The Structure of Imprimitive Non-symmetric 3-Class Association Schemes

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In this paper we present a classification into three categories of the imprimitive non-symmetric association schemes with three classes. For two of the three categories we present complete solutions, while for the third one we find partial results.


1. Introduction

According to [2, p. 52] the imprimitivity of an association scheme can be recognized from its parameters. So we begin to characterize the imprimitivity of non-symmetric 3-schemes (we call an n-class association scheme an n-scheme) in terms of its parameters. Here we utilize the fact that an association scheme is imprimitive if its symmetric closure is imprimitive. The symmetric closure of a non-symmetric 3-scheme is a symmetric 2-scheme. All imprimitive symmetric 2-schemes belong to a family of a simple structure: the group-divisible schemes. If \((X, \mathcal{R})\) is the symmetric closure of a non-symmetric 3-scheme \((X, \mathcal{R})\), then we say that \((X, \mathcal{R})\) is a splitting of \((X, \mathcal{R})\).

It appears that one can divide the imprimitive non-symmetric 3-schemes into three disjoint categories. For two of the three categories there are obvious ways to construct all the schemes belonging to these categories (see the Theorems 5.5 and 7.1), but for the last one the schemes are harder to construct (case (4) of Theorem 5.5) and we only find partial results.

Every imprimitive non-symmetric 3-scheme, of which the construction is described in this paper, can be derived in a simple way from an Hadamard matrix of a certain form (e.g., skew-Hadamard matrices) in such a way that two such 3-schemes are isomorphic iff the connected Hadamard matrices are graph equivalent (for a definition see just before Example 5.2). We refer to the Corollaries 7.2 and 8.3 and Theorem 8.5. So, in fact, we reduce the construction of imprimitive non-symmetric 3-schemes to the construction of Hadamard matrices.

Recently Sung Y. Song published paper [11]. It seems appropriate that we discuss briefly the contents of Song’s paper and compare it with our results. Apart from a few non-existence results for primitive non-symmetric 3-schemes the paper deals, in essence, with the same subjects as are considered in the present paper. However, since the link between imprimitive non-symmetric 3-schemes and Hadamard matrices is not considered, the approach has to be different (use of permutation groups and, to quote Song, ‘brute force’). Because of this and another choice of parameters (see below) only partial results are reached in cases where we found complete solutions.

Song is concerned, as we are, with splitting imprimitive symmetric 2-schemes of which one of the graphs is a trivial strongly regular graph with parameters \((n, k, 2k - n, k)\). That is, in the terminology of this paper (cf. Remark 3.3), Song considers the splitting of GD-schemes of type \((g, h) = (n - k, n/(n - k))\). Song mainly finds results for the pairs \((g, h) = (2, h), (3, h), (7, h), (11, h)\) and \((g, 3)\).

In [6, 8] we laid the foundation for our investigation of non-symmetric 3-schemes.
For the general theory on association schemes we refer to [2] and the papers mentioned there. We shall use the notation of Delsarte as it was introduced for association schemes in [4]. This implies the use of a few peculiar notations: if \( P \) is any complex entity (number, vector, etc.) then \( P^* \) denotes the complex conjugate of \( P \), and if \( S \) is a set then \( S^* \) denotes the set of all complex conjugates of the elements of \( S \).

Always, \( n \in \mathbb{N}\backslash\{0\} \), and \( u \in \mathbb{N}\backslash\{0, 1\} \). For any association scheme \((X, R)\) we denote \([X]\) by \( v \).

\( I \) (or simply \( I \)) denotes the \((t \times t)\)-identity matrix and \( J \) (or \( J \)) denotes the \((t \times t)\)-all-one matrix.

The \textit{Kronecker product} of two matrices \( A \) and \( B \) (consisting of the blocks \( a_{ij}B \)) will be denoted by \( A \otimes B \), while the \textit{direct sum} of the matrices \( A_1, A_2, \ldots, A_m \) (which is a block matrix with on the main diagonal the matrices \( A_1, A_2, \ldots, A_m \) and the other blocks equal to 0) is denoted by \( A_1 \oplus A_2 \oplus \cdots \oplus A_m \).

2. Preliminaries

\textbf{Definition 2.1.} Let \( X \) be a set with \( v \) elements. Let \( R = \{R_0, R_1, \ldots, R_u\} \) be a family of \( n + 1 \) binary relations on \( X \). The pair \((X, R)\) will be called an \textit{association scheme with \( n \) classes} (also called an \textit{n-scheme}) if the following conditions are satisfied:

(1) the family \( R \) is a partition of \( X^2 \) and \( R_0 \) is the diagonal (equality) relation;
(2) for any \( i, j \in \{0, 1, \ldots, n\} \) the inverse \( R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\} \) of the relation \( R_i \) belongs to \( R \) (the index of the relation \( R_i^{-1} \) is denoted by \( i_R \));
(3) for \( i, j, k \in \{0, 1, \ldots, n\} \) the so-called \textit{intersection numbers}

\[ p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| \]

are independent of the choice of \((x, y) \in R_i\);
(4) for all \( i, j, k \in \{0, 1, \ldots, n\} \) we have \( p_{ij}^k = p_{ji}^k \).

Association schemes as they are used in this paper are called ‘commutative association schemes’ in [1].

For every \( i \) the number \( p_{0i}^0 \) is called the \textit{valency} of \( R_i \) and is denoted by \( v_i \). An association scheme \((X, R)\) is called \textit{symmetric} if all its relations are symmetric, i.e. \( i = i_R \) for all \( i \); otherwise it is called \textit{non-symmetric}. Let \((X, R)\) be an association scheme. The association scheme \((X, \bar{R})\), in which \( R = \{R \cup R^{-1} \mid R \in R\} \), is said to be the \textit{symmetric closure} of \((X, R)\).

The \textit{adjacency matrix} of the relation \( R_i \) is denoted by \( D_i \). If \( D = \{D_0, D_1, \ldots, D_u\} \), then we say that \( D \) \textit{represents} \((X, R)\). The \( n + 1 \) \textit{maximal common eigenspaces} of \((X, R)\) are denoted by \( V_k \). The eigenvalue of \( D_i \) on \( V_k \) is denoted by \( \lambda_i \), and we denote \( \dim(V_k) \) by \( \mu_k \): the \textit{multiplicities} of \((X, R)\). The \textit{co-intersection numbers} (or \textit{Krein parameters}) are denoted by \( q_{ij}^k \).

Now we define the following \((n + 1) \times (n + 1)\)-matrices: \( P \) with \((i, j)\)-entry \( P_{ij} \), \( Q \) with \((i, j)\)-entry \( Q_{ij} \), \( L \), with \((k, j)\)-entry \( P_{kj}^k \) and \( \lambda_j \) with \((k, j)\)-entry \( q_{kj}^k \). Hence \( P \) and \( Q \) are the first and the second eigenvalue matrices, while the \( L \) and \( \lambda_j \) are the intersection and co-intersection matrices.

Let \((X, R)\) and \((X, \bar{R})\) be two association schemes with eigenvalue matrices \( P, Q \) and \( \bar{P}, \bar{Q} \), respectively. The schemes are called:

(i) \textit{isomorphic} if there is a permutation matrix \( \Pi \) such that the matrices \( \Pi D \Pi^T \) are the adjacency matrices of \((X, \bar{R})\);
(ii) \textit{isospectral}, if \( P = \bar{P} \) for a certain numbering of the relations and the eigenspaces of both schemes; and
(iii) formally dual, if $P = \hat{Q}^*$ for a certain numbering of the relations and the eigenspaces of both schemes.

An association scheme $(X, R)$ is called primitive if the union of some of its relations is an equivalence relation distinct from $R_0$ and $X \times X$; otherwise $(X, R)$ is called primitive.

**Theorem 2.2.** An $n$-scheme $(X, R)$ is imprimitive if its symmetric closure $(X, \tilde{R})$ is imprimitive.

**Proof.** Let $(X, R)$ be imprimitive and let $R$ be a union of relations of $R$ which is an equivalence relation. $R_i \subseteq R$ implies $R_i^{-1} \subseteq R$ and so we see that $R$ is also a union of relations of $\tilde{R}$. So $(X, \tilde{R})$ is imprimitive.

The proof of the converse is also trivial. $\square$

From now on in this paper, $(X, \tilde{R})$ denotes a symmetric 2-scheme and its parameters are provided with a bar. $(X, R)$ denotes, unless otherwise stated, a non-symmetric 3-scheme. Two of the three non-trivial relations of $(X, R)$ are not symmetric. We assume throughout this paper that $R_2 = R_1^{-1}$ and $V_2^\perp = V_1$.

In this paper we shall use the following shorthand notation for the parameters of $(X, R)$: $u = v_1/v_2$, $u' = \mu_1/\mu_3$ and

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<th>$\alpha = p_{11}^1$</th>
<th>$\beta = p_{11}^2$</th>
<th>$\gamma = p_{13}^1$</th>
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For a proof of the next theorem we refer to [6, 8].

**Theorem 2.3.** The intersection matrices and the first eigenvalue matrix of $(X, R)$ have the following forms. $L_0 = I$ and

$$L_1 = \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 1 & \alpha & \alpha \delta \\ 0 & \beta & \alpha \varepsilon \\ 0 & \mu \varepsilon & \mu \delta & \mu \gamma \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & \alpha & \beta & \varepsilon \\ 1 & \alpha & \alpha \delta \\ 0 & \mu \delta & \mu \varepsilon & \mu \gamma \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & v_3 \\ 0 & \delta & \varepsilon & \gamma \\ 0 & \varepsilon & \delta & \gamma \\ 1 & \mu \gamma & \mu \gamma & \lambda \end{pmatrix}, \quad P = \begin{pmatrix} 1 & v_1 & v_1 & v_3 \\ 1 & \Lambda & \Lambda^* & \Phi \\ 1 & \Lambda^* & \Lambda & \Phi \\ 1 & \Psi & \Psi & \Omega \end{pmatrix}.$$  

It holds that $\Lambda \in \mathbb{C} \setminus \mathbb{R}$, while the other first eigenvalues are real.

For $i = 0, 1, 2, 3$, the co-intersection matrix $M_i$ can be found from the matrix $L_i$ by replacing $v_i$ by $\mu_i$ and by providing the respective intersection numbers with an accent.

The second eigenvalue matrix $Q$ can be derived from $P$ by replacing, for $i = 1, 3$, the $v_i$ by $\mu_i$ and by providing the eigenvalues $\Lambda, \Phi, \Psi$ and $\Omega$ with an accent. Again $\Lambda' \in \mathbb{C} \setminus \mathbb{R}$ and the other second eigenvalues are real.
The parameters of \((X, R)\) are determined once \(v, v_1\), \(\alpha\) and \(\beta\) are given. In particular,

\[
\Lambda = \frac{1}{2} \left( \alpha - \beta + i \sqrt{\frac{v v_1}{\mu_1}} \right) \quad \text{and} \quad \Psi = u \gamma - \varepsilon.
\]

From \(L_i L_j = L_j L_i\) and \(M_i M_j = M_j M_i\), the fact that the row-sum of \(L_j\) is \(v_j\) and that the row-sum of \(M_i\) is \(\mu_i, \mu_i P(i) = v_j Q^*(j)\) and \(P Q = v I\) several conditions on the parameters of \((X, R)\) can be derived. These conditions all follow from the properties of the symmetric closure of \((X, R)\). We mention in Lemma 2.4 some of the most useful of these conditions on the parameters of \((X, R)\).

**Lemma 2.4.** For \((X, R)\) the following relations hold:

1. \((\alpha - \beta)\varepsilon = u \gamma (\varepsilon - \delta)\) and \((\alpha' - \beta')\varepsilon' = u' \gamma' (\varepsilon' - \delta')\);
2. \(\alpha - \beta + \delta - \varepsilon = -1\) and \(\alpha' - \beta' + \delta' - \varepsilon' = -1\);
3. \(\frac{\delta - \varepsilon}{v_3} = \frac{u' \gamma - \varepsilon'}{\mu_1}\) and \(\frac{u \gamma - \varepsilon}{v_3} = \frac{\delta' - \varepsilon'}{\mu_3}\);
4. \(\frac{\alpha - \beta}{v_1} = \frac{\alpha' - \beta'}{\mu_1}\) and \(\frac{2 \varepsilon - 2 u \gamma - 1}{v_3} = \frac{2 \varepsilon' - 2 u' \gamma' - 1}{\mu_3}\).

**Proof.** For the proof of the first formula of (1) consider the \((1, 3)\)-entry of both \(L_1 L_2\) and \(L_2 L_1\) (we assume the rows and the columns of the \(L_j\)-matrices to be numbered by the relations).

\(\alpha - \beta + \delta - \varepsilon = -1\) comes from the fact that the sum of every row of \(L_1\) is equal to \(v_1\), while the first formula of (3) can be derived from \(\mu_i P(i) = v_j Q^*(j)\).

The rest of the proof of the lemma is left to the reader. \(\square\)

**Definition 2.5.** If \((X, \bar{R})\) is the symmetric closure \((X, R)\), then we call \((X, R)\) a splitting of \((X, \bar{R})\).

The indices \(s, S, n\) and \(N\) of the relations of \((X, \bar{R})\) are defined as follows. \(\bar{R} = R_1 \cup R_2, \bar{V}_S = V_1 \oplus V_2, \bar{R}_S = R_3\) and \(\bar{V}_N = V_3\), and we say that:

1. the splitting of \((X, \bar{R})\) is according to case I if \(s = S = 1\);
2. the splitting of \((X, \bar{R})\) is according to case II if \(s = 1\) and \(S = 2\);
3. the splitting of \((X, \bar{R})\) is according to case III if \(s = 2\) and \(S = 1\);
4. the splitting of \((X, \bar{R})\) is according to case IV if \(s = S = 2\).

For the proof of the next theorem we must again refer to [6, 8].

**Theorem 2.6.** Let \((X, R)\) be a non-symmetric 3-scheme which is a splitting of the symmetric 2-scheme \((X, \bar{R})\). The parameters of \((X, R)\) expressed in those of \((X, \bar{R})\) are as follows:

1. \(v, v_1 = \frac{1}{2} \bar{v}_i, \bar{v}_3 = \bar{v}_n, \mu_1 = \frac{1}{2} \bar{\mu}_i, \mu_2 = \bar{\mu}_N;\)
2. \(\alpha = \frac{1}{2} (\bar{P} + \bar{P}(S)), \beta = \frac{1}{2} (\bar{P} - 3 \bar{P}(S)), \gamma = \bar{P}, \delta = \frac{1}{2} (\bar{P} - \bar{P}(S)), \varepsilon = \frac{1}{2} (\bar{P} - \bar{P}(S)), \lambda = \bar{P}_N, \\lambda' = \bar{P}_N;\)
3. \(\alpha' = \frac{1}{2} (\bar{Q}^T + \bar{Q}(s)), \beta' = \frac{1}{2} (\bar{Q}^T - 3 \bar{Q}(s)), \gamma' = \bar{Q}^T, \delta' = \frac{1}{2} (\bar{Q}^T + \bar{Q}(s)), \varepsilon' = \frac{1}{2} (\bar{Q}^T - \bar{Q}(s)), \lambda' = \bar{Q}_N;\)
(iv) $\Lambda = \frac{1}{2} \left( \tilde{P}_3(S) + i \sqrt{\frac{\nu_1}{\mu_3}} \right)$, $\Phi = \tilde{P}_3(S)$, $\Psi = \frac{1}{2} \tilde{P}_3(N)$, $\Omega = \tilde{P}_3(N)$;

(v) $\Lambda' = \frac{1}{2} \left( \tilde{Q}_3(s) - i \sqrt{\frac{\nu_1}{\mu_3}} \right)$, $\Phi' = \tilde{Q}_3(s)$, $\Psi' = \frac{1}{2} \tilde{Q}_3(n)$, $\Omega' = \tilde{Q}_3(n)$.

For later use we prove the following lemma.

**Lemma 2.7.** For $(X, R)$ the following hold:

1. Both $\gamma = 0$ and $\delta + \varepsilon = 0$ is not possible; nor is it possible that both $\gamma' = 0$ and $\delta' + \varepsilon' = 0$.
2. Both $\alpha = \beta$ and $\varepsilon = 0$ is not possible; nor is it possible that both $\alpha' = \beta'$ and $\varepsilon' = 0$.
3. Both $\beta = 0$ and $\varepsilon = 0$ is not possible; nor is it possible that both $\beta' = 0$ and $\varepsilon' = 0$.
4. $\alpha = \beta$ implies $\gamma = 0$, while $\alpha' = \beta'$ implies $\gamma' = 0$.
5. $\varepsilon = 0$ but $\delta \neq 0$ implies $\gamma = 0$, while $\varepsilon' = 0$ but $\delta' \neq 0$ implies $\gamma' = 0$.
6. $\delta = \varepsilon$ implies $\delta = \varepsilon = 0$, while $\delta' = \varepsilon'$ implies $\delta' = \varepsilon' = 0$.
7. $\varepsilon = u \gamma$ implies $\varepsilon = \gamma = 0$, while $\varepsilon' = u' \gamma'$ implies $\varepsilon' = \gamma' = 0$.

**Proof.** We shall use the symmetric closure $(X, \tilde{R})$ of $(X, R)$.

If $\delta + \varepsilon = \gamma = 0$, then $\tilde{P}_{11} = \tilde{P}_{22} = 0$, which is not possible. In the same way, $\delta + \varepsilon' = \gamma' = 0$ leads to a contradiction.

The rest of the lemma follows from (1) and (2) of Lemma 2.4, and the Krein conditions ($q_i^n \geq 0$).

3. **Imprimitive Symmetric 2-Schemes**

Because of Theorem 2.2, we first characterize the imprimitive symmetric 2-schemes in terms of their parameters.

**Lemma 3.1.** For $(X, \tilde{R})$ the following hold:

1. $R_0 \cup \tilde{R}_1$ is an equivalence relation iff $\tilde{P}_{12} = 0$.
2. $R_0 \cup \tilde{R}_2$ is an equivalence relation iff $\tilde{P}_{12} = 0$.

**Proof.** The proof is obvious, since $\tilde{P}_{11} = \tilde{P}_{12}$.

**Theorem 3.2.** The following statements are equivalent for $(X, \tilde{R})$:

1. $(X, \tilde{R})$ is imprimitive.
2. $\tilde{P}_{12} = 0$.
3. $\tilde{q}_{12} = 0$.
4. $\tilde{P}_1(1)\tilde{P}_2(2)\tilde{P}_2(1)\tilde{P}_2(2) = 0$.
5. $\tilde{Q}_1(1)\tilde{Q}_2(2)\tilde{Q}_2(1)\tilde{Q}_2(2) = 0$.

**Proof.** By Lemma 3.1, (1) and (2) are equivalent. Calculating the determinants of $L_1$, $L_2$, $M_1$ and $M_2$ and using $\tilde{\mu}_i \tilde{P}_i(i) = \tilde{v}_i \tilde{Q}_i(j)$, one easily derives

$$\tilde{P}_{12} \tilde{P}_{22} = \tilde{P}_1(1)\tilde{P}_2(2)\tilde{P}_2(1)\tilde{P}_2(2) = \left( \frac{\tilde{v}_1 \tilde{v}_2}{\mu_1 \mu_2} \right)^2 \tilde{Q}_1(1)\tilde{Q}_2(2)\tilde{Q}_2(1)\tilde{Q}_2(2) = \left( \frac{\tilde{v}_1 \tilde{v}_2}{\mu_1 \mu_2} \right)^2 \tilde{a} \tilde{q}_{12} \tilde{q}_{22},$$

implying the equivalence of the rest of the assertions.
REMARK 3.3. It is easily seen that an imprimitive symmetric 2-scheme has the following simple structure. There are natural numbers $g$ and $h$ (both $\neq 0, 1$) such that the adjacency matrices can be put in the following form:

$$
\bar{D}_0 = I_h \otimes I_h, \quad \bar{D}_1 = I_h \otimes (J_h - I_h) \quad \text{and} \quad \bar{D}_2 = (J_h - I_h) \otimes J_h.
$$

The scheme is said to be a \textit{group-divisible} 2-scheme (of type $(g, h)$), also called a GD-scheme.

For the above numbering of the relations and a suitable numbering of the eigenspaces, the intersection matrices and the first eigenvalue matrix of a GD-scheme of type $(g, h)$ have the following form:

$$
\bar{L}_0 = I, \quad \bar{L}_1 = \begin{pmatrix}
0 & g - 1 & 0 \\
1 & g - 2 & 0 \\
0 & 0 & g - 1
\end{pmatrix}, \quad \bar{L}_2 = \begin{pmatrix}
0 & 0 & g(h - 1) \\
0 & 0 & g(h - 1) \\
1 & g - 1 & g(h - 2)
\end{pmatrix},
$$

$$
\bar{P} = \begin{pmatrix}
1 & g - 1 & g(h - 1) \\
1 & g - 1 & -g \\
1 & -1 & 0
\end{pmatrix}.
$$

The co-intersection matrices and the second eigenvalue matrix can be found by interchanging $g$ and $h$ in the above matrices. From this it follows that a GD-scheme of type $(g, h)$ and a GD-scheme of type $(h, g)$ are formally dual.

Throughout this paper we assume that the numbering of the relations and the eigenspaces of a GD-scheme of type $(g, h)$ is in accordance with the setting of this remark.

The next corollary to Theorem 3.2 gives a few simple conditions for the imprimitivity of a symmetric 2-scheme.

COROLLARY 3.4. \textbf{(X, $\bar{R}$) is imprimitive if one of the following conditions are met:}

1. $\gcd(\bar{v}_1, \bar{v}_2) = 1$.
2. There is a prime $p$ such that $\nu = p + 1$.

PROOF. Since $\bar{p}_{12}^1 = \bar{p}_{22}^1 \bar{v}_1 / \bar{v}_2 \in \mathbb{N}$ and $\gcd(\bar{v}_1, \bar{v}_2) = 1$ we have either $\bar{p}_{12}^1 = 0$ or $\bar{p}_{22}^1 = \bar{v}_2$. But $\bar{p}_{12}^1 + \bar{p}_{22}^1 = \bar{v}_2$ and so if $\bar{p}_{22}^1 \neq 0$ then $\bar{p}_{12}^1 = \bar{v}_2$ and $\bar{p}_{12}^1 = 0$. Theorem 3.2 now implies the first assertion.

$\nu - 1 = \bar{v}_1 + \bar{v}_2 = p$ implies $\gcd(\bar{v}_1, \bar{v}_2) = 1$. \hfill $\square$

4. \textbf{Imprimitivity Conditions for Non-symmetric 3-Schemes}

THEOREM 4.1. The next statements are equivalent:

1. \textbf{(X, $\bar{R}$) is imprimitive}.
2. $\gamma(\delta + \varepsilon) = 0$.
3. $\gamma'(\delta' + \varepsilon') = 0$.
4. $(\alpha - \beta)\varepsilon = 0$.
5. $(\alpha' - \beta')\varepsilon' = 0$.

PROOF. Again we shall use the symmetric closure $(\mathbf{X}, \bar{R})$ of $(\mathbf{X}, R)$. 

Plainly, \((\delta + \varepsilon)\gamma = \tilde{p}_{s\alpha}^{\text{impr}}\tilde{p}_{\text{sym}}^{\text{impr}}\) and \((\delta' + \varepsilon')\gamma' = \tilde{q}_{s\alpha}^{\text{sym}}\tilde{q}_{\text{sym}}^{\text{impr}}\) (Theorem 2.6). Theorems 3.2 and 2.2 imply the equivalence of (1), (2) and (3). The rest of the theorem is easily derived from Lemmas 2.7 and 2.4.

Expressed in the parameters \(v, v_1, \alpha, \beta\) using \(y_3 = y - 2v_1 - 1\), one sees that \((X, R)\) is imprimitive iff \(\alpha - \beta \in \{0, -1, -v_3 - 1\}\) (use Lemmas 2.4 and 2.7 and the fact that \(\delta = v_3\) if \(\gamma = \varepsilon = 0\)).

**Theorem 4.2.** The following hold for \((X, R)\):

1. \(R_0 \cup R_1 \cup R_2\) is an equivalence relation iff \(\delta = \varepsilon\).
2. \(R_0 \cup R_1\) is an equivalence relation iff either \(uv = \varepsilon\) or \(\alpha = \beta\).

**Proof.** By Theorem 2.6, \(\delta = \varepsilon\) is equivalent to \(\tilde{p}_1(S) = 0\). So \((X, R)\) can be considered as the splitting according to case II of an imprimitive symmetric 2-scheme.

\(\delta = \varepsilon\) is also equivalent to \(\delta = \varepsilon = 0\) (Lemma 2.7) which, by Theorem 2.6, is equivalent to \(\tilde{p}_{12} = 0\) (\(s = 1\) and \(n = 2\), splitting according to case II), and by Lemma 3.1 this is equivalent to the fact that \(R_0 \cup R_1 \cup R_2\) is an equivalence relation. This deals with (1).

If \(\delta \neq \varepsilon\) then \(\delta + \varepsilon \neq 0\) and so \(\gamma = 0\) if the scheme is imprimitive, implying either \(uv = \varepsilon\) or \(\alpha = \beta\) (Lemma 2.4). Now it is not too difficult to complete the proof of the theorem. \(\square\)

**Theorem 4.3.** For an imprimitive non-symmetric 3-scheme \((X, R)\), which is the splitting of \((X, \bar{R})\), there are the following possibilities, each one excluding the other two:

1. \(u\gamma = \varepsilon\) (or, equivalently, \(u'\gamma' = \varepsilon'\)) and the splitting is according to case II.
2. \(u\gamma = \varepsilon\) (or, equivalently, \(\delta' = \varepsilon'\)) and the splitting is according to case III.
3. \(\alpha = \beta\) (or, equivalently, \(\alpha' = \beta'\)) and the splitting is according to case IV.

**Proof.** From Theorem 4.2 one derives that there are exactly three possibilities, given in the theorem, such that a non-symmetric 3-scheme \((X, R)\) is imprimitive. They are mutually exclusive by Lemma 2.7.

The rest of the proof follows easily from Lemma 2.4 and the proof of Theorem 4.2. \(\square\)

For completeness, we state the following theorem. Its proof is analogous to that of Corollary 3.4 and is left to the reader.

**Theorem 4.4.** The scheme \((X, R)\) is imprimitive if one of the following conditions holds:

1. \(\gcd(v_1, v_3) = 1\).
2. There is a prime \(p\) such that \(v = p + 1\).

5. A Classification

In this section we use the notions of subscheme and quotient scheme of an imprimitive scheme as described in [1, p. 140 ff.], [2, pp. 51, 52] (for symmetric schemes only) and [6, pp. 45, 46].

First we discuss two simple ways of constructing new association schemes from smaller ones.
The direct sum of isospectral association schemes. Let \( m, n \in \mathbb{N} \setminus \{0\} \). For \( k \in \{1, 2, \ldots, n\} \) let \( D^{(k)} = \{D_0^{(k)}, \ldots, D_m^{(k)}\} \) be an ordered set of \( m + 1 \) square matrices with entries 0 and 1 of order \( v \), while \( D_0^{(k)} = I_v \). Let \( D \) be the ordered set \( D = \{D_0, \ldots, D_{m+1}\} \) of \( m + 2 \) square matrices of order \( mn \) defined by \( D_j = D_j^{(1)} \oplus \cdots \oplus D_j^{(m)} \) for \( 0 \leq i \leq m \), \( D_{m+1} = (I_v - J_v) \otimes J_v \). It is easy to show (use Theorem 2.6.1 in [2]) that in the above-described situation \( D \) represents an \((m + 1)\)-scheme iff the \( D^{(k)} \), \( 1 \leq k \leq n \), represent a set of isospectral \( m \)-schemes, where for \( 0 \leq i \leq m \), \( D^{(k)} \) is mapped on \( D^{(k)} \) by \( D^{(k)} \rightarrow D^{(k)} \).

The \((m + 1)\)-scheme \((X, R)\) represented by \( D \) is said to be the direct sum of the \( m \)-schemes represented by \( D^{(k)} \). Let \((X^{(k)}, R^{(k)})\) be the scheme represented by \( D^{(k)} \), then by \((X^{(1)}, R^{(1)}) \oplus \cdots \oplus (X^{(m)}, R^{(m)})\) we shall denote the direct sum of the schemes \((X^{(k)}, R^{(k)})\) \( (k = 1, 2, \ldots, m) \).

**Lemma 5.1.** In the above-described setting the following hold:

1. \((X, R)\) is symmetric iff the schemes represented by \( D^{(k)} \) are symmetric.
2. \((X, R)\) is imprimitive.
3. \((X, R)\) has the schemes \((X^{(1)}, R^{(1)}), \ldots, (X^{(m)}, R^{(m)})\) as subschemes, and a 1-scheme as quotient scheme with respect to \( \bigcup_{k=0}^{m} R^{(k)} \).

For the construction of several non-symmetric (imprimitive) 3-schemes, we shall need non-symmetric 2-schemes.

As noticed in [4], it is easy to show that if \( \{D_0, D_1, D_2\} \) are the adjacency matrices of a non-symmetric 2-scheme, then the skew-symmetric matrix \( D = D_1 - D_2 \) satisfies \( DJ = 0 \) and \( D^2 = J - vI \). Hence \( D \) is the kernel of a skew-symmetric Hadamard matrix of order \( v + 1 \), implying \( v = 3 \) (mod 4).

If we call two Hadamard matrices \( H_1 \) and \( H_2 \) graph equivalent, or \( G \)-equivalent, if there is a permutation matrix \( \Pi \) such that \( H_2 = \Pi H_1 \Pi^T \), then we see that two non-symmetric 2-schemes are isomorphic iff the corresponding Hadamard matrices are \( G \)-equivalent.

**Example 5.2.** Let \((X, R)\) be the non-symmetric 2-scheme represented by \( D = \{I_v, D, D^T\} \), where

\[
D = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

The scheme \((X, R)\) is \((X, R) \oplus (X, R)\) is represented by \( D_0 = I_v \oplus I_v = I_v \), \( D_1 = D \oplus D \), \( D_2 = D^T \oplus D^T \) and \( D_3 = (I_v - I_v) \otimes J_v \). The scheme \((X, R)\) is a non-symmetric 3-scheme; it has the parameters \( v = 6 \), \( v_1 = 1 \), \( \alpha = 0 \), \( \beta = 1 \). (It is, in fact, the splitting of the GD-scheme of type (3.2) according to case II.)

The restricted Kronecker product of association schemes. Let \( m_1, m_2, w_1, w_2 \) be natural numbers \( \neq 0 \), and let \( D = \{D_0, D_1, \ldots, D_m\} \) be a set of square matrices with entries 0 and 1 of order \( w_1 \) and suppose that \( H = \{H_0, H_1, \ldots, H_m\} \) is a set of square matrices with entries 0 and 1 of order \( w_2 \), with \( D_0 = I_{w_1} \) and \( H_0 = I_{w_2} \). We define the following set of matrices:

\[
D \otimes H = \{D_0 \otimes H_j, 0 \leq j \leq m_2\} \cup \{D_i \otimes J_{w_2}, 1 \leq i \leq m_1\}.
\]

In the situation described above, \( D \otimes H \) represents an \((m_1 + m_2)\)-scheme iff \( D \) represents an \( m_1 \)-scheme and \( H \) represents an \( m_2 \)-scheme; use Theorem 2.6.1 in [2].
Imprimitive association schemes

If $D$ and $H$ represent the association schemes $(X_D, R_D)$, $(X_H, R_H)$, respectively, then the scheme represented by $D \otimes H$ is said to be the restricted Kronecker product of both schemes, and it will be denoted by $(X_D, R_D) \otimes_l (X_H, R_H)$.

**Lemma 4.3.** In the above setting, if $(X, R) = (X_D, R_D) \otimes_l (X_H, R_H)$ we have:

1. $(X, R)$ is symmetric iff both $(X_D, R_D)$ and $(X_H, R_H)$ are symmetric.
2. $(X, R)$ is imprimitive.
3. $(X, R)$ has with respect to the union of the relations corresponding to the matrices $D_0 \otimes H$, the scheme $(X_D, R_D)$ as quotient scheme; its subschemes all are isomorphic to $(X_H, R_H)$.

**Example 5.4.** Let $(X, R)$ be the scheme used in Example 5.2 and let $(Y, S)$ be the 1-scheme with adjacency matrices $I_2$ and $J_2 - I_2$. Then the scheme $(X_2, R_2) = (X, R) \otimes_l (Y, S)$ is a non-symmetric 3-scheme with adjacency matrices $D_0 = I_2 \otimes I_2 = I_6$, $D_1 = D \otimes J_2$, $D_2 = D^T \otimes J_2$ and $D_3 = I_2 \otimes (J_2 - I_2)$.

$(X_2, R_2)$ has the parameters $v = 6$, $v_1 = 2$, $\alpha = 0$, $\beta = 2$ and is the splitting of the GD-scheme of type $(2,3)$ according to case III. The scheme has as symmetric closure the triangular scheme $\Delta(4)$. Song [11] notices that this scheme comes from the action of the alternating group $A_4$ on the set of the two-element subsets of a four-element set.

The graph of the first relation of $(X_2, R_2)$ is as follows (here $X_2 = \{1, 2, 3, 4, 5, 6\}$):

![Graph of the first relation of $(X_2, R_2)$](image)

Note that the scheme $(Y, S) \otimes_l (X, R)$ is the non-symmetric 3-scheme $(X_1, R_1)$ found in Example 5.2, showing that a restricted Kronecker product is not commutative, in general.

**Theorem 5.5.** For a non-symmetric 3-scheme $(X, R)$ with $R_2 = R_1^{-1}$ there are the following possibilities:

1. $(X, R)$ is primitive.
2. $(X, R)$ is imprimitive and $\delta = \varepsilon$; in this case $R_0 \cup R_1 \cup R_2$ is an equivalence relation and $(X, R)$ is the direct sum of its subschemes, which are isospectral non-symmetric 2-schemes.
3. $(X, R)$ is imprimitive and $u\gamma = \varepsilon$; in this case $R_0 \cup R_3$ is an equivalence relation, while its quotient scheme is a non-symmetric 2-scheme and $(X, R)$ is the restricted Kronecker product of the mentioned non-symmetric 2-scheme and a 1-scheme.
4. $(X, R)$ is imprimitive and $\alpha = \beta$; in this case $R_0 \cup R_3$ is an equivalence relation, while its subschemes and its quotient scheme all are 1-schemes.

**Proof.** We denote by $\hat{0}$ a set of indices such that $\bigcup_{i \in \hat{0}} R_i$ is an equivalence relation. By Theorem 4.2, $\hat{0} = \{0, 1, 2\}$ iff $\delta = \varepsilon$ (and so $\delta = \varepsilon = 0$) and $\hat{0} = \{0, 3\}$ iff either $u\gamma = \varepsilon$ or $\alpha = \beta$.

From Lemma 2.7 we derive if $u\gamma = \varepsilon$ then $\gamma = \varepsilon = 0$, and if $\alpha = \beta$ then $\gamma = 0$ but $\varepsilon \neq 0$.

If $\hat{0} = \{0, 1, 2\}$ the assertions of (2) follow from the results of this section. The details are left to the reader.
The relation $\sim$, introduced in [1, p. 140], can be characterized by the fact that $a \sim b$ if the $(a, b)$-entry of $L = \sum_{i=0}^{m} L_i$ is $\neq 0$.

If $u \gamma = e$, then

$$L = \begin{pmatrix}
1 & 0 & 0 & v_3 \\
0 & v_3 + 1 & 0 & 0 \\
0 & 0 & v_3 + 1 & 0 \\
1 & 0 & 0 & v_3
\end{pmatrix}. $$

Hence the quotient scheme of $(X, R)$ is a 2-scheme, which is easily seen to be non-symmetric (calculate its parameters).

Let $X_1, X_2, \ldots, X_h$ be the classes of the equivalence relation $R_0 \cup R_3$ and let $|X_m| = v_0 + v_3 = g$ for $m = 1, 2, \ldots, h$. Let $\mathcal{S}_0 = R_0 \cup R_3$, $\mathcal{S}_1 = R_1$, and $\mathcal{S}_2 = R_2$. Put $X = \{X_1, X_2, \ldots, X_h\}$ and define on $X$ the relations $\tilde{R}_0, \tilde{R}_1$ and $\tilde{R}_2$ as follows:

$$(X_i, X_j) \in \tilde{R}_a \text{ iff for some } x_i \in X_i \text{ and some } y_j \in X_j \text{ we have } (x_i, y_j) \in \mathcal{S}_a.$$ 

**Observation.** If $x_i, y_j \in X_i$ and $y_j, y_j \in X_i$, then $(x_i, y_j) \in \mathcal{S}_a$.

If $\tilde{R} = \{\tilde{R}_0, \tilde{R}_1, \tilde{R}_2\}$ then, by definition, $(X, \tilde{R})$ is the quotient scheme of $(X, R)$ with respect to $R_0 \cup R_3$. So $(X, \tilde{R})$ is a non-symmetric 2-scheme. Let $\tilde{D}_0 = I_h$, $\tilde{D}_1$ and $\tilde{D}_2$ be the adjacency matrices of $(X, \tilde{R})$.

By the above observation, it follows that the adjacency matrices $D_i (i = 0, 1, 2, 3)$ of $(X, R)$ can be put into the following block form:

$$D_i = \tilde{D}_i \otimes I_a \quad D_1 = D_1 \otimes (J_k - I_k) \quad D_2 = D_2 \otimes (J_k - I_k) \quad D_3 = D_3 \otimes (J_k - I_k).$$

Hence $(X, R)$ is the restricted Kronecker product of the non-symmetric 2-scheme $(X, \tilde{R})$ and a 1-scheme on $g$ elements.

If $\alpha = \beta$, then

$$\tilde{L} = \begin{pmatrix}
1 & 0 & 0 & v_3 \\
0 & \delta + 1 & \varepsilon & 0 \\
0 & \varepsilon & \delta + 1 & 0 \\
1 & 0 & 0 & v_3
\end{pmatrix},$$

implying (4). \hfill $\square$

## 6. Feasibility

In [6, 8] the following theorem has been shown.

**Theorem 6.1.** Let $(X, \tilde{R})$ be an imprimitive symmetric 2-scheme. Then the conditions stated below are necessary conditions in order that $(X, \tilde{R})$ can be split into a (imprimitive) non-symmetric 3-scheme:

1. $\tilde{u}_a \equiv 0 \pmod{2}$.
2. $\tilde{u}_a \equiv 0 \pmod{2}$.
3. $\tilde{P}_a(N) = 0 \pmod{2}$.
4. $\tilde{P}_a(N) = 0 \pmod{2}$.
5. $\tilde{P}_a(N) = 0 \pmod{4}$.
6. $-\tilde{P}_a \equiv \tilde{P}_a \equiv \tilde{P}_a \equiv \tilde{P}_a$.
7. $-\tilde{Q}_a \equiv \tilde{Q}_a \equiv \tilde{Q}_a \equiv \tilde{Q}_a$.
8. $-\tilde{Q}_a \equiv \tilde{Q}_a \equiv \tilde{Q}_a \equiv \tilde{Q}_a$.

**Definition 6.2.** Let $(X, \tilde{R})$ be an imprimitive symmetric 2-scheme. Then it is said that the splitting of $(X, \tilde{R})$ into a non-symmetric 3-scheme according to one of the
cases I, II, III or IV is feasible if the parameters of \((X, R)\) satisfy the conditions mentioned for the case concerned in Theorem 6.1.

Let \((X, \bar{R})\) be a symmetric 2-scheme. Then it is said that the splitting of \((X, \bar{R})\) into a non-symmetric 3-scheme \((X, R)\) is realizable if \((X, R)\) exists.

The conditions mentioned in Theorem 6.1 are called the feasibility conditions.

In the next theorem we consider the feasibility of cases I and IV, whereas in Theorem 7.1 we completely describe the construction for cases II and III.

**Theorem 6.3.** Let \((X, \bar{R})\) be a GD-scheme of type \((g, h)\). Then the following hold:

1. The splitting of \((X, \bar{R})\) according to case I is not feasible.

2. The conditions for the feasibility of the splitting of \((X, \bar{R})\) according to case IV are:

\[ g = h = 0 \mod 2 \quad \text{and} \quad h - 1 = 0 \mod g - 1. \]

**Proof.** In case I, \(\bar{P}_1(2) = \bar{P}_1(N) = -1 \neq 0 \mod 2\) and so the splitting according to case I is not feasible.

Now we consider case IV. \(\bar{P}_1(1) = \bar{P}_1(N) = -g\), so by condition (3) of Theorem 6.1 we must have \(g = 0 \mod 2\). Also, \(\bar{\mu}_2 = h(g - 1) = 0 \mod 2\) (condition (2)) and so \(h = 0 \mod 2\), necessarily. Now one sees that the condition (5) is also fulfilled.

Condition (4) becomes \(\bar{P}_2 = 0 \mod 2\) and therefore \(g(h - 1) = 0 \mod 2\), but this implies \(h - 1 = 0 \mod g - 1\).

The conditions (6) and (7) are trivially fulfilled, while condition (8) becomes \(-h - 1 \leq -1 \leq h - 1\), which is satisfied by definition. \(\square\)

It is easily checked that for imprimitive non-symmetric 3-schemes the Neumaier conditions (see [1, Theorem II.4.8]) imply no new restrictions for such schemes.

7. CONSTRUCTION FOR CASES II AND III

**Theorem 7.1.** Let \((X, R)\) be a non-symmetric 3-scheme. Then the following hold with \(g, h \in \mathbb{N}\setminus\{0, 1\}\):

1. \((X, R)\) is the direct sum of \(h\) non-symmetric 2-schemes on \(g\) elements iff \(\delta = \varepsilon\). In this instance \((X, R)\) is a splitting according to case II of the GD-scheme of type \((g, h)\), \(g = 3 \mod 4\) and \((X, R)\) has the parameters \(v = gh\), \(v_1 = \frac{1}{2}(g - 1)\), \(\alpha = \frac{1}{4}(g - 3)\) and \(\beta = \frac{1}{4}(g + 1)\).

2. \((X, R)\) is the restricted Kronecker product of a non-symmetric 2-scheme on \(h\) elements and the 1-scheme on \(g\) elements iff \(\omega\gamma = \varepsilon\). In this instance \((X, R)\) is a splitting according to case III of a GD-scheme of type \((g, h)\), \(h = 3 \mod 4\) and \((X, R)\) has the parameters \(v = gh\), \(v_1 = \frac{1}{2}g(h - 1)\), \(\alpha = \frac{1}{4}g(h - 3)\) and \(\beta = \frac{1}{4}g(h + 1)\).

**Proof.** The theorem follows directly from Theorems 5.5 and 4.3, the structure of the non-symmetric 2-schemes and, for the parameters, from Theorem 2.6. \(\square\)

**Corollary 7.2.** The following hold:

1. A scheme as described in case (1) of Theorem 5.5 exists iff a skew-Hadamard matrix of order \(g + 1\) exists. Two schemes, which are splittings according to case II of the GD-scheme of type \((g, h)\), are isomorphic iff the corresponding Hadamard matrices are \(G\)-equivalent.
A scheme as described in case (2) of Theorem 5.5 exists iff a skew-Hadamard matrix of order \( h + 1 \) exists. Two schemes, which are splittings according to case III of the GD-scheme of type \((g, h)\), are isomorphic iff the corresponding Hadamard matrices are \( G \)-equivalent.

**Proof.** The proof is a direct consequence of the structure of non-symmetric 2-schemes. \(\square\)

Note that a splitting according to case II of the GD-scheme of type \((g, h)\) and a splitting according to case III of the GD-scheme of type \((h, g)\) are formally dual.

8. Constructions for Case IV

Case (4) of Theorem 5.5 \((\alpha = \beta)\) seems more difficult to tackle: for example, note that by Theorem 5.5 one cannot apply the method of construction used in Theorem 7.1 to the present case.

As \(\alpha = \beta\) we are now considering the splitting of a GD-scheme of type \((g, h)\) according to case IV (Theorem 4.3). By Theorem 6.3, \(2 \leq g \leq h\) holds. We shall here discuss the cases \(g = 2\) and \(g = h\). For \(2 < g < h\) no constructions or non-existence theorems have yet been found (the ‘first’ pair is \((g, h) = (4, 10)\)).

We have

\[
v = gh, \quad v_1 = \frac{1}{2}g(h - 1), \quad \alpha = \beta = \frac{1}{2}g(h - 2) \quad \text{and} \quad u = \frac{g(h - 1)}{2(g - 1)}.
\]

The next theorem is an adaptation to the present case of a theorem shown in [6, 7].

**Theorem 8.1.** A splitting according to case IV of the GD-scheme \((X, \tilde{R})\) of type \((g, h)\) is realizable iff there are two matrices \(D_1\) and \(D_2\) of order \(v\) and with entries 0 and 1 such that:

1. \(D_2 = D_1^T\);
2. \(\tilde{D}_2 = D_1 + D_2\);
3. \((D_1 + D_2)(D_1 - D_2) = 0\);
4. \((D_1 - D_2)(D_1 - D_2)^T = g(h - 1)D_0 - \frac{g(h - 1)}{g - 1}D_1\).

Since \(\tilde{D}_2 = (J_n - I_n) \otimes J_g\), there are, for \(i, j \in \{1, 2, \ldots, h\}\), square matrices \(A_{ij}\) of order \(g\), \(A_{ii} = 0\) for all \(i\) and \(A_{ij}\) has entries \(\pm 1\) if \(i \neq j\) such that

\[
D_1 - D_2 = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1h} \\
\vdots & \ddots & \ddots & \vdots \\
A_{h1} & A_{h2} & \cdots & A_{hh}
\end{pmatrix}.
\]

Note that by (3) of Theorem 8.1 we have \(A_{ij}J_g = J_gA_{ij} = 0\) for \(i, j = 1, 2, \ldots, h\), while the fourth condition of Theorem 8.1 can be rewritten as \((D_1 - D_2)(D_1 - D_2)^T = 2u[I_h \otimes (gI_g - J_g)]\).

The next lemma is applicable to all splittings of a GD-scheme according case IV. The lemma can be checked by straightforward calculation.
**Lemma 8.2.** Let $(X, R)$ be an imprimitive non-symmetric 3-scheme, which is the splitting according to case IV of a GD-scheme of type $(g, h)$. If

$$H = D_1 - D_2 + \left( I_h \otimes \frac{h-1}{g-1} J_g \right)$$

then, $HH^T = H^TH = 2gul_{gh}$.

**Corollary 8.3.** In the setting of Lemma 8.2, $H$ is an Hadamard matrix iff $g = h$. Two imprimitive non-symmetric 3-schemes, which are splittings according to case IV of the GD-scheme of type $(g, g)$, are isomorphic iff the corresponding Hadamard matrices are $G$-equivalent.

The Hadamard matrix in Corollary 8.3 has to be a Hadamard matrix of a special block form. Hence the case $g = h$ has been reduced to the problem of finding Hadamard matrices of a special form. This is considered in [9]. In [5] a infinite family of non-symmetric 3-schemes which are splittings of GD-schemes of type $(4, 4')$ are constructed: the non-symmetric cyclotomic 3-schemes over 1-rings.

In the next lemma we consider the case that in (1), $A_{ij} = \pm A$ for a certain fixed matrix $A$.

**Lemma 8.4.** If we assume $D_1 - D_2 = M \otimes A$ for certain square matrices $M$ and $A$, $M$ with main diagonal entries 0 and the other entries $\pm 1$ and $A$ with all entries $\pm 1$, then $g = 2$, $M + I_h$ is a skew-Hadamard matrix and

$$A = \pm \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}.$$

**Proof.** We have

$$(D_1 - D_2)(D_1 - D_2)^T = MM^T \otimes AA^T = 2u(I_h \otimes (gg_i - J_g)).$$

If $MM^T = (k_{ij})$, then the $(i, j)$-th block entry of $MM^T \otimes AA^T$ is $k_{ij}AA^T$. Hence $k_{ii}AA^T = 2u(gg_i - J_g)$ so $AA^T \neq 0$ and $k_{ii} = k_{11}$ for all $i$. Also, $k_{ij}AA^T = 0$ for $i \neq j$; hence $k_{ij} = 0$ for $i \neq j$, implying $MM^T = k_{11}I_h = (h-1)I_h$. From this formula we derive $AA^T = [g/(g-1)](gg_i - J_g)$, yielding $g = 2$.

Therefore $A$ is a $(2 \times 2)$-matrix of rank 1 with elements $\pm 1$ and so $A$ has the prescribed form. $D_1 - D_2 = -(D_1 - D_2)^T$ implies that $M + I_h$ is skew-Hadamard. This completes the proof of the lemma.

**Theorem 8.5.** Let $(X, R)$ be a GD-scheme of type $(2, h)$. Then $(X, R)$ can be split according to case IV iff a skew-Hadamard matrix of order $h$ exists.

In the case that a splitting exists and $H$ is a skew-Hadamard matrix of order $h$, while $D_0 = I_{2h}$, $D_1 = \hat{D}_1$, $D_1 + D_2 = \hat{D}_2$ and

$$D_1 - D_2 = (H - I_h) \otimes \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix},$$

then $\{D_0, D_1, D_2, D_3\}$ represents the resulting non-symmetric 3-scheme.

Two imprimitive non-symmetric 3-schemes, which are splittings according to case IV of a GD-scheme of type $(2, h)$, are isomorphic iff the corresponding Hadamard matrices are $G$-equivalent.

**Proof.** The ‘if’-part of the theorem is easily verified.
For the ‘only if’-part, note that since $g = 2$, the $A_{ij}$ ($i \neq j$) all must have the form $\pm A$, where $A$ is given in Lemma 8.4. But now Lemma 8.4 implies the desired result.

The last part of the theorem is evident. □

Using $g = h = 2$ in Theorem 8.5 one finds the non-symmetric 3-scheme with the smallest number of elements possible. The graph of the first relation is a digraph on 4 elements.

9. Final Remarks

By the results of [6–8] and of this paper, we have the following situation, as far as the existence of non-symmetric 3-schemes (primitive or imprimitive) is concerned:

1. There exists no non-symmetric 3-scheme if $v$ is prime.
2. For composite $v$ within the range $2 \leq v \leq 50$, no non-symmetric 3-schemes exist for $v = 10, 20, 25, 26, 34$.
3. There exist no imprimitive non-symmetric 3-schemes for $v = 50$, while the case for primitive ones is undecided.
4. For every value of $v$ within the range $2 \leq v \leq 50$, not excluded above, there exists at least one imprimitive non-symmetric 3-scheme.
5. If $2 \leq v < 50$ only for $v = 36$ does there exist a primitive non-symmetric 3-scheme, which is the splitting of a scheme of type $NL_2(6)$, while the splitting of a scheme of type $L_3(6)$ is still undecided.

In [11] it is conjectured that (in our terminology) a GD-scheme of type $(2, h)$ can be split iff $h = 2$ or $h = 0, 3 \pmod 4$. The results of this paper imply that:

1. for $h = 2$ and $h = 0 \pmod 4$ such a splitting exists iff a skew-Hadamard matrix of order $h$ exists;
2. for $h = 3 \pmod 4$ such a splitting exists iff a skew-Hadamard matrix of order $h + 1$ exists; and
3. for $h = 1, 2 \pmod 4$ but $h \neq 2$, such a splitting does not exist.

Since 1-schemes and non-symmetric 2-schemes exist, it is easy to see that by repeatedly using the restricted Kronecker product and the direct sum of schemes one can prove the next theorem.

**Theorem 9.1.** Let $m \in \mathbb{N}\setminus\{0\}$ and let $r \in \mathbb{N}$ such that $r \leq \frac{1}{2}m$. Then there exist $m$-schemes with exactly $2r$ non-symmetric relations.

For non-symmetric 3-schemes we have used this technique in Theorem 7.1. The schemes found in Theorem 9.1 all are necessarily imprimitive. However, note that using this method one does not find all imprimitive schemes. For example, the imprimitive non-symmetric 3-schemes with $\alpha = \beta$ cannot be found in this simple way from 1- and 2-schemes.

Theorem 9.1 leads to the following question.

**Question.** Do there exist primitive $m$-schemes with exactly $2r$ non-symmetric relations for given $m \in \mathbb{N}\setminus\{0\}$ and $r \in \mathbb{N}$ such that $r \leq \frac{1}{2}m$?

$m = 1$ is evident. In [5] we have shown, using cyclotomic schemes over finite fields, the existence of primitive schemes $m$-schemes with either $m$ is odd and $r = 0$ or $m$ is even and $r = 0, \frac{1}{2}m$. The case $m = 3$ and $r = 1$ has been solved by the results of [10] and [7]. In the latter paper we constructed a primitive non-symmetric 3-scheme on 36 elements.
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