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Exponential Families of Non-Isomorphic Triangulations of Complete Graphs

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We prove that the number of non-isomorphic face 2-colourable triangulations of the complete graph K_n in an orientable surface is at least $2^{n^2/54 - O(n)}$ for n congruent to 7 or 19 modulo 36, and is at least $2^{2n^2/81 - O(n)}$ for n congruent to 19 or 55 modulo 108. © 2000 Academic Press

1. INTRODUCTION

It is known [5] that a complete graph, K_n, triangulates some orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$. It triangulates some non-orientable surface if and only if $n \equiv 0$ or $1 \pmod 3$, $n \geqslant 6$ and $n \neq 7$. For the embedding to be face 2-colourable it is necessary for the vertex degrees to be even and, consequently, for n to be odd. As pointed out in [1], at least one face 2-colourable orientable embedding does exist for each $n \equiv 3$ or $7 \pmod{12}$; the case $n \equiv 3$ is dealt with in Ringel's book [5] and the case $n \equiv 7$ is dealt with by material in Youngs' paper [8]. The proof techniques employed by these authors use the theory of current and voltage graphs. Face 2-colourable embeddings are of particular interest because the sets of three vertices which form the triangles in each of the two colour classes

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themselves form two Steiner triple systems of order n, STS(n)s. We recall here that an STS(n) may be formally defined as being an ordered pair $(V, \mathscr{B})$, where V is an n-element set (the points) and $\mathscr{B}$ is a family of 3-element subsets of V (the blocks) such that every 2-element subset of V appears in precisely one block.

In two earlier papers [1, 2], three of the present authors presented constructions of face 2-colourable triangular embeddings of K_n for various values of n. These constructions focused much more closely on the design-theoretical aspect of the problem than did the earlier work of Ringel and Youngs. Based partly on these constructions and partly on the existing work of Ringel and Youngs, two non-isomorphic triangular embeddings of K_n in an orientable surface are given in [1] for each $n \equiv 7 \pmod{12}$ and $n \neq 7$, and two non-isomorphic triangular embeddings of K_n in a non-orientable surface are given for half (in arithmetic set-density terms) of the residue class $n \equiv 1 \pmod 6$. In each case one of the two embeddings is face 2-colourable and the other is not.

It appears that remarkably few examples are known of non-isomorphic triangular embeddings of K_n. The paper [4] (see also [6]) gives three non-isomorphic orientable triangulations for $n = 19$, two of which are not face 2-colourable. The third of these, together with a further seven non-isomorphic orientable triangulations (all eight of which have a cyclic automorphism of order 19 and are face 2-colourable) are given in [2]. For $n = 31$, [2] gives seven non-isomorphic face 2-colourable orientable triangulations of the complete graph, and it is there remarked that computational evidence suggests the existence of many more. However, to the best of our knowledge no other explicit examples have been given of non-isomorphic triangular embeddings of K_n and there has been no lower bound established (other than the figures given above) for the number of non-isomorphic triangular embeddings.

The primary purpose of this paper is to establish that for $n \equiv 7$ or $19 \pmod{36}$, there are at least $2^{n^2/54 - O(n)}$ non-isomorphic triangular embeddings of K_n in an orientable surface, all of which are face 2-colourable. When $n \equiv 19$ or $55 \pmod{108}$ this estimate can be increased to $2^{2n^2/81 - O(n)}$. We also establish a similar estimate for non-orientable embeddings when $n \equiv 1$ or $7 \pmod{18}$ (and an improved estimate in the cases when $n \equiv 1$ or $19 \pmod{54}$).

In the remainder of this paper, when we speak of isomorphisms and automorphisms we will restrict ourselves to colour-preserving mappings; this makes statements of some of the results a little simpler and in counting the number of non-isomorphic systems there is only a factor 2 involved in moving between colour-preserving mappings and non-colour-preserving mappings. In the course of the proof we will exhibit collections of such embeddings with a range of (to us) extraordinary and unexpected properties.

Henceforth we will use the term *2to-embedding* to refer to a face 2-*colourable triangular* embedding in an *orientable* surface. The colour classes will be called "black" and "white". We refer the reader to [1, 2] for reviews of the basic facts about graph embeddings, the connection with biembeddings of Steiner triple systems, and for some of the historical background.

The recursive construction which appears in Theorems 1 and 2 of [1] in a topological form and again in Theorems 2 and 3 of [2] in a design-theoretical form takes a 2to-embedding of K_n and produces a 2to-embedding of K_{3n-2}. This construction plays a key role in the current paper. We therefore now give an informal review of this construction and show how it can be further extended in a fashion suitable for our current purposes.

The construction commences with a given 2to-embedding of K_n. We fix a particular vertex z^* of K_n and, from the embedding, we delete z^*, all open edges incident with z^* and all the open triangular faces incident with z^*. The resulting surface S now has a hole whose boundary is an oriented Hamiltonian cycle in $G = K_n - z^* \simeq K_{n-1}$. We next take three disjoint copies of the surface S, all with the same colouring and orientation; we denote these by S^0, S^1 and S^2, and use superscripts in a similar way to identify corresponding points on the three surfaces. For each white triangular face (uvw) of S, we "bridge" S^0, S^1 and S^2 by gluing a torus to the triangles $(u^i v^i w^i)$ for $i = 0, 1, 2$ in the following manner. We take a 2to-embedding in a torus of the complete tripartite graph $K_{3,3,3}$ with the three vertex parts $\{u^i\}$, $\{v^i\}$ and $\{w^i\}$ and with black faces $(u^i w^i v^i)$, for $i = 0, 1, 2$ (see Fig. 1). The orientation of the torus must induce the opposite cyclic permutation of $\{u^i, v^i, w^i\}$ to that induced by the surfaces S^i; this is important for the integrity of the gluing operation where black faces $(u^i w^i v^i)$ on the torus are glued to the white faces $(u^i v^i w^i)$ on S^0, S^1 and S^2 respectively.

After all the white triangles have been bridged we are left with a new connected triangulated surface with a boundary. We denote this surface by $\hat{S}$. It has $(3n-3)$ vertices and the boundary comprises three disjoint cycles, each of length $(n-1)$. In order to complete the construction to obtain a 2to-embedding of K_{3n-2} we must construct an auxiliary triangulated bordered surface $\bar{S}$ and paste it to $\hat{S}$ so that all three holes of $\hat{S}$ will be capped. To do this, suppose that $D = (u_1 u_2, ..., u_{n-1})$ is our oriented Hamiltonian cycle in $G = K_n - z^*$. Since n is odd, every other edge of D is incident with a white triangle in S; let these edges be $u_2 u_3, u_4 u_5, ..., u_{n-1} u_1$.

The surface $\bar{S}$ has, as vertices, the points u_j^i for $i = 0, 1, 2$ and $j = 1, 2, ..., n-1$ together with one additional point which we here call ∞. Suppose initially that $n \equiv 3 \pmod{12}$. We may then construct $\bar{S}$ from the oriented triangles listed below (Table 1). The reason for the classification of the triangles into types 1 and 2 will become apparent shortly. Precisely how

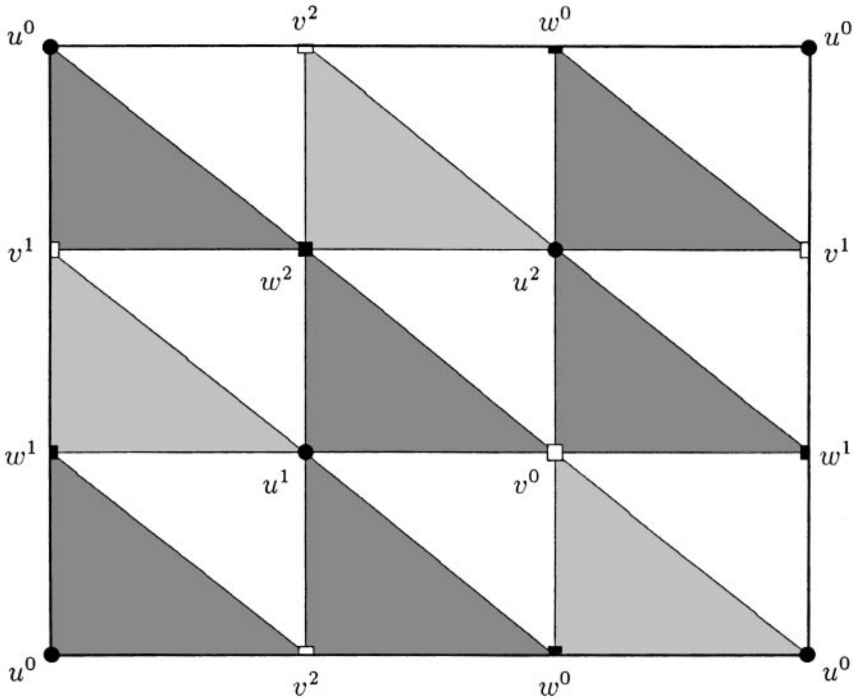

FIG. 1. Toroidal embedding of $K_{3,3,3}$.

$\bar{S}$ is constructed is described in more detail in [1] where it is also proved that the final graph that triangulates the final surface is indeed a complete graph of order $3n-2$.

The significance of the condition $n \equiv 3 \pmod{12}$ is that it ensures that the resulting surface is a closed surface and not a pseudosurface. (A pseudosurface is obtained from a collection of closed surfaces by making a finite number of identifications, each of finitely many points, so that the resulting topological space is connected.) Equivalently, it ensures that the point ∞ has a single cycle of $3n-3$ points surrounding it and not a union of shorter cycles. As it appears above, the construction does not work in the case $n \equiv 7 \pmod{12}$; however we can modify the construction by taking a single value of $j \in \{1, 3, 5, ..., n-2\}$ and applying a "twist" to the type 1 triangles associated with this value of j. To do this we replace them by those shown in Table 2.

Again, for an explanation of why this works, see [1]. It is also there remarked that we may apply any number, say k, of such twists provided that, if $n \equiv 3 \pmod{12}$ we select $k \equiv 0$ or $1 \pmod 3$, while if $n \equiv 7 \pmod{12}$ we select $k \equiv 1$ or $2 \pmod 3$.

TABLE 1

Subscript Arithmetic Cycles Modulo $n-1$

Type 1 oriented triangles ($j=1, 3, 5, ..., n-2$)	
White	Black
$(u_j^0 u_{j+1}^0 u_{j+1}^2)$	$(u_j^0 u_j^2 u_{j+1}^1)$
$(u_j^1 u_{j+1}^1 u_{j+1}^0)$	$(u_j^1 u_j^0 u_{j+1}^2)$
$(u_j^2 u_{j+1}^2 u_{j+1}^1)$	$(u_j^2 u_j^1 u_{j+1}^0)$
$(u_j^0 u_j^1 u_j^2)$	$(u_{j+1}^0 u_{j+1}^1 u_{j+1}^2)$
$(u_j^0 u_{j+1}^1 \infty)$	
$(u_j^1 u_{j+1}^2 \infty)$	
$(u_j^2 u_{j+1}^0 \infty)$	
Type 2 oriented triangles ($j=1, 3, 5, ..., n-2$)	
	Black
	$(u_{j+1}^0 u_{j+2}^0 \infty)$
	$(u_{j+1}^1 u_{j+2}^1 \infty)$
	$(u_{j+1}^2 u_{j+2}^2 \infty)$

We now make two new observations about the construction which enable us to extend it. The proof of the original construction given in [1] continues to hold good for the extended version with minor and obvious modifications.

Firstly, the toroidal embedding of $K_{3,3,3}$ given in Fig. 1 may be replaced by one in which the cyclic order of the three superscripts is reversed. The reversed embedding of $K_{3,3,3}$ is isomorphic with the original but is labelled

TABLE 2

Oriented triangles	
White	Black
$(u_j^0 u_{j+1}^0 u_{j+1}^1)$	$(u_j^0 u_j^1 u_{j+1}^2)$
$(u_j^1 u_{j+1}^1 u_{j+1}^2)$	$(u_j^1 u_j^2 u_{j+1}^0)$
$(u_j^2 u_{j+1}^2 u_{j+1}^0)$	$(u_j^2 u_j^0 u_{j+1}^1)$
$(u_j^0 u_j^2 u_j^1)$	$(u_{j+1}^0 u_{j+1}^2 u_{j+1}^1)$
$(u_j^0 u_{j+1}^2 \infty)$	
$(u_j^1 u_{j+1}^0 \infty)$	
$(u_j^2 u_{j+1}^1 \infty)$	

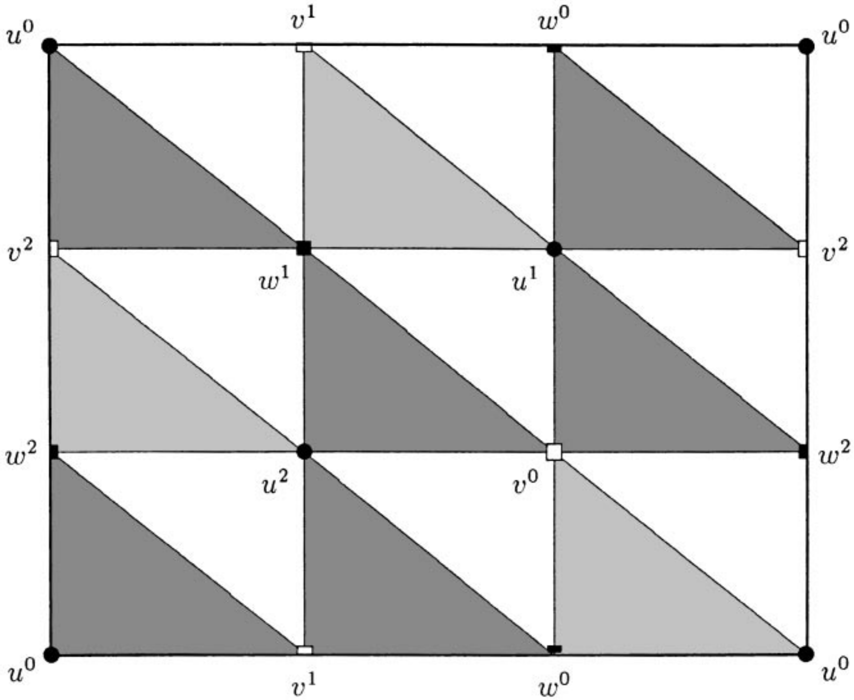

FIG. 2. Reversed toroidal embedding of $K_{3,3,3}$.

differently (see Fig. 2). For each original white triangular face (uvw) of S we may carry out the bridging operation across S^0, S^1, S^2 using either the original $K_{3,3,3}$ embedding or the reversed embedding. The choice of which of the two $K_{3,3,3}$ embeddings to use can be made *independently* for each white triangle (uvw).

Secondly, it is not necessary for S^0, S^1 and S^2 to be three copies of the same surface S. All that the construction requires is that the three surfaces have the "same" white triangular faces and the "same" cycle of $(n-1)$ points around the border, all with the "same" orientations. To be more precise, by the term "same" we mean that there is a mapping from the vertices of each surface onto the vertices of each of the other surfaces which preserves the white triangular faces, the border and the orientation. The sceptical reader may feel dubious that we can satisfy this requirement without in fact having three identically labelled copies of a single surface S. However, we shall see that not only is it possible to arrange this by other means but it can often be done in a very large number of ways. We continue to use the notation x^0, x^1, x^2 to denote corresponding points on the three surfaces.

Throughout the remainder of this paper we shall use the term "the construction" to refer to the most general form of our construction allowing the possibility of:

(a) different surfaces S^0, S^1 and S^2 (with the "same" white triangles, etc.),

(b) use of either of the two labelled toroidal embeddings of $K_{3,3,3}$ independently for each white triangle (uvw), and

(c) use of k twists in constructing the cap $\bar{S}$ for any value of k satisfying the admissibility condition modulo 3.

We shall have occasion to use the term "Pasch configuration" in connection with Steiner triple systems. A Pasch configuration is a set of four blocks whose union has cardinality six, i.e. a set of four triangles isomorphic to $\{\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}\}$.

2. THE MAIN RESULTS

Suppose that we have a *particular* 2to-embedding of K_{3n-2} obtained from the construction. Our first goal is to show that we can identify the point ∞. We then show that it is also possible to identify the entire cap $\bar{S}$ and the three surfaces S^0, S^1 and S^2. In order to identify ∞ we specify a property which is shared by all vertices other than ∞. To do this we consider the following operation and its result.

Let $\{X, Y, Z\}$ be a non-facial triangle on a face 2-coloured triangulated surface. (We are not concerned about its orientation.) By *severing* this triangle, we mean that the surface is cut along the edges XY, YZ, and ZX. Our interest focuses on the arrangement of edges and coloured facial triangles incident with the points X, Y and Z on either side of the cut. We will say that the non-facial triangle $\{X, Y, Z\}$ gives *configuration* C if, when severed, it has on one side of the cut the following arrangement of edges and facial triangles:

(a) including the edges XY, YZ, and ZX, precisely six edges emanate from X, from Y and from Z,

(b) these edges define faces, in sequence, around each of the three vertices which are coloured white, black, white, black, white respectively.

The configuration C is illustrated in Fig. 3.

We now say that a point X of the K_{3n-2} embedding has *property P* if there exists a non-facial triangle $\{X, Y, Z\}$ giving configuration C.

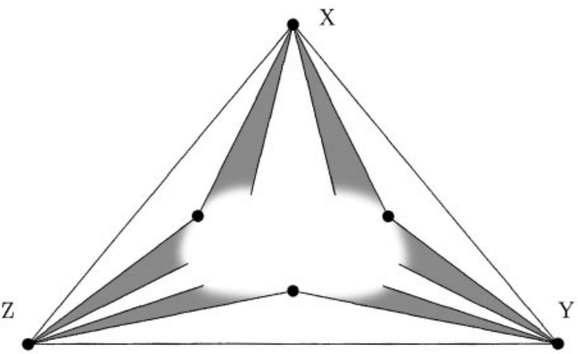

FIG. 3. Configuration C.

LEMMA 1. *If $X \neq \infty$, then X has property P.*

Proof. Since $X \neq \infty$, $X = x^i$ for $i = 0, 1$ or 2. Pick an original white triangle (xuv), i.e. $(x^j u^j v^j)$ is a white triangle on S^j for each $j = 0, 1$, or 2. Then the triangle $\{x^i, u^i, v^i\}$ is non-facial in the K_{3n-2} embedding and, from Figs. 1 and 2, we see that x^i has property P. (Note that this is the case whether or not the $K_{3,3,3}$ bridge applied to (xuv) is as shown in Fig. 1 or is its reverse as shown in Fig. 2.) ∎

LEMMA 2. *The point ∞ does not have property P.*

Proof. Suppose that ∞ does have property P, so that there is a non-facial triangle $\{\infty, X, Y\}$ giving configuration C. Suppose firstly that, using the lettering given in the Introduction, $X = u_j^i$ for some $j \in \{1, 3, 5, ..., n-2\}$ and $i = 0, 1$ or 2. Then, following the sequence of facial triangles about ∞ given in Tables 1 and 2 above, we find that $Y = u_{j+5}^h$ for some value of $h = 0, 1$ or 2. But then following the sequence of facial triangles about Y we find $X = u_{j+4}^h$. Thus we obtain $u_j = u_{j+4}$, which is a contradiction because $n \neq 5$. The second alternative is that $X = u_j^i$ for some $j \in \{2, 4, 6, ..., n-1\}$ and $i = 0, 1$ or 2. In this case we obtain $Y = u_{j-5}^h$ for some value of $h = 0, 1$ or 2. Then, by reversing the roles of X and Y, this alternative reduces to the former case and again provides a contradiction. (For the reader who is happier with rotation schemes, it may be helpful to examine the rotation schemes about the points ∞, a_i and b_i given in Theorem 2 of [2] but note that these need amendment as described in Theorem 3 of that paper for any twists in the construction.) ∎

Having now identified the point ∞ in the given K_{3n-2} embedding we can proceed to identify the entire cap $\bar{S}$ and consequently the surface $\hat{S}$. To do this, start with an arbitrary white face containing ∞. Label the other two vertices of this triangle 1 and 2. Using the orientation established by this labelling, label the remaining vertices around ∞ with integers 3, 4, ..., $3n-3$.

Note that the white triangles incident with ∞ are $\{\infty, 1, 2\}$, $\{\infty, 3, 4\}$, ..., $\{\infty, 3n-4, 3n-3\}$. For each $j \in \{1, 2, ..., (n-1)/2\}$ we will refer to the seven points $\{\infty, 2j-1, 2j, 2j-1+(n-1), 2j+(n-1), 2j-1+2(n-1), 2j+2(n-1)\}$ as the *jth crevice*. Apart from their numbering, the crevices are defined independently of the choice of starting vertex (i.e. the vertex numbered 1) and the orientation of the rotation about ∞.

As a consequence of the construction, the points of each crevice define seven white facial triangles forming an STS(7). For each of the three of these white triangles which are incident with ∞, the points of the crevice define a neighbouring black triangle not containing ∞. The points of the crevice further define a unique fourth black triangle forming a Pasch configuration with the other three. In effect the crevices generate the type 1 triangles given in Table 1 (or the alternative triangles given in Table 2).

If we now remove all eleven of these (open) triangles for each crevice, together with all (open) black triangles incident with ∞, all (open) edges forming the common boundary to any two of these triangles, and the point ∞ itself, then we will have removed the cap (i.e. $\bar{S}$) and we obtain the surface $\hat{S}$. It follows that the surface with which we are now left is a bordered surface, the border comprising three disjoint oriented cycles each of length $n-1$. We now wish to recover the three original surfaces S^0, S^1 and S^2; to do this we define *levels*. We say that the points X, Y, $(\neq \infty)$ are on the same level if they lie on the same $(n-1)$-cycle. There are therefore three levels, each consisting of $(n-1)$ points. We may label these levels A, B, and C. The points at level A necessarily *are* the points $\{x_j^i : j \in \{1, 2, ..., n-1\}\}$ for one value of $i \in \{0, 1, 2\}$; and likewise for B and C.

Again, from the construction, two points x^i, y^i at the same level which are not adjacent on the $(n-1)$-cycle boundary will define a black triangle, say $(x^i y^i z^i)$, where z^i also lies on the same level. The two points will also define a white triangle, say $(y^i x^i w^h)$, $h \neq i$. The surface S^i may now be reconstructed from these black triangles $(x^i y^i z^i)$ and the derived white triangles $(y^i x^i w^i)$. Note that triangles with an edge forming part of the boundary are covered by this process since the other two edges will not lie on the boundary. Thus we may recover the three original surfaces S^0, S^1 and S^2 (although their labelling as 0, 1, and 2 is indeterminate).

We will now pay attention to the levels of the points encountered in traversing the $(3n-3)$-cycle around the point ∞ in any given K_{3n-2} embedding obtained from the construction. The construction ensures that we obtain a (circular, ordered) list of the following form.

points	1	2	3	4	5	6	$\cdots$	$3n-4$	$3n-3$
levels	a_0	a_1	a_1	a_2	a_2	a_3	$\cdots$	$a_{(3n-5)/2}$	a_0

where the letters a_i identify the levels of the points to which they correspond, i.e. each a_i is one of 0, 1 or 2. Moreover, $a_i \neq a_{i+1}$ (subscript arithmetic modulo $(3n-3)/2$).

We may firstly compress this list of levels to a circular list of $(3n-3)/2$ symbols by omitting every other entry. We may then derive a circular list of $(3n-3)/2$ 0's and 1's from this list of levels by recording for each adjacent pair of levels (a_i, a_{i+1}) a "0" if $a_{i+1} \equiv a_i - 1 \pmod 3$ or a "1" if $a_{i+1} \equiv a_i + 1 \pmod 3$. It follows from the construction method that this list will be periodic with a period $(n-1)/2$ (each crevice is encountered three times in the $(3n-3)$-cycle). We now take $(n-1)/2$ consecutive terms from this list of 0's and 1's. We will call this (circular) list of $(n-1)/2$ 0's and 1's a *twist* list; it records the pattern of twists in a K_{3n-2} embedding as described in the Introduction. Not all strings of $(n-1)/2$ 0's and 1's can arise from the construction. In fact, if k is the number of 1's in such a list then, as previously mentioned, for $n \equiv 3 \pmod{12}$ we require $k \equiv 0$ or $1 \pmod 3$, while for $n \equiv 7 \pmod{12}$ we require $k \equiv 1$ or $2 \pmod 3$.

Two twist lists will be called *equivalent* if one can be obtained from the other by a combination of

(a) rotation (i.e. starting the list at a different position in the cycle),

(b) reversal (i.e. writing the list in the reverse order), and

(c) negation (i.e. permuting the 0 and 1 entries).

Rotation corresponds to choosing a different point to serve as the initial vertex numbered 1, reversal corresponds to reversing the direction of rotation, and negation corresponds to a renumbering of the levels which reverses the cyclic ordering (0, 1, 2). However, an equivalence class of twist lists is an invariant of any K_{3n-2} embedding produced by the construction: the equivalence class is independent of the labelling of points and the choice of orientation.

Suppose now that we choose three fixed initial surfaces S^0, S^1 and S^2 and perform two versions of the construction using a fixed distribution of the two alternative types of $K_{3,3,3}$ bridges but *different* distributions of twists. If the two resulting K_{3n-2} embeddings have non-equivalent twist lists then there can be no colour-preserving isomorphism between them. From this observation alone, and using three copies of the same initial 2to-embedding of K_n to form our three surfaces S^0, S^1 and S^2, it is possible to deduce the existence of exponentially many non-isomorphic 2to-embeddings of K_{3n-2}. However, we can do much better than this.

For $n \geqslant 19$ we now examine those K_{3n-2} embeddings which arise, by varying the selection of $K_{3,3,3}$ bridges, from three fixed initial surfaces

S^0, S^1 and S^2 and a fixed distribution of twists with a (representative) twist list

$$T_0 = (1, 1, 1, 0, 1, 0, 0, 0, 0, a, b, c, ..., z)$$

where *all* of the entries $a, b, c, ..., z$ are zeros (so that T_0 contains at least four consecutive zeros). Note that such a list has four 1's in total and is therefore a valid twist list both for $n \equiv 3$ and for $n \equiv 7 \pmod{12}$. Note also that such a twist list does not map to itself under any combination of the three operations (a), (b), (c) described above, a fact which is important for the subsequent argument. Consequently, if we consider the cycle around ∞, then by choosing an appropriate starting point and direction and an appropriate numbering of the levels, we may assume that the cycle has the form

points	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	···
levels	0	1	1	2	2	0	0	2	2	0	0	2	2	1	1	0	0	···

The pattern of levels ..., 2, 2, 1, 1, 0, 0, ... continues to complete $n-1$ entries in the table and then the entire pattern of rising and falling levels is repeated twice more to form a $(3n-3)$-cycle. The pattern of rising and falling levels determines that there are only three choices for the vertex to be numbered "1" (one at each level). Having chosen the vertex "1", there is only one choice of direction (i.e. which vertex to number "2") and, if we take vertex 1 to define level 0, one choice of subsequent level numbering. It follows that the only possible colour-preserving mappings between a pair of such embeddings are of the form $u_j^i \to u_j^{i+k}$ where k is independent of both i and j, and superscript arithmetic is modulo 3; we will express this property by saying that the mapping cyclically permutes levels.

THEOREM 1. *Suppose that for $n \equiv 3$ or $7 \pmod{12}$ and $n \geqslant 19$ we take three fixed orientable surfaces S^0, S^1 and S^2, and that we choose a fixed labelling of the $3n-3$ points to generate the twist list T_0. If we then apply the construction twice, using two* different *selections of the $K_{3,3,3}$ bridges, then the two resulting 2to-embeddings of K_{3n-2} are non-isomorphic.*

Proof. From the remarks above, we see that the twist list T_0 only permits isomorphisms which cyclically permute levels. If there were an isomorphism between the two embeddings then, because the $K_{3,3,3}$ bridges and the cap $\bar{S}$ are invariant under a cyclic permutation of levels, the two embeddings would have identical white triangles, which they do not because we have used *different* selections of the $K_{3,3,3}$ bridges. Consequently the resulting K_{3n-2} embeddings are non-isomorphic. ∎

There are $(n-1)(n-3)/6$ white triangles on each of the surfaces S^0, S^1 and S^2. Using the construction, we may therefore generate $2^{(n-1)(n-3)/6}$ non-isomorphic 2to-embeddings of K_{3n-2}. Since there is a 2to-embedding of K_n for every $n \equiv 3$ or 7 (mod 12), we may use this to produce the initial surfaces S^0, S^1 and S^2. Consequently we have the following.

COROLLARY 1. *For every $n \equiv 3$ or 7 (mod 12) with $n \geqslant 19$ there are at least $2^{(n-1)(n-3)/6}$ non-isomorphic 2to-embeddings of K_{3n-2}.*

In fact the result also holds for $n = 15$. The argument is similar but uses the twist list $T_1 = (1, 1, 0, 1, 0, 0, 0)$. It is also clear that the only automorphisms of these embeddings are those which cyclically permute the three levels, i.e. the embeddings have C_3 as their (full) automorphism group. ∎

In terms of the order of growth we may state the result in the following form.

COROLLARY 2. *For $n \equiv 7$ or 19 (mod 36) there are at least $2^{n^2/54 - O(n)}$ non-isomorphic 2to-embeddings of K_n.* ∎

We now make an observation about the black triangles of the embeddings generated as described in the above Theorem and the first Corollary. Given any two such embeddings, the black triangles which they contain are identical and have the same orientations. To see this we note that the black triangles come from three sources. Those lying on the surfaces S^0, S^1 and S^2 are unaltered during the construction and therefore are common to both embeddings. Those lying on the $K_{3,3,3}$ bridges are the same whether or not the bridges are reversed (see Figs. 1 and 2). Those lying on the surface $\bar{S}$ are common to both embeddings. It follows that the $2^{(n-1)(n-3)/6}$ non-isomorphic 2to-embeddings of K_{3n-2} each contain identical black triangles with the same orientations. In particular, the STS($3n-2$) defined by the black triangles is identical for each of the $2^{(n-1)(n-3)/6}$ non-isomorphic embeddings. We find these observations startling. Furthermore, we can put them to good use.

We now take these $2^{(n-1)(n-3)/6}$ non-isomorphic embeddings of K_{3n-2} and reverse the colours. From each, we then delete the point ∞ together with all (open) edges and all (open) triangular faces incident with ∞. This produces a plentiful supply of non-isomorphic surfaces S^i on which to base a reapplication of the construction to produce 2to-embeddings of K_{9n-8}. All of these surfaces S^i have the "same" white triangles and the "same" Hamiltonian cycle of points forming the border, all with the "same" orientation. We can select three different surfaces from this collection to form

S^0, S^1, S^2 (in some order) in $\binom{N}{3}$ ways, where $N = 2^{(n-1)(n-3)/6}$. We will again use a fixed selection of twists giving rise to the twist list T_0. The $K_{3,3,3}$ bridges may be selected in $2^{(3n-3)(3n-5)/6}$ different ways. Any two of the resulting 2to-embeddings of K_{9n-8} (obtained by varying the surfaces S^0, S^1 and S^2, and the $K_{3,3,3}$ bridges, but with a fixed selection of twists) will be non-isomorphic. To see this, note firstly that embeddings based on two different selections of the surfaces S^0, S^1 and S^2 cannot be isomorphic. For those based on a common selection, T_0 only permits isomorphisms which cyclically permute levels. Because the three surfaces S^0, S^1 and S^2 are not isomorphic, the only possible isomorphism is then the identity mapping. However the use of different selections of the $K_{3,3,3}$ bridges precludes this possibility. We also observe that all of the resulting 2to-embeddings of K_{9n-8} are automorphism-free (i.e. have only the trivial automorphism). Again this follows from the structure of T_0, which only permits automorphisms which cyclically permute levels, together with the fact that S^0, S^1 and S^2 are selected to be non-isomorphic. We may summarise these results in the statement of the following Theorem.

THEOREM 2. *Suppose $n \geqslant 15$ and $n \equiv 3$ or 7 (mod 12). Put $N = 2^{(n-1)(n-3)/6}$. Then there are at least $\binom{N}{3}\, 2^{(3n-3)(3n-5)/6}$ non-isomorphic 2to-embeddings of K_{9n-8}, all of which are automorphism-free.* ▮

In terms of the order of growth we may state the result in the following form.

COROLLARY 3. *For $n \equiv 19$ or 55 (mod 108) there are at least $2^{2n^2/81 - O(n)}$ non-isomorphic 2to-embeddings of K_n.* ▮

3. THE NON-ORIENTABLE CASE

An inspection of the proofs given above shows that, in essence, they apply also to the non-orientable case. We now briefly discuss this aspect. We form S^0, S^1 and S^2 from three face 2-colourable embeddings (having the "same" white triangles and the "same" cycle of points around z^*) of K_n in a non-orientable surface. The white triangles are bridged using the toroidal embeddings given in Figs. 1 and 2. The construction is completed, to form a face 2-coloured triangular embedding of K_{3n-2} in a non-orientable surface, by forming a cap $\bar{S}$ having k twists in the manner previously described. The number k must satisfy the congruence $k \equiv 1$ or 2 (mod 3) if $n \equiv 1$ (mod 6), or $k \equiv 0$ or 1 (mod 3) if $n \equiv 3$ (mod 6).

In any embedding generated by this construction we may, as previously, identify firstly the point ∞, then the cap $\bar{S}$, the surface $\hat{S}$, and finally the original surfaces S^0, S^1, and S^2 (although their labelling as 0, 1 and 2 remains indeterminate). A twist list can be defined as before and we can then examine those K_{3n-2} embeddings which arise from three fixed (non-orientable) surfaces S^0, S^1, and S^2, and a fixed distribution of twists giving rise to the twist list T_0. The analogue of Theorem 1 is Theorem 3 below.

THEOREM 3. *Suppose that for* $n \equiv 1$ *or* 3 (*mod* 6) *and* $n \geqslant 19$ *we take three fixed non-orientable surfaces* S^0, S^1 *and* S^2, *and that we choose a fixed labelling of the* $3n-3$ *points to generate the twist list* T_0. *If we then apply the construction twice, using two* different *selections of the* $K_{3,3,3}$ *bridges, then the two resulting face* 2*-colourable triangular non-orientable embeddings of* K_{3n-2} *are non-isomorphic.* ▮

Since there is a face 2-colourable triangular embedding of K_n in a non-orientable surface for every $n \equiv 1$ or 3 (mod 6) with $n \neq 7$, we may use this to produce the initial surfaces S^0, S^1 and S^2. This enables us to state the following.

COROLLARY 4. *For every* $n \equiv 1$ *or* 3 (*mod* 6) *with* $n \geqslant 19$, *there are at least* $2^{(n-1)(n-3)/6}$ *non-isomorphic face* 2*-colourable triangular embeddings of* K_{3n-2} *in a non-orientable surface.* ▮

As before it is the case that the automorphism group of each of these embeddings is C_3. Once again we can make a colour reversal and then reapply the construction to form a face 2-colourable triangular embedding of K_{9n-8} in a non-orientable surface. Similar arguments to those given previously lead to the following Theorem.

THEOREM 4. *Suppose* $n \geqslant 19$ *and* $n \equiv 1$ *or* 3 (*mod* 6). *Put* $N = 2^{(n-1)(n-3)/6}$. *Then there are at least* $\binom{N}{3} 2^{(3n-3)(3n-5)/6}$ *non-isomorphic face* 2*-colourable triangular embeddings of* K_{9n-8} *in a non-orientable surface.* ▮

Again, these embeddings are all automorphism-free.

4. CONCLUDING REMARKS

It is clear that a refinement of some of the arguments given above, such as consideration of different twist lists, would lead to an improvement in the $O(n)$ term in the orders of growth. In the opposite direction a weaker version of Corollary 2 may be obtained without recourse to Theorem 1. To see this, note that the recursive construction produces, from a given

2to-embedding of K_n with a fixed vertex set $V = V(K_n)$, at least $2^{(n-1)(n-3)/6}$ distinct labelled 2to-embeddings of K_{3n-2}. Each isomorphism class of these embeddings can contain at most $(3n-2)!$ embeddings. The number of non-isomorphic embeddings is therefore at least $2^{(n-1)(n-3)/6}/(3n-2)!$. Writing m for $3n-2$ and estimating the factorial term gives $2^{m^2/54 - O(m \log m)}$. However, Theorem 1 continues to be important because it identifies a representative of each isomorphism class as well as providing a better estimate for the order of growth.

It seems highly likely that for **all** $n \equiv 3$ or 7 (mod 12) there will be at least 2^{an^2} non-isomorphic 2to-embeddings of K_n for some value of a. Perhaps the most interesting question is whether this is the true order. We can obtain an upper estimate by using the known upper bound for the number of labelled Steiner triple systems of order n, namely $(e^{-1/2}n)^{n^2/6}$ (see [7]). Each labelled 2to-embedding of K_n gives rise to a pair of labelled STS(n)s, "white" and "black". There are $2^{n(n-1)/6}$ possible choices for the orientations of the white triangles (i.e. the blocks of the white system). Any one such choice will determine the orientation of the corresponding black triangles. Thus the number of labelled 2to-embeddings of K_n is at most $(e^{-1/2}n)^{n^2/6} \cdot (e^{-1/2}n)^{n^2/6} \cdot 2^{n(n-1)/6} < n^{n^2/3}$. Consequently, the number of non-isomorphic 2to-embeddings of K_n is less than $n^{n^2/3}$. Unfortunately there seems to be no simple way of using this type of argument to establish a lower bound because an arbitrary pair of labelled STS(n)s will not, in general, be biembeddable as the black and white systems of a 2to-embedding of K_n no matter what orientations are chosen for the blocks (for example, the systems may have a common triple). Indeed, it is far from clear whether or not every STS(n) is biembeddable (i.e. forms the black system of a 2to-embedding). If the rate of growth of the number of non-isomorphic 2to-embeddings of K_n were of the order 2^{an^2} then this would imply that almost all STS(n)s are **not** biembeddable. At various times, various combinations of the present authors have felt that 2^{an^2} may be the correct order of growth, that n^{an^2} may be correct, or that the truth lies in some intermediate order.

Note. The referee has drawn to our attention a recent paper by V. P. Korzhik and H. J. Voss [3] containing different results on non-isomorphic embeddings of complete graphs.

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REFERENCES

1. M. J. Grannell, T. S. Griggs, and J. Širáň, Face 2-colourable triangular embeddings of complete graphs, *J. Combin. Theory Ser. B* **74** (1998), 8–19.
2. M. J. Grannell, T. S. Griggs, and J. Širáň, Surface embeddings of Steiner triple systems, *J. Combin. Des.* **6** (1998), 325–336.
3. V. P. Korzhik and H. J. Voss, On the number of non-isomorphic orientable regular embeddings of complete graphs, submitted for publication.
4. S. Lawrencenko, S. Negami, and A. T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.* **135** (1994), 367–369.
5. G. Ringel, "Map Color Theorem," Springer, 1974.
6. A. T. White, Block designs and graph imbeddings, *J. Combin. Theory Ser. B* **25** (1978), 166–183.
7. R. M. Wilson, Nonisomorphic Steiner triple systems, *Math. Z.* **135** (1974), 303–313.
8. J. W. T. Youngs, The mystery of the Heawood conjecture, *in* "Graph Theory and its Applications" (B. Harris, Ed.), pp. 17–50, Academic Press, 1970.