now what about the 4-rank of $K_2(\mathcal{O}_F)$?

In [6], Conner and Hurrelbrink characterize the 4-rank of $K_2(\mathcal{O})$ for certain quadratic number fields in terms of positive definite binary quadratic forms. This characterization led to a connection between densities of certain sets of primes and 4-rank values. Specifically, the author in [8] considers the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$. In [6], it was shown that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank}\, K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$

$$\text{4-rank}\, K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

The idea in [8] is to fix $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \left\{ l \text{ rational prime } : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1 \right\}.$$

In [8], the following was proved.

Theorem 1.1. *For the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, 4-rank 1 and 2 each appear with natural density $\frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $\frac{1}{2}$ in Ω.*

In this paper, we consider the 4-rank of $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$ and $\mathbb{Q}(\sqrt{pl})$ for primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$. We will see that for the primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$,

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \text{ or } 2,$$

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \text{ or } 2.$$

For the primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$, we will see

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \text{ or } 1.$$

Let us fix a prime $p \equiv 1 \bmod 8$ and consider the sets

$$A = \left\{ l \text{ rational prime } : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1 \right\},$$

$$B = \left\{ l \text{ rational prime } : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = -1 \right\}.$$

The goal of this paper is to prove two theorems analogous to Theorem 1.1, namely:

Theorem 1.2. *For the field* $\mathbb{Q}(\sqrt{pl})$, 4-*rank* 1 *and* 2 *appear with natural density* $\frac{3}{4}$ *and* $\frac{1}{4}$ *in A. For the field* $\mathbb{Q}(\sqrt{-pl})$, 4-*rank* 1 *and* 2 *each appear with natural density* $\frac{1}{2}$ *in A.*

Theorem 1.3. *For the field* $\mathbb{Q}(\sqrt{pl})$, 4-*rank* 0 *and* 1 *each appear with natural density* $\frac{1}{2}$ *in B.*

Now for squarefree, odd integers d, consider the sets

$$X = \{d : d = pl\}$$

and

$$Y = \{d : d = -pl\}$$

for distinct primes p and l.

We have computed the following: For $15 \leqslant d < 10^6$, there are 168 331 d's in X. Among them, there are 35 787 d's (21.26%) yielding 4-rank 0, 128 468 d's (76.32%) yielding 4-rank 1, and 4076 d's (2.42%) yielding 4-rank 2.

For $-10^6 < d \leqslant -15$, there are 168 330 d's in Y. Among them, there are 104 056 d's (61.82%) yielding 4-rank 0, 63 054 d's (37.46%) yielding 4-rank 1, and 1220 d's (0.72%) yielding 4-rank 2. As a consequence of Theorems 1.2, 1.3 and Tables I and II in [9,10], we obtain:

Corollary 1.4. *For the fields* $\mathbb{Q}(\sqrt{pl})$, 4-*rank* 0, 1, *and* 2 *appear with natural density* $\frac{13}{64}$, $\frac{97}{128}$, $\frac{5}{128}$, *respectively in X.*

Corollary 1.5. *For the fields* $\mathbb{Q}(\sqrt{-pl})$, 4-*rank* 0, 1, *and* 2 *appear with natural density* $\frac{37}{64}$, $\frac{13}{32}$, *and* $\frac{1}{64}$, *respectively in Y.*

2. Preliminaries

Let $\mathcal{D}$ be a Galois extension of $\mathbb{Q}$, and $G = Gal(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ denote the center of G and $\mathcal{D}^{Z(G)}$ denote the fixed field of $Z(G)$. Let p be a rational prime which is unramified in $\mathcal{D}$ and β be a prime of $\mathcal{D}$ containing p. Let $(\frac{\mathcal{D}/\mathbb{Q}}{p})$ denote the Artin symbol of p and $\{g\}$ the conjugacy class containing one element $g \in G$. In Sections 5 and 6 we use the following elementary lemma from [8].

Lemma 2.1. $(\frac{\mathcal{D}/\mathbb{Q}}{p}) = \{g\}$ *for some* $g \in Z(G)$ *if and only if* p *splits completely in* $\mathcal{D}^{Z(G)}$.

Thus, if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G, then we know the associated Artin symbol is a conjugacy class containing one element. Note that determining the order of $Z(G)$ gives us the number of possible choices for the Artin symbol. The order of $Z(G)$ can be computed using the following setup.

Let G_1 and G_2 be finite groups and A a finite abelian group. Suppose $r_1 : G_1 \to A$ and $r_2 : G_2 \to A$ are two epimorphisms and $\mathscr{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2\colon r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \to A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathscr{G} = ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathscr{G}) = \mathscr{G} \cap Z(G_1 \times G_2)$. From [8], we provide:

Lemma 2.2. *$Z(\mathscr{G}) = Z(G_1) \times Z(G_2) \Leftrightarrow r_1|_{Z(G_1)}$ and $r_2|_{Z(G_2)}$ are both trivial.*

We will use the following definition throughout this paper.

Definition 2.3. For primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$, $\mathscr{K} = \mathbb{Q}(\sqrt{2p})$, and $h^+(\mathscr{K})$ the narrow class number of $\mathscr{K}$, we say:

l satisfies $\langle 1, 32 \rangle$ if and only if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$;

l satisfies $\langle p, -2 \rangle$ if and only if $l^{\frac{h^+(\mathscr{K})}{4}} = pn^2 - 2m^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$;

l satisfies $\langle 1, -2p \rangle$ if and only if $l^{\frac{h^+(\mathscr{K})}{4}} = n^2 - 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

3. First extension

Consider the fixed prime $p \equiv 1 \bmod 8$. Note p splits completely in $\mathscr{L} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so

$$p\mathcal{O}_{\mathscr{L}} = \mathfrak{B}\mathfrak{B}'$$

for some primes $\mathfrak{B} \neq \mathfrak{B}'$ in $\mathscr{L}$. The field $\mathscr{L}$ has narrow class number $h^+(\mathscr{L}) = 1$ as $h(\mathscr{L}) = 1$ and $N_{\mathscr{L}/\mathbb{Q}}(\varepsilon) = -1$ where $\varepsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$. Similar to Lemma 2.1 in [6],

Lemma 3.1. *The prime $\mathfrak{B}$ which occurs in the decomposition of $p\mathcal{O}_{\mathscr{L}}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathscr{L}}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}^*_{\mathscr{L}}$ for which $N_{\mathscr{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = p$.*

The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p})$ as $N_{\mathscr{L}/\mathbb{Q}}(\pi) = p$. Set

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}).$$

Then N is Galois over $\mathbb{Q}$ and $[N:\mathbb{Q}]=8$. By Corollary 24.5 in [4], 4 divides the narrow class number of $\mathbb{Q}(\sqrt{2p})$. Moreover, N over $\mathbb{Q}(\sqrt{2p})$ is unramified at all finite primes. Similar to Lemma 2.3 in [6], N is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$.

Consider the rational primes $l \equiv 1 \bmod 8$ for which $(\frac{l}{p})=1$. These primes split completely in $\mathbb{Q}(\sqrt{2},\sqrt{p})$ over $\mathbb{Q}$. We characterize such primes l that split completely in N over $\mathbb{Q}$. As N is the unique unramified cyclic degree 4 extension of $\mathbb{Q}(\sqrt{2p})$, mimicing Lemma 3.3 in [6] yields

Lemma 3.2. *Let* $l \equiv 1 \bmod 8$ *be a prime such that* $(\frac{l}{p})=1$. *Then*:
l splits completely in N if and only if l satisfies $\langle 1,-2p\rangle$.

Similar to Lemma 3.4 in [6], with 2 (respectively, $\mathfrak{D}$, the unique dyadic prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{2p})}$) replaced by p (respectively $\mathfrak{p}$, the prime over p whose class is the unique element of order 2 in the narrow ideal class group of $\mathbb{Q}(\sqrt{2p})$), we obtain

Lemma 3.3. *Let* $l \equiv 1 \bmod 8$ *be a prime such that* $(\frac{l}{p})=1$. *Then*:
l does not split completely in N if and only if l satisfies $\langle p,-2\rangle$.

We now relate the characterizations of Lemmas 3.2 and 3.3 to the quadratic symbol $(\frac{\pi}{l})$. From Lemma 3.1, we have a presentation $\pi = a+b\sqrt{2}\in\mathcal{O}_{\mathscr{L}}$ with $N_{\mathscr{L}/\mathbb{Q}}(\pi)=p$. Let $\mathfrak{P}$ be a prime above l in $\mathcal{O}_{\mathscr{L}}$. As l splits in $\mathscr{L}$ over $\mathbb{Q}$, then the residue field $\mathcal{O}_{\mathscr{L}}/\mathfrak{P}$ is isomorphic to $\mathbb{Z}/l\mathbb{Z}=\mathbb{F}_l$, the field with l elements. As 2 is a square modulo l, we have $2\equiv \alpha^2 \bmod l$ for some $\alpha\in\mathbb{F}_l^*$. Thus, we can identify $\pi = a+b\sqrt{2}\in\mathcal{O}_{\mathscr{L}}$ with $a+b\alpha\in\mathbb{F}_l$. When we write the symbol $(\frac{\pi}{l})$, it is understood that we mean $(\frac{a+b\alpha}{l})$. From the discussion in Section 3 of [6], the symbol $(\frac{\pi}{l})$ is well defined and l splits completely in N over $\mathbb{Q}$ if and only if $(\frac{\pi}{l})=1$. Combining this discussion with Lemmas 3.2 and 3.3, we have:

Proposition 3.4. *Let* $l \equiv 1 \bmod 8$ *be a prime with* $(\frac{l}{p})=1$. *Then*:
l satisfies $\langle 1,-2p\rangle \Leftrightarrow (\frac{\pi}{l})=1$,
l satisfies $\langle p,-2\rangle \Leftrightarrow (\frac{\pi}{l})=-1$.

4. Matrices and symbols

Hurrelbrink and Kolster [7] generalize Qin's approach in [9,10] and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Let us be more specific. Let $F=\mathbb{Q}(\sqrt{d})$, $d\neq 0,1$, squarefree. Let $p_1,p_2,\dots,p_t$ denote the odd primes dividing d. Recall 2 is a norm from $F \Leftrightarrow$ all p_i's are $\equiv \pm 1 \bmod 8$. If so, then d

is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider two matrices:
If $d < 0$,

$$M'_{F/\mathbb{Q}} = \begin{pmatrix} (-d,p_1)_2 & (-d,p_1)_{p_1} & \cdots & (-d,p_1)_{p_t} \\ (-d,p_2)_2 & (-d,p_2)_{p_1} & \cdots & (-d,p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d,p_{t-1})_2 & (-d,p_{t-1})_{p_1} & \cdots & (-d,p_{t-1})_{p_t} \\ (-d,v)_2 & (-d,v)_{p_1} & \cdots & (-d,v)_{p_t} \\ (-d,-1)_2 & (-d,-1)_{p_1} & \cdots & (-d,-1)_{p_t} \end{pmatrix}.$$

If $d > 0$,

$$M_{F/\mathbb{Q}} = \begin{pmatrix} (-d,p_1)_2 & (-d,p_1)_{p_1} & \cdots & (-d,p_1)_{p_t} \\ (-d,p_2)_2 & (-d,p_2)_{p_1} & \cdots & (-d,p_2)_{p_t} \\ \vdots & \vdots & & \vdots \\ (-d,p_{t-1})_2 & (-d,p_{t-1})_{p_1} & \cdots & (-d,p_{t-1})_{p_t} \\ (-d,v)_2 & (-d,v)_{p_1} & \cdots & (-d,v)_{p_t} \\ (d,-1)_2 & (d,-1)_{p_1} & \cdots & (d,-1)_{p_t} \end{pmatrix}.$$

If 2 is not a norm from F, set $v = 2$. Otherwise, set $v = u + w$. Replacing the 1's by 0's and the -1's by 1's, we calculate the matrix rank over $\mathbb{F}_2$. Why look at these matrices? From [7],

Lemma 4.1. *Let $F = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree. Then*

(i) *If $d < 0$, then 4-rank $K_2(\mathcal{O}_F) = t - rk(M'_{F/\mathbb{Q}})$,*
(ii) *If $d > 0$, then 4-rank $K_2(\mathcal{O}_F) = t - rk(M_{F/\mathbb{Q}}) + a' - a$,*

where

$$a = \begin{cases} 0 & \text{if } 2 \text{ is a norm from } F, \\ 1 & \text{otherwise} \end{cases}$$

and

$$a' = \begin{cases} 0 & \text{if both } -1 \text{ and } 2 \text{ are norms from } F, \\ 1 & \text{if exactly one of } -1 \text{ or } 2 \text{ is a norm from } F, \\ 2 & \text{if none of } -1 \text{ or } 2 \text{ are norms from } F. \end{cases}$$

Recall that our cases are:

- $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ where $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$,
- $\mathbb{Q}(\sqrt{pl})$ for $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$.

In both cases 2 is a norm from $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{-pl})$. Before we view the matrices for our cases, we characterize the symbol $(-d, v)_2$ for $d = pl, -pl$ (see [7, Lemmas 5.3 and 5.15]).

- $(-pl, v)_2 = 1 \Leftrightarrow$ both p, l satisfy $\langle 1, 32 \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$,
- $(pl, v)_2 = 1$.

Also, v is an l-adic unit and hence

$$(-pl, v)_l = (l, v)_l = \left(\frac{v}{l}\right).$$

Similarly, $(-pl, v)_p = (\frac{v}{p})$. In the entries of the matrices below, we write $(-pl, v)_2$, $(\frac{v}{l})$, and $(\frac{v}{p})$ remembering to first evaluate the symbols, make the substitutions 1 for 0 and -1 for 1, and then calculate the matrix rank over $\mathbb{F}_2$. Now what are the matrices in our situations?

- For $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ (-pl, v)_2 & (\frac{v}{p}) & (\frac{v}{l}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\frac{v}{p}) & (\frac{v}{l}) \\ 0 & 0 & 0 \end{pmatrix}.$$

- For $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$, we have:

$$M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = \begin{pmatrix} 0 & 1 & 1 \\ (-pl, v)_2 & (\frac{v}{p}) & (\frac{v}{l}) \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 4.2. For $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$, we have:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \Leftrightarrow \operatorname{rank} M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \Leftrightarrow (-pl, v)_2 = 1$, $(\frac{v}{l}) = -1$ or $(-pl, v)_2 = -1 \Leftrightarrow$ both p, l satisfy $\langle 1, 32 \rangle$, $(\frac{v}{l}) = -1$ or neither p, l satisfy $\langle 1, 32 \rangle$, $(\frac{v}{l}) = -1$, or exactly one of p, l satisfies $\langle 1, 32 \rangle$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \Leftrightarrow \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 0 \Leftrightarrow (-pl, v)_2 = 1$, $(\frac{v}{l}) = 1 \Leftrightarrow$ both p, l satisfy $\langle 1, 32 \rangle$, $(\frac{v}{l}) = 1$ or neither p, l satisfy $\langle 1, 32 \rangle$, $(\frac{v}{l}) = 1$.
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \Leftrightarrow$ rank $M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 1 \Leftrightarrow (\frac{v}{l}) = -1$.
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \Leftrightarrow$ rank $M'_{\mathbb{Q}(\sqrt{-pl})/\mathbb{Q}} = 0 \Leftrightarrow (\frac{v}{l}) = 1$.

Remark 4.3. For $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \Leftrightarrow \text{rank } M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 1 \Leftrightarrow (-pl, v)_2 = 1 \Leftrightarrow$ both p, l satisfy $\langle 1, 32 \rangle$ or neither p, l satisfy $\langle 1, 32 \rangle$.
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \Leftrightarrow$ rank $M_{\mathbb{Q}(\sqrt{pl})/\mathbb{Q}} = 2 \Leftrightarrow (-pl, v)_2 = -1 \Leftrightarrow$ exactly one of p, l satisfies $\langle 1, 32 \rangle$.

We can now prove Theorem 1.3.

Proof. Consider the sets

$$\mathcal{A}_1 = \{l \text{ prime } : l \equiv 1 \bmod 8 \text{ and } l \text{ satisfies } \langle 1, 32 \rangle\},$$

$$\mathcal{A}_2 = \{l \text{ prime } : l \equiv 1 \bmod 8 \text{ and } l \text{ does not satisfy} \langle 1, 32 \rangle\}.$$

By the discussion before Corollary 24.2 in [4], $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{2}$ in the set of all primes $l \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, $\mathcal{A}_1$ and $\mathcal{A}_2$ each have density $\frac{1}{8}$ in the set of all primes l. Note that for primes $p \equiv 1 \bmod 8$, the sets

$$\mathcal{B}_1 = \left\{l \text{ prime } : l \equiv 1 \bmod 8, \ \left(\frac{l}{p}\right) = -1, \right.$$
$$\left. \text{and } l \text{ satisfies } \langle 1, 32 \rangle \right\},$$

$$\mathcal{B}_2 = \left\{l \text{ prime } : l \equiv 1 \bmod 8, \ \left(\frac{l}{p}\right) = -1, \right.$$
$$\left. \text{and } l \text{ does not satisfy } \langle 1, 32 \rangle \right\}$$

each have density $\frac{1}{2}$ in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ have densities $\frac{1}{16}$ in the set of all primes l. If p satisfies $\langle 1, 32 \rangle$, then by Remark 4.3:

$$\mathcal{B}_1 = \left\{l \text{ prime } : l \equiv 1 \bmod 8, \ \left(\frac{l}{p}\right) = -1, \right.$$
$$\left. \text{and 4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \right\},$$

$$\mathcal{B}_2 = \left\{l \text{ prime } : l \equiv 1 \bmod 8, \ \left(\frac{l}{p}\right) = -1, \right.$$
$$\left. \text{and 4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 0 \right\}.$$

For each $\mathscr{B}_i$, $i = 1, 2$, we have:

$$\left\{\begin{array}{l}\text{Density of } \mathscr{B}_i \text{ in the} \\ \text{set of all primes } l\end{array}\right\} = \left\{\begin{array}{l}\text{Density of} \\ \mathscr{B}_i \text{ in } B\end{array}\right\}\left\{\begin{array}{l}\text{Density of } B \text{ in the} \\ \text{set of all primes } l\end{array}\right\},$$

where B has density $\frac{1}{8}$ in the set of all primes l. Thus 4-rank 0 and 4-rank 1 each appear with natural density $\frac{1}{2}$ in B. A similar argument works if p does not satisfy $\langle 1, 32 \rangle$. $\square$

For the primes $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$, let us relate the Legendre symbol $(\frac{v}{l})$ to the quadratic symbol $(\frac{\pi}{l})$. For primes $l \equiv 1 \bmod 8$, the quadratic symbol $(\frac{1+\sqrt{2}}{l})$ is well defined and satisfies, see [1],

$$\left(\frac{1+\sqrt{2}}{l}\right) = 1 \Leftrightarrow l \text{ satisfies } \langle 1, 32 \rangle.$$

Proposition 4.4. *Let* $d = \pm pl$ *be as above*, $d = u^2 - 2w^2$ *with* $u, w \in \mathbb{Z}$. *Then*:

$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right)\left(\frac{1+\sqrt{2}}{l}\right) \quad \textit{if } d = pl,$$
$$\left(\frac{v}{l}\right) = \left(\frac{\pi}{l}\right) \quad \textit{if } d = -pl.$$

Proof. From the proof of Proposition 4.6 in [6], we use the identity

$$\left(\frac{v}{l}\right) = \left(\frac{\gamma + \delta\sqrt{2}}{l}\right)\left(\frac{1+\sqrt{2}}{l}\right),$$

where $\frac{d}{l} = N_{\mathscr{L}/\mathbb{Q}}(\gamma + \delta\sqrt{2})$ for $\gamma, \delta \in \mathbb{Z}$. For $d = pl$, we have $\frac{d}{l} = p = N_{\mathscr{L}/\mathbb{Q}}(\pi)$ and thus $\gamma + \delta\sqrt{2} = \pi$, up to squares. For $d = -pl$, we have $\frac{d}{l} = -p = -N_{\mathscr{L}/\mathbb{Q}}(\pi)$ and so $\gamma + \delta\sqrt{2} = (1 + \sqrt{2})\pi$, up to squares. $\square$

In view of Proposition 3.4, Remark 4.2, and Proposition 4.4, we can determine the 4-rank of the tame kernel in terms of quadratic forms.

Proposition 4.5. *For* $p \equiv l \equiv 1 \bmod 8$ *with* $(\frac{l}{p}) = 1$:

- 4-*rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \Leftrightarrow$ *both* p, l *satisfy* $\langle 1, 32 \rangle$, l *satisfies* $\langle p, -2 \rangle$ *or neither* p, l *satisfy* $\langle 1, 32 \rangle$, l *satisfies* $\langle p, -2 \rangle$ *or exactly one of* p, l *satisfies* $\langle 1, 32 \rangle$.
- 4-*rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \Leftrightarrow$ *both* p, l *satisfy* $\langle 1, 32 \rangle$, l *satisfies* $\langle 1, -2p \rangle$ *or neither* p, l *satisfy* $\langle 1, 32 \rangle$, l *satisfies* $\langle 1, -2p \rangle$.

- 4-*rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 1 \Leftrightarrow l$ *satisfies* $\langle p, -2 \rangle$.
- 4-*rank* $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 2 \Leftrightarrow l$ *satisfies* $\langle 1, -2p \rangle$.

It should be noted that Qin Yue has obtained characterizations of 4-rank values, similar to Proposition 4.5, by additionally assuming that the fundamental unit of $\mathbb{Q}(\sqrt{2p}),\ p \equiv 1 \bmod 8$, has norm -1, see [11].

5. Two Artin symbols

5.1. First Artin symbol

Consider $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\varepsilon = 1 + \sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then ε is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1. The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1}).$$

Note that $Gal(N_1/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N_1/\mathbb{Q})) = Gal(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$ (see [8, Section 3.2]).

Only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\varepsilon})$, and so only the prime 2 ramifies in the compositum N_1 over $\mathbb{Q}$. Now as $l \in A$ is unramified in N_1 over $\mathbb{Q}$, the Artin symbol $(\frac{N_1/\mathbb{Q}}{\beta})$ is defined for primes β of $\mathcal{O}_{N_1}$ containing l. Let $(\frac{N_1/\mathbb{Q}}{l})$ denote the conjugacy class of $(\frac{N_1/\mathbb{Q}}{\beta})$ in $Gal(N_1/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and $N_1^{Z(Gal(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have that $(\frac{N_1/\mathbb{Q}}{l}) = \{g\}$ for some $g \in Z(Gal(N_1/\mathbb{Q}))$. As $Z(Gal(N_1/\mathbb{Q}))$ has order 2, there are two possible choices for $(\frac{N_1/\mathbb{Q}}{l})$. Combining this statement with Addendum (3.7) from [6], we have

Remark 5.1.

$$\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{id\} \Leftrightarrow l \text{ splits completely in } N_1$$
$$\Leftrightarrow l \text{ satisfies } \langle 1, 32 \rangle.$$

5.2. Second Artin symbol

In Section 3, we considered

$$N = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{p}),$$

the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{2p})$. Similar to the extension N_1, we have $Gal(N/\mathbb{Q})$ is the dihedral group of order 8 and $Z(Gal(N/\mathbb{Q})) = Gal(N/\mathbb{Q}(\sqrt{2}, \sqrt{p}))$.

Proposition 5.2. *If $l \in A$, then l is unramified in N over $\mathbb{Q}$.*

Proof. Since $p \equiv 1 \bmod 8$, the discriminant of $\mathbb{Q}(\sqrt{2p})$ is $8p$. For $l \in A$, we have $(\frac{2p}{l}) = 1$ and so l is unramified in $\mathbb{Q}(\sqrt{2p})$. We conclude that l is unramified in N over $\mathbb{Q}$. □

As $l \in A$ is unramified in N over $\mathbb{Q}$, the Artin symbol $(\frac{N/\mathbb{Q}}{\beta})$ is defined for primes β of $\mathcal{O}_N$ containing l. Let $(\frac{N/\mathbb{Q}}{l})$ denote the conjugacy class of $(\frac{N/\mathbb{Q}}{\beta})$ in $Gal(N/\mathbb{Q})$. The primes $l \in A$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and $N^{Z(Gal(N/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{p})$. By Lemma 2.1, we have that $(\frac{N/\mathbb{Q}}{l}) = \{h\}$ for some $h \in Z(Gal(N/\mathbb{Q}))$. As $Z(Gal(N/\mathbb{Q}))$ has order 2, there are two possible choices for $(\frac{N/\mathbb{Q}}{l})$. Combining this statement and Lemmas 3.2 and 3.3, we have

Remark 5.3.

$$\left(\frac{N/\mathbb{Q}}{l}\right) = \{id\} \Leftrightarrow l \text{ splits completely in } N$$
$$\Leftrightarrow l \text{ satisfies } \langle 1, -2p \rangle,$$

$$\left(\frac{N/\mathbb{Q}}{l}\right) \neq \{id\} \Leftrightarrow l \text{ does not split completely in } N$$
$$\Leftrightarrow l \text{ satisfies } \langle p, -2 \rangle.$$

6. A composite and proof of Theorem 1.2

In this section, we consider the composite field $N_1 N$. Set

$$\mathfrak{N} = N_1 N.$$

Note that $[\mathfrak{N} : \mathbb{Q}] = 32$. As N_1 and N are normal extensions of $\mathbb{Q}$, $\mathfrak{N}$ is a normal extension of $\mathbb{Q}$.

For $l \in A$, l is unramified in $\mathfrak{N}$ as it is unramified in N_1 and N. The Artin symbol $(\frac{\mathfrak{N}/\mathbb{Q}}{\beta})$ is now defined for some prime β of $\mathcal{O}_{\mathfrak{N}}$ containing l. Let $(\frac{\mathfrak{N}/\mathbb{Q}}{l})$ denote the conjugacy class of $(\frac{\mathfrak{N}/\mathbb{Q}}{\beta})$ in $Gal(\mathfrak{N}/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p}) \subset \mathfrak{N}$, we prove

Lemma 6.1. *$Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$ is elementary abelian of order* 4.

Proof. For $\sigma \in Gal(\mathfrak{N}/M)$, σ can only change the sign of $\sqrt{\varepsilon}$ and $\sqrt{\pi}$ as $\varepsilon \in M$. Since $\mathfrak{N} = M(\sqrt{\varepsilon}, \sqrt{\pi})$, $Gal(\mathfrak{N}/M)$ is elementary abelian of order 4. Now consider the restrictions $r_1 : G_1 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \to Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = Gal(N_1/\mathbb{Q})$ and $G_2 = Gal(N/\mathbb{Q})$. Clearly $r_1|_{Z(G_1)}$ and $r_1|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2, $Z(\mathscr{G})$ is elementary abelian of order 4 where $\mathscr{G} = Gal(\mathfrak{N}/\mathbb{Q})$. Thus, $Z(Gal(\mathfrak{N}/\mathbb{Q})) = Gal(\mathfrak{N}/M)$. □

Now for $l \in A$, l splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. From Lemma 6.1, $\mathfrak{N}^{Z(Gal(\mathfrak{N}/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{p})$. So by Lemma 2.1, we have

$$\left(\frac{\mathfrak{N}/\mathbb{Q}}{l}\right) = \{k\} \quad \text{for some } k \in Gal(\mathfrak{N}/\mathbb{Q}).$$

As $Z(Gal(\mathfrak{N}/\mathbb{Q}))$ has order 4, there are four possible choices for $(\frac{\mathfrak{N}/\mathbb{Q}}{l})$. Using Remarks 5.1 and 5.3, we now make the following one to one correspondences.

Remark 6.2. (i) $(\frac{\mathfrak{N}/\mathbb{Q}}{l}) = \{id\} \Leftrightarrow l$ splits completely in $\mathfrak{N} \Leftrightarrow \left\{\begin{matrix} l \text{ splits completely in} \\ N_1 \text{ and } N \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{matrix}\right\}$.

(ii) $(\frac{\mathfrak{N}/\mathbb{Q}}{l}) \neq \{id\} \Leftrightarrow l$ does not split completely in $\mathfrak{N}$. Now there are three cases:

(1) $\left\{\begin{matrix} l \text{ splits completely in } N_1, \\ \text{but does not in } N \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{matrix}\right\}$,

(2) $\left\{\begin{matrix} l \text{ splits completely in } N \\ \text{but does not in } N_1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{matrix}\right\}$,

(3) $\left\{\begin{matrix} l \text{ does split completely} \\ \text{in } N_1 \text{ or } N \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} l \text{ does not satisfy } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{matrix}\right\}$.

We can now prove Theorem 1.2

Proof. Consider the set $X = \{l$ prime : l is unramified in $\mathfrak{N}$ and $(\frac{\mathfrak{N}/\mathbb{Q}}{l}) = \{k\}\}$ for some $k \in \mathrm{Gal}(\mathfrak{N}/\mathbb{Q})$. By the Čebotarev Density Theorem, the set X has natural density $\frac{1}{32}$ in the set of all primes l. Recall

$$A = \left\{ l \text{ rational prime } : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = 1 \right\}$$

for some fixed prime $p \equiv 1 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, A has natural density $\frac{1}{8}$ in the set of all primes l. Thus, X has natural density $\frac{1}{4}$ in A. If p satisfies $\langle 1, 32 \rangle$, then by Proposition 4.5,

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1 \Leftrightarrow \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle p, -2 \rangle \end{array} \right\} \text{ or } \left\{ \begin{array}{c} l \text{ does not} \\ \text{satisfy } \langle 1, 32 \rangle \end{array} \right\}.$$

and

$$\text{4-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 2 \Leftrightarrow \left\{ \begin{array}{l} l \text{ satisfies } \langle 1, 32 \rangle \\ l \text{ satisfies } \langle 1, -2p \rangle \end{array} \right\}.$$

Using Remark 6.2, we see that for $\mathbb{Q}(\sqrt{pl})$, 4-rank 1 and 4-rank 2 appear with natural density $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ and $\frac{1}{4}$, respectively. A similar argument works if p does not satisfy $\langle 1, 32 \rangle$. For $\mathbb{Q}(\sqrt{-pl})$, use Proposition 4.5 and Remark 6.2 to obtain that 4-rank 1 and 2 each appear with natural density $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ in A. □

7. Proof of two corollaries

For squarefree, odd integers d, recall the sets $X = \{d : d = pl\}$ and $Y = \{d : d = -pl\}$ for distinct primes p and l. Now consider the sets

$$X_i = \{d : d = pl,\ p \equiv i \bmod 8\},$$

$$Y_i = \{d : -pl,\ p \equiv i \bmod 8\}.$$

Thus, $X = X_1 \cup X_3 \cup X_5 \cup X_7$ and $Y = Y_1 \cup Y_3 \cup Y_5 \cup Y_7$. Additionally consider the sets

$$X_{i,j} = \{d : d = pl,\ p \equiv i \bmod 8,\ l \equiv j \bmod 8\},$$

$$Y_{i,j} = \{d : d = -pl,\ p \equiv i \bmod 8,\ l \equiv j \bmod 8\}.$$

Thus, for example, $X_1 = X_{1,1} \cup X_{1,3} \cup X_{1,5} \cup X_{1,7}$ and $Y_7 = Y_{7,1} \cup Y_{7,3} \cup Y_{7,5} \cup Y_{7,7}$.

In Tables 1 and 2, for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})})$, we provide cases in which densities of 4-rank values follow from congruence conditions on p and l, a condition on the Legendre symbol $(\frac{l}{p})$ (if any), and Dirichlet's theorem on primes in arithmetic progressions. In Tables 3 and 4, we provide the same information for $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})})$ (compare with [5] or [9, Tables I and II, 10]).

Table 1
$\mathbb{Q}(\sqrt{pl})$

$p, l \bmod 8$	4-rank	Densities
3, 3	0	$\frac{1}{4}$ in X_3
5, 5	1	$\frac{1}{4}$ in X_5
7, 7	1	$\frac{1}{4}$ in X_7
3, 5	1	$\frac{1}{4}$ in X_3 and X_5
3, 7	1	$\frac{1}{4}$ in X_3 and X_7
5, 7	1	$\frac{1}{4}$ in X_5 and X_7

Table 2
$\mathbb{Q}(\sqrt{pl})$

$p, l \bmod 8$	Legendre symbols	4-rank	Densities
1, 3	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in X_1 and X_3
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in X_1 and X_3
1, 5	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in X_1 and X_5
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in X_1 and X_5
1, 7	$(\frac{l}{p}) = -1$	1	$\frac{1}{8}$ in X_1 and X_7
	$(\frac{l}{p}) = 1$	1	$\frac{1}{16}$ in X_1 and X_7
		2	$\frac{1}{16}$ in X_1 and X_7

Table 3
$\mathbb{Q}(\sqrt{-pl})$

$p, l \bmod 8$	4-rank	Densities
3, 3	1	$\frac{1}{4}$ in Y_3
5, 5	1	$\frac{1}{4}$ in Y_5
7, 7	1	$\frac{1}{4}$ in Y_7
3, 5	0	$\frac{1}{4}$ in Y_3 and Y_5
3, 7	0	$\frac{1}{4}$ in Y_3 and Y_7
5, 7	0	$\frac{1}{4}$ in Y_5 and Y_7

Remark 7.1. By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$ yields 4-rank 1 and 2 with densities $\frac{3}{32}$ and $\frac{1}{32}$, respectively in X_1. By Theorem 1.3, $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = -1$ yields 4-rank 0 and 1 each with density $\frac{1}{16}$ in X_1. We can now prove Corollary 1.4.

Table 4
$\mathbb{Q}(\sqrt{-pl})$

p, $l \bmod 8$	Legendre symbols	4-rank	Densities
1, 1	$(\frac{l}{p}) = -1$	1	$\frac{1}{8}$ in Y_1
1, 3	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_3
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_3
1, 5	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_5
	$(\frac{l}{p}) = 1$	1	$\frac{1}{8}$ in Y_1 and Y_5
1, 7	$(\frac{l}{p}) = -1$	0	$\frac{1}{8}$ in Y_1 and Y_7
	$(\frac{l}{p}) = 1$	0	$\frac{1}{16}$ in Y_1 and Y_7
		1	$\frac{1}{16}$ in Y_1 and Y_7

Proof. Regarding the set X_1:

- 4-rank 0, 1, and 2 appear with natural densities $\frac{1}{16}$, $\frac{3}{32} + \frac{1}{16} = \frac{5}{32}$, and $\frac{1}{32}$ in $X_{1,1}$;
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,3}$;
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $X_{1,5}$;
- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $X_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{5}{16}$, $\frac{19}{32}$, and $\frac{3}{32}$ in X_1. For the set X_3:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $X_{3,1}$;
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $X_{3,3}$;
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,5}$;
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $X_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{3}{8}$ and $\frac{5}{8}$ in X_3. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{7}{8}$ in X_5 and 4-rank 1 and 2 appear with natural densities $\frac{15}{16}$ and $\frac{1}{16}$ in X_7. As each X_i has density $\frac{1}{4}$ in X,

- 4-rank 0 appears with natural density $\frac{5}{64} + \frac{3}{32} + \frac{1}{32} = \frac{13}{64}$ in X;
- 4-rank 1 appears with natural density $\frac{19}{128} + \frac{5}{32} + \frac{7}{32} + \frac{15}{64} = \frac{97}{128}$ in X;
- 4-rank 2 appears with natural density $\frac{3}{128} + \frac{1}{64} = \frac{5}{128}$ in X. □

Remark 7.2. By Theorem 1.2, $p \equiv l \equiv 1 \bmod 8$ with $(\frac{l}{p}) = 1$ yields 4-rank 1 and 2 each with density $\frac{1}{16}$ in Y_1. We can now prove Corollary 1.5.

Proof. Regarding the set Y_1:

- 4-rank 1 and 2 appear with natural densities $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ and $\frac{1}{16}$ in $Y_{1,1}$;

- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,3}$;
- 4-rank 0 and 1 each appear with natural densities $\frac{1}{8}$ in $Y_{1,5}$;
- 4-rank 0 and 1 appear with natural densities $\frac{1}{8}$ and $\frac{1}{16}+\frac{1}{16}=\frac{1}{8}$ in $Y_{1,7}$.

Thus 4-rank 0, 1, and 2 appear with natural densities $\frac{3}{8}$, $\frac{9}{16}$, and $\frac{1}{16}$ in Y_1. For the set Y_3:

- 4-rank 0 and 1 each appear with natural density $\frac{1}{8}$ in $Y_{3,1}$;
- 4-rank 1 appears with natural density $\frac{1}{4}$ in $Y_{3,3}$;
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,5}$;
- 4-rank 0 appears with natural density $\frac{1}{4}$ in $Y_{3,7}$.

So 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in Y_3. Similarly, 4-rank 0 and 1 appear with natural densities $\frac{5}{8}$ and $\frac{3}{8}$ in Y_5 and 4-rank 0 and 1 appear with natural densities $\frac{11}{16}$ and $\frac{5}{16}$ in Y_7. As each Y_i has density $\frac{1}{4}$ in Y,

- 4-rank 0 appears with natural density $\frac{3}{32}+\frac{5}{32}+\frac{5}{32}+\frac{11}{64}=\frac{37}{64}$ in Y;
- 4-rank 1 appears with natural density $\frac{9}{64}+\frac{3}{32}+\frac{3}{32}+\frac{5}{64}=\frac{13}{32}$ in Y;
- 4-rank 2 appears with natural density $\frac{1}{64}$ in Y. □

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Appendix

The approach of Hurrelbrink and Kolster [7] led us to write a program in GP/PARI [2] which generates the numerical values in Tables 5–8. The aim is to motivate possible density results for tame kernels of quadratic number fields. In Tables 5 and 6, p, l, and r are distinct odd primes. In Tables 7 and 8, d is odd and squarefree.

Table 5

Cardinality	$105 \leqslant d = plr < 10^6$	(%)
|4-rank 0|	8247	6.827
|4-rank 1|	92 544	76.605
|4-rank 2|	20 000	16.555
|4-rank 3|	16	0.013

Table 6

Cardinality	$-10^6 < d = -plr \leqslant -105$	(%)
|4-rank 0|	67 970	56.2633
|4-rank 1|	50 147	41.5100
|4-rank 2|	2688	2.2250
|4-rank 3|	2	0.0017

Table 7

Cardinality	$3 \leqslant d < 10^6$	(%)
|4-rank 0|	93 736	23.1284
|4-rank 1|	278 138	68.6278
|4-rank 2|	33 148	8.1789
|4-rank 3|	263	0.0649

Table 8

Cardinality	$-10^6 < d \leqslant -3$	(%)
|4-rank 0|	251 884	62.14985
|4-rank 1|	148 669	36.68258
|4-rank 2|	4730	1.16708
|4-rank 3|	2	0.00049

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