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An Elastoplasticity Model for Antiplane Shearing with a Non-associative Flow Rule: Genuine Nonlinearity of Plastic Waves

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1. INTRODUCTION

In elastoplasticity models, there is a stress threshold or *yield condition* that plays a role in determining whether the material is deforming elastically or plastically. If the stress is below the threshold, then the deformation is *elastic*, and is typically modeled by linear elasticity. If the stress reaches the threshold, it is said to be *at yield*, and the deformation is considered to be *plastic*. In models of plastic deformation in which the material hardens with increasing stress, the stress-strain constitutive law is typically nonlinear. Since the equations are hyperbolic (at least up to some maximum stress), nonlinearities can in principle lead to the formation of

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shocks. However, the nonlinear constitutive law for plastic deformation is in effect only under conditions of continued plastic loading. That is, the stress-strain law switches to linear elasticity unless the stress is at yield and is increasing in time. This additional nonlinearity is thought to prevent the formation of shock waves in many contexts.

The equations for plastic deformation are a system of nonlinear hyperbolic partial differential equations. Plastic waves associated with non-zero wave speeds are typically genuinely nonlinear in the sense of Lax [2]. For genuinely nonlinear plastic waves, shocks will form only if the magnitude of the characteristic speed increases in time. However, for continued plastic deformation, the stress must also increase. Thus, the possibility of formation of plastic shocks would be ruled out if increases in stress during plastic deformation necessarily gave rise to decreases in the characteristic speed.

To formulate precise results, we analyze an elastoplasticity model in the context of antiplane shearing. Antiplane shearing is a simplification of full two or three dimensional deformation in that, although the model applies to a three dimensional body of material, the deformation depends on only two of the space variables (i.e., x, y) plus time, and the velocity has only one component (the z component). The stress tensor reduces to a pair of unknown stress variables.

The model we consider includes non-associativity in the flow rule. The degree of non-associativity is measured by a parameter $\alpha \geq 0$. For $\alpha = 0$, the model is associative, and we show that plane plastic waves are genuinely nonlinear. Moreover, the characteristic speed decreases in magnitude with respect to time across plane plastic waves. Consequently, plane plastic waves are rarefaction waves, in which the characteristics spread out rather than focus.

For $\alpha > 0$, however, there can be a balance between hardening and the degree of non-associativity that makes the issue of genuine nonlinearity much more delicate. As is well understood, constitutive laws in plasticity models for granular materials that have a non-associative flow rule are in better agreement with experiments [4]. However, it is also well understood that these models are less stable than models with an associated flow rule [5, 6]. The possibility of plastic shock waves is a further destabilizing feature of these models. In Theorem 3.1, the main result of this paper, we prove that provided the hardening function is not too stiff, a condition depending upon the degree of non-associativity measured by α, then the system of partial differential equations is genuinely nonlinear everywhere, i.e., for all stress fields. Consequently, for models satisfying the extra condition, plastic shock waves cannot appear.

An examination of the possibility of shock waves for a different elastoplasticity model in a different configuration was undertaken by Nouri and

Chossat [3]. In their paper, weak plastic shock waves are realized as inviscid limits of smooth solutions of an elasto-viscoplastic model, when the shear stress on one side of the wave is zero.

In Section 4, we formulate a constitutive law that violates the main hypothesis of Theorem 3.1, and for which genuine nonlinearity does not hold. In this case, plastic shocks may form. The example is arrived at by choosing unrealistically extreme (but physically possible) values of parameters in the constitutive relation.

In Section 5, we give an algorithm that tests directly for genuine nonlinearity; the algorithm is implemented as a MAPLE procedure in the Appendix. The model and it's characteristic structure are presented in Section 2.

2. EQUATIONS OF MOTION

In this section, we present the specific equations to be studied, and analyze the characteristic structure of the equations.

2.1. *Antiplane Shear Model*

The model, derived in [7], represents dynamic antiplane shearing. Such deformations occur in three space dimensions perpendicular to the x, y-plane. Thus, the first and second components of velocity vanish. The fundamental variables are v, the z component of velocity, and the stress vector $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$, whose components correspond to the T_{zx} and T_{zy} components of the full stress tensor. The density (taken to be one) and the other components of the stress tensor are assumed constant, and the three unknown functions depend on $\mathbf{x} = (x, y)^T$, and t, but not on z.

Conservation of momentum is expressed by the equation

$$\partial_t v = \nabla^T \boldsymbol{\tau}, \tag{2.1}$$

where $\nabla = (\partial_x, \partial_y)^T$ is the gradient operator, and ∇^T is the divergence operator. In antiplane shearing, the strain rate tensor reduces to ∇v. In the theory of elastoplasticity, during plastic deformation the strain rate may be decomposed into elastic and plastic components:

$$\nabla v = D^{(e)} + D^{(p)}. \tag{2.2}$$

We assume that elastic strain rate satisfies the constitutive relation for linear elasticity,

$$\partial_t \boldsymbol{\tau} = D^{(e)}, \tag{2.3}$$

in which the elastic wave speed has been normalized to have magnitude one.

Plastic deformation occurs only when the stress satisfies the plastic yield condition, which is taken to be

$$|\boldsymbol{\tau}(\mathbf{x}, t)| = \gamma(\mathbf{x}, t), \tag{2.4}$$

where $\gamma(\mathbf{x}, t) = \max_{0 \leq s \leq t} |\boldsymbol{\tau}(\mathbf{x}, s)|$. Notice that the yield surface in stress space is a circle which may expand as time evolves, but never contracts. We note that by Lemma 2.1 of [7], the material is deforming plastically at $(\mathbf{x}, t)$ if and only if (2.4) is satisfied and

$$\boldsymbol{\tau}(\mathbf{x}, t)^T \partial_t \boldsymbol{\tau}(\mathbf{x}, t) \geq 0. \tag{2.5}$$

We assume that the plastic strain rate satisfies the constitutive relation

$$\frac{\boldsymbol{\tau}^T \partial_t \boldsymbol{\tau}}{h(\gamma)} R(\alpha) \boldsymbol{\tau} = D^{(p)}, \tag{2.6}$$

called the *flow rule*, where

$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

and $\alpha \in (0, \pi/2)$ is a parameter. Note that the plastic component of strain makes an angle α, measured clockwise, with respect to the normal to the yield surface. Thus α characterizes the degree of non-associativity (if $\alpha = 0$, the flow rule is associated). The function h is called the hardening modulus; as in [7], we take h to be a monotonically decreasing[1] differentiable function on the interval $[0, 1]$, with $h(1) = 0$.

Remark 2.1. The yield condition (2.4) is related to the more conventional yield condition with hardening as follows: Write

$$\gamma = H(\xi),$$

where H is the yield stress, depending on the shear strain ξ, and defined only for values of ξ less than some maximum value of shear strain, $\xi \leq \xi_{\max}$. H is typically monotonically increasing and concave: $H' > 0$, $H'' < 0$. Also H' approaches zero as H approaches its maximum $(H_{\max})$ at $\xi_{\max}$. It is shown in [7, Appendix B] that h is related to H through the

[1] It would be more accurate to assume h is decreasing on some interval $[\delta, 1]$ with $\delta > 0$ small. However, the results here are concerned with anomalous behavior that arises for larger values of γ (i.e., closer to $\gamma = 1$), so we do not include this additional complication.

identity

$$h(H(\xi)) = 2G[H(\xi)]^2 H'(\xi), \tag{2.7}$$

where G is the elastic modulus. In this paper, we have normalized H so that $H_{\max} = 1$. Correspondingly, $\gamma = H(\xi)$ lies in the interval [0, 1].

Combining (2.1)–(2.3), we have the system describing elastic deformations (in which $D^{(p)}$ is 0),

$$\begin{aligned} \partial_t v &= \nabla^T \boldsymbol{\tau} \\ \partial_t \boldsymbol{\tau} &= \nabla v. \end{aligned} \tag{2.8}$$

Combining (2.1)–(2.3), and (2.6), we have the system describing plastic deformations,

$$\begin{aligned} \partial_t v &= \nabla^T \boldsymbol{\tau} \\ \left[I + \frac{R(\alpha)\boldsymbol{\tau}\boldsymbol{\tau}^T}{h(\gamma)} \right] \partial_t \boldsymbol{\tau} &= \nabla v, \end{aligned} \tag{2.9}$$

where $\gamma = |\boldsymbol{\tau}|$.

2.2. *Characteristic Speeds*

In this subsection, we calculate the characteristic speeds of plane wave solutions of the systems (2.8) and (2.9). We consider solutions which depend only on x and t (rotational invariance of the equations assures the same result for other directions in the x, y-plane). Then the equations reduce to the system

$$\begin{aligned} \partial_t v &= \partial_x \tau_1 \\ \partial_t \boldsymbol{\tau} &= \begin{pmatrix} \partial_x v \\ 0 \end{pmatrix} \end{aligned} \tag{2.10}$$

for elastic deformation, and the system

$$\begin{aligned} \partial_t v &= \partial_x \tau_1 \\ \left[I + \frac{R(\alpha)\boldsymbol{\tau}\boldsymbol{\tau}^T}{h(\gamma)} \right] \partial_t \boldsymbol{\tau} &= \begin{pmatrix} \partial_x v \\ 0 \end{pmatrix} \end{aligned} \tag{2.11}$$

for plastic deformation. We write (2.10) and (2.11) in the form

$$\begin{pmatrix} v \\ \boldsymbol{\tau} \end{pmatrix}_t + A\begin{pmatrix} v \\ \boldsymbol{\tau} \end{pmatrix}_x = 0, \tag{2.12}$$

where

$$A = -\begin{bmatrix} 0 & 1 & 0 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix}.$$

Letting $\mathbf{a} = (a_1, a_2)^T$, we have

$$\mathbf{a} = \mathbf{a}^{(e)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.13}$$

for elastic deformations, and

$$\mathbf{a} = \mathbf{a}^{(p)} = \left[I - \frac{R(\alpha)\boldsymbol{\tau}\boldsymbol{\tau}^T}{h(\gamma) + \gamma^2 \cos\alpha} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.14}$$

for plastic deformations.

By computing the eigenvalues of A, we find that the characteristic speeds of the system (2.12) are $0, \pm\sqrt{a_1}$. For future reference, we note that the corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \mp\sqrt{a_1} \\ a_1 \\ a_2 \end{pmatrix}. \tag{2.15}$$

System (2.9) is hyperbolic (in fact strictly hyperbolic), i.e., $a_1^{(p)} > 0$ for all values of θ, if

$$h(\gamma) > \gamma^2 \sin^2(\alpha/2). \tag{2.16}$$

(Compare [7].) We define $\gamma_c = \gamma_c(\alpha)$ to be the smallest value of γ at which (2.16) fails, i.e., γ_c is the unique number in $(0, 1]$ satisfying

$$h(\gamma_c) = \gamma_c^2 \sin^2(\alpha/2). \tag{2.17}$$

In what follows, we shall make extensive use of planar polar coordinates for $\boldsymbol{\tau}$: $\boldsymbol{\tau} = |\boldsymbol{\tau}|(\cos\theta, \sin\theta)$. In particular, during plastic deformation, we

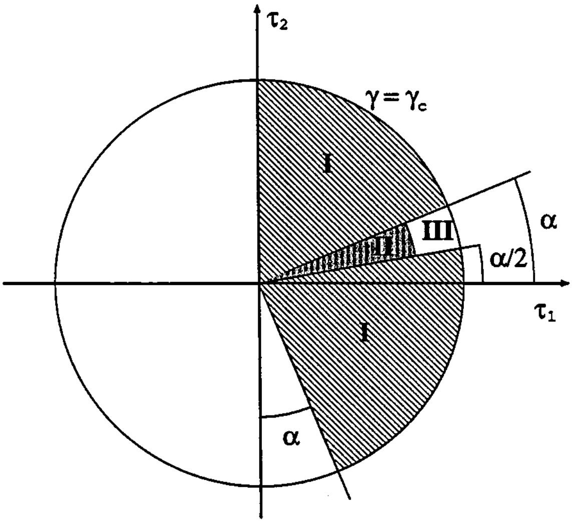

FIG. 1. Regions.

have

$$\boldsymbol{\tau} = \gamma(\cos\theta, \sin\theta).$$

Notice that the speed of plastic waves exceeds the speed of elastic waves exactly when $a_1^{(p)} > a_1^{(e)} = 1$. This occurs only in the intervals $\pi/2 < \theta < \pi/2 + \alpha$ and $-\pi/2 < \theta < -\pi/2 + \alpha$ (cf. [7]). Since such an ordering of the elastic and plastic wave speeds is associated with a loss of well-posedness of initial value problems [5], we specifically exclude these ranges of θ in what follows (see Fig. 1). A more complete discussion of this issue in the context of two dimensional deformations is given in the forthcoming paper [1].

3. GENUINELY NONLINEAR LOADING WAVES

In this section we give conditions on the hardening function $h(\gamma)$ in relation to the degree of non-associativity measured by α, that guarantee the genuine nonlinearity of plane plastic waves, and thereby establishing that plane plastic waves are necessarily rarefaction waves. For the entirety of this section, we impose the restriction $0 \leq \alpha < \pi/3$.

Consider a rarefaction wave solution of the system (2.12) for plastic deformation. Then the stress $\boldsymbol{\tau}$ is constrained to follow an integral curve of an eigenvector (2.15), in the direction of increasing total stress γ (so as to maintain plastic deformation). Thus, the stress rate $\boldsymbol{\tau}_t$ is a positive multiple of the stress component $\mathbf{a}$ of the eigenvector. Moreover, since the characteristics spread out in a rarefaction wave, the characteristic speed $\lambda = \sqrt{a_1(\boldsymbol{\tau}(x,t))}$ must decrease in time. Consequently, the directional derivative $\dot{\lambda}$ of λ along the integral curve must be negative. (This condition is more stringent than genuine nonlinearity, which requires only $\dot{\lambda} \neq 0$.) From the above discussion, we see that the condition $\dot{\lambda} < 0$ is equivalent to

$$\mathbf{a} \cdot \nabla_{\boldsymbol{\tau}} a_1 < 0 \qquad \text{for } -\frac{\pi}{2} + \alpha \leq \theta \leq \frac{\pi}{2}, 0 < \gamma \leq \gamma_c. \tag{3.18}$$

(Note that $\mathbf{a} \cdot \nabla_{\boldsymbol{\tau}} a_1 < 0$ for $\pi/2 + \alpha \leq \theta \leq 3\pi/2$, $0 < \gamma \leq \gamma_c$, then follows by symmetry.) This condition is derived in more detail in Ref. [1].

The main result of this paper is as follows.

THEOREM 3.1. (i) *For* $\alpha = 0$,

$$\mathbf{a} \cdot \nabla_{\boldsymbol{\tau}} a_1 < 0 \qquad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(ii) *For* $0 < \alpha < \pi/3$, (3.18) *holds if h satisfies condition* (A);

$$\text{(A)} \quad -h(\gamma)h'(\gamma)/\gamma^3 > B(\alpha) \qquad \text{for } \tilde{\gamma} < \gamma < \gamma_c(\alpha), \tag{3.19}$$

where $\tilde{\gamma}$ *is defined by*

$$h(\tilde{\gamma}) = \tilde{\gamma}^2, \tag{3.20}$$

and

$$B(\alpha) = \frac{1 - \cos(\alpha/2)}{\cos(\alpha/2)\cos\alpha}. \tag{3.21}$$

Remark 3.2. (a) Note that $B(\alpha)$ is monotonically increasing in the interval $0 \leq \alpha < \pi/3$, and $B(\pi/3) < .31$.

(b) Since $B(0) = 0$, part (ii) is consistent with part (i). However, the points $\theta = \pm\pi/2$ are special when $\alpha = 0$, and must be excluded.

(c) In terms of the yield function H of Remark 2.1, condition (A) becomes

$$2H'(t)^2 + H(t)H''(t) > B(\alpha),$$

for $\tilde{t} < t < t_{\max}$, where $H(\tilde{t}) = \tilde{\gamma}$ and $H(t_{\max}) = 1$.

To prove the theorem, it is convenient to express (3.18) in terms of the auxilliary function

$$\Phi(\theta, \gamma, \alpha) = \frac{(h + \gamma^2 \cos \alpha)^3}{\gamma} \mathbf{a} \cdot \nabla_\tau a_1. \tag{3.22}$$

Since $h \geq 0$ for $\gamma \leq \gamma_c$ and $0 \leq \alpha \leq \pi/3$, we see that (3.18) is equivalent to the condition

$$\Phi(\theta, \gamma, \alpha) > 0, \qquad \text{for } -\frac{\pi}{2} + \alpha \leq \theta \leq \frac{\pi}{2} \text{ and } 0 < \gamma \leq \gamma_c. \tag{3.23}$$

A straightforward calculation shows that

$$\begin{aligned} \Phi(\theta, \gamma, \alpha) = {} & h^2(\cos(\theta - \alpha) + \cos\theta \cos\alpha) \\ & - \gamma h h' \cos^2\theta \cos(\theta - \alpha) \\ & + \gamma^2 h \sin(2\theta - \alpha)(\sin(\theta - \alpha) + \sin\theta\cos\alpha) \\ & + \gamma^4 \sin(2\theta - \alpha)\sin(\theta - \alpha)\cos\alpha. \end{aligned} \tag{3.24}$$

Considering Φ as the sum of four terms, we make the following observations, restriction θ to the interval $-\pi/2 + \alpha \leq \theta \leq \pi/2$, and considering $\gamma \in (0, \gamma_c]$:

(1) The first two terms in (3.24) are nonnegative. (Note that $h'(\gamma) < 0$.)

(2) The third and fourth terms can be negative only in the range $\alpha/2 < \theta < \alpha$.

First consider the associative case, for which $\alpha = 0$. Then all four terms are nonnegative. In fact, they are all zero at $\theta = \pm\pi/2$, and positive elsewhere. The singular behavior at $\theta = \pm\pi/2$ occurs because the projection of the right eigenvector onto the stress plane is tangent to the yield surface at $\theta = \pm\pi/2$. The characteristic speed, restricted to the integral curve, therefore necessarily has a critical point (a maximum) at $\theta = \pm\pi/2$. This completes the proof of part (i) of Theorem 3.1.

The remainder of this section is devoted to proving part (ii) of the theorem. That is, we establish that (3.23) holds for $\alpha > 0$, under condition (A). The argument is based upon assessing the signs of the four terms in (3.24), and analyzing cancellations when one or more of the terms are negative.

We define the regions (see Fig. 1)

$$\mathrm{I} = \left\{(\theta, \gamma, \alpha) : 0 < \gamma \leq \gamma_c, -\frac{\pi}{2} + \alpha \leq \theta \leq \frac{\pi}{2} \text{ and } \left|\theta - \frac{3\alpha}{4}\right| \geq \frac{\alpha}{4}\right\}$$

$$\mathrm{II} = \left\{(\theta, \gamma, \alpha) : 0 < \gamma \leq \tilde{\gamma}, \frac{\alpha}{2} \leq \theta \leq \alpha\right\}$$

$$\mathrm{III} = \left\{(\theta, \gamma, \alpha) : \tilde{\gamma} \leq \gamma \leq \gamma_c, \frac{\alpha}{2} \leq \theta \leq \alpha\right\},$$

where $\gamma_c = \gamma_c(\alpha)$ is given by (2.17), and $\tilde{\gamma}$ is given by (3.18).

In Propositions 3.3 and 3.5 we prove that (3.23) is satisfied in Regions I and II without any added hypothesis. In Proposition 3.8 we prove (3.23) is satisfied in Region III under the extra assumption (3.19). This seems reasonable since for γ close to γ_c, $h(\gamma)$ could be very small and the fourth term in (3.24) (negative for $\alpha/2 < \theta < \alpha$) can only be cancelled if the second term in (3.24) is sufficiently large in comparison.

In the next section we construct an example for which relation (3.23) is not satisfied. This proves that some conditions on h and h' are certainly necessary in order to rule out the possibility of plastic shock waves.

PROPOSITION 3.3. *For* $0 < \alpha \leq \pi/3$, $\Phi(\theta, \gamma, \alpha) > 0$ *in region I.*

Proof. From observations (1), (2) above, all four terms in (3.24) are nonnegative in Region I. To show strict inequality, it is enough to note that the first term in (3.24) can be zero only if $\alpha = 0, \theta = \pi/2$. ∎

In regions II and III, the third and fourth terms in (3.24) may be negative, and there is some cancellation with the first and second terms of (3.24). In Region II, we are able to show that $\Phi(\theta, \gamma, \alpha) > 0$ (see Proposition 3.5), whereas in Region III, we need the additional condition (3.19) to guarantee $\Phi(\theta, \gamma, \alpha) > 0$ (see Theorem 3.1). We begin with two elementary trigonometric inequalities.

LEMMA 3.4. *For* $0 < \alpha/2 \leq \theta \leq \alpha$, *we have*

$$\cos(\theta - \alpha) + \sin(2\theta - \alpha)(\sin(\theta - \alpha) + \sin\theta\cos\alpha) > 0,$$

and

$$\sin(2\theta - \alpha)\sin(\theta - \alpha) \geq \cos\frac{\alpha}{2} - 1 > -\frac{1}{2}.$$

Proof. Since $\sin(\theta - \alpha) + \sin\theta\cos\alpha$ is an increasing function of θ, the minimum is attained at $\theta = \alpha/2$, where it takes the value

$$-\sin\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\alpha = -2\sin^3\frac{\alpha}{2}.$$

Now using the observation that $\sin(2\theta - \alpha) \le \sin\alpha$ for $\alpha/2 \le \theta \le \alpha$, we have

$$\begin{aligned}
&\cos(\theta - \alpha) + \sin(2\theta - \alpha)(\sin(\theta - \alpha) + \sin\theta\cos\alpha) \\
&\quad \ge \cos(\theta - \alpha) + \sin\alpha\left(-2\sin^3\frac{\alpha}{2}\right) \\
&\quad = \cos\theta\cos\alpha + \sin\theta\sin\alpha - 2\sin\alpha\sin^3\frac{\alpha}{2} \\
&\quad \ge \cos\theta\cos\alpha + \sin\frac{\alpha}{2}\sin\alpha - 2\sin\alpha\sin^3\frac{\alpha}{2} \\
&\quad = \cos\theta\cos\alpha + \sin\alpha\sin\frac{\alpha}{2}\cos\alpha > 0.
\end{aligned}$$

To prove the second inequality, we write

$$\begin{aligned}
&\sin\left(2\left(\theta - \frac{\alpha}{2}\right)\right)\sin(\alpha - \theta) \\
&\quad = 2\cos\left(\theta - \frac{\alpha}{2}\right)\sin\left(\theta - \frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2} - \left(\theta - \frac{\alpha}{2}\right)\right) \\
&\quad \le 2\sin\left(\theta - \frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2} - \left(\theta - \frac{\theta}{2}\right)\right).
\end{aligned}$$

Since the function $\sin y\sin(\alpha/2 - y)$ has a maximum at $y = \alpha/4$, we conclude that

$$\sin(2\theta - \alpha)\sin(\alpha - \theta) \le 2\sin^2\frac{\alpha}{4} = 1 - \cos\frac{\alpha}{2}. \tag{3.25}$$

This completes the proof of the lemma. ∎

Note that from (3.25), we have

$$\sin(2\theta - \alpha)\sin(\alpha - \theta) < 1/2 \qquad \text{for } \alpha/2 < \theta < \alpha < \pi/3. \tag{3.26}$$

PROPOSITION 3.5. *In Region II*, $\Phi(\theta, \gamma, \alpha) > 0$.

Proof.

$$\begin{aligned}\Phi(\theta,\gamma,\alpha) &\geq h^2(\cos(\theta-\alpha)+\cos\theta\cos\alpha)\\ &\qquad + \gamma^2 h\sin(2\theta-\alpha)(\sin(\theta-\alpha)+\sin\alpha\cos\alpha)\\ &\qquad + \gamma^4\sin(2\theta-\alpha)\sin(\theta-\alpha)\cos\alpha\\ &= h\{h\cos(\theta-\alpha)+\gamma^2\sin(2\theta-\alpha)\\ &\qquad \times(\sin(\theta-\alpha)+\sin\theta\cos\alpha)\}\\ &\qquad + \cos\alpha\{h^2\cos\theta+\gamma^4\sin(2\theta-\alpha)\sin(\theta-\alpha)\}.\end{aligned}$$

The proposition now follows from Lemma 3.4, the inequality $h \geq \gamma^2$ in Region II, and (3.26). ∎

Finally, we consider Region III.

LEMMA 3.6. *For* $0 < \alpha/2 \leq \theta \leq \alpha \leq \pi/3$, *and* $0 < \gamma \leq \gamma_c(\alpha)$, *we have*

$$\begin{aligned}&h^2(\cos(\theta-\alpha)+\cos\theta\cos\alpha)\\ &\qquad + \gamma^2 h\sin(2\theta-\alpha)(\sin(\theta-\alpha)+\sin\theta\cos\alpha) > 0.\end{aligned}$$

Proof. As in Lemma 3.4

$$\sin(2\theta-\alpha)(\sin(\theta-\alpha)+\sin\theta\cos\alpha) \geq -2\sin\alpha\sin^3\frac{\alpha}{2}.$$

We use this fact together with $h \geq \gamma^2\sin^2(\alpha/2)$ (since $\gamma \leq \gamma_c$) to write

$$\begin{aligned}&h^2(\cos(\theta-\alpha)+\cos\theta\cos\alpha)\\ &\qquad + \gamma^2 h\sin(2\theta-\alpha)(\sin(\theta-\alpha)+\sin\theta\cos\alpha)\\ &\qquad \geq h\gamma^2\left[\sin^2\frac{\alpha}{2}\cos(\theta-\alpha\cos\theta\cos\alpha)-2\sin\alpha\sin^3\frac{\alpha}{2}\right]\\ &\qquad = h\gamma^2\sin^2\frac{\alpha}{2}\left\{\cos(\theta-\alpha)+\cos\theta\cos\alpha-2\sin\alpha\sin\frac{\alpha}{2}\right\}\\ &\qquad = h\gamma^2\sin^2\frac{\alpha}{2}\left\{2\cos\theta\cos\alpha+\sin\alpha\sin\theta-2\sin\alpha\sin\frac{\alpha}{2}\right\}\\ &\qquad \geq h\gamma^2\sin^2\frac{\alpha}{2}\left\{1\cos\theta\cos\alpha-\sin\alpha\sin\frac{\alpha}{2}\right\}.\end{aligned}$$

For $\theta \leq \alpha \leq \pi/3$, we have $\cos\theta \geq 1/2$, $\cos\alpha \geq 1/2$, $\sin\alpha \leq \sqrt{3}/2$, and $\sin(\alpha/2) \leq 1/2$. This proves the lemma. ∎

LEMMA 3.7. *In Region III, if h satisfies condition* (A), *then*

$$\frac{-hh'}{\gamma^3}\cos^2\theta\cos(\theta-\alpha) + \sin(2\theta-\alpha)\sin(\theta-\alpha)\cos\alpha > 0.$$

Proof. From Lemma 3.4 we write

$$\begin{aligned}
&\frac{-hh'}{\gamma^3}\cos^2\theta\cos(\theta-\alpha) + \sin(2\theta-\alpha)\sin(\theta-\alpha)\cos\alpha\\
&\quad\geq \frac{-hh'}{\gamma^3}\cos^2\theta\cos(\theta-\alpha) - \cos\alpha\left(1-\cos\frac{\alpha}{2}\right)\\
&\quad\geq \frac{-hh'}{\gamma^3}\cos^2\alpha\cos\frac{\alpha}{2} - \cos\alpha\left(1-\cos\frac{\alpha}{2}\right)\\
&\quad= \cos\alpha\left\{\frac{-hh'}{\gamma^3}\cos\alpha\cos\frac{\alpha}{2} - \left(1-\cos\frac{\alpha}{2}\right)\right\}.
\end{aligned}$$

This last term will be positive if

$$\frac{-hh'}{\gamma^3} > \frac{1-\cos(\alpha/2)}{\cos\alpha\cos(\alpha/2)},$$

i.e., if h satisfies condition (A). This proves the lemma. ∎

PROPOSITION 3.8. *Let* $\alpha \in (0, \pi/3)$. *If h satisfies condition* (A), *then* $\Phi(\theta, \gamma, \alpha) > 0$ *for* θ, γ *in Region III*.

Proof.

$$\begin{aligned}
\Phi(\theta,\gamma,\theta) = h\{&h(\cos(\theta-\alpha) + \cos\theta\cos\theta)\\
&+\gamma^2\sin(2\theta-\alpha)(\sin(\theta-\alpha) + \sin\theta\cos\alpha)\}\\
&+\gamma^4\left\{-\frac{hh'}{\gamma^3}\cos^2\theta\cos(\theta-\alpha)\right.\\
&\left.\qquad+\sin(2\theta-\alpha)\sin(\theta-\alpha)\cos\alpha\right\}.
\end{aligned}$$

The proposition follows from Lemmas 3.6 and 3.7. ∎

Proof of Theorem 3.1. The result follows from (3.22), together with Propositions 3.3, 3.5, and 3.8.

4. EXAMPLE FOR WHICH GENUINE NONLINEARITY FAILS

In this section we construct an example of a hardening function for which the plastic rarefactions are not always genuinely nonlinear. Specifically, we show that for any given $\alpha \in (0, \pi/4)$, there are functions $h(\gamma)$ for which $\mathbf{a} \cdot \nabla a_1 > 0$ at some values of stress, i.e., $\Phi(\theta, \gamma, \alpha) < 0$ for some values of γ, θ.

Let f be a C^2 function on the interval $[0, 1]$, and satisfy $f(0) = 1$, $f(1) = 0$, and $f'(0) = \beta < 0$. For $\nu > 0$ we define the one parameter family of functions $h_\nu(\gamma)$ by $h_\nu(\gamma) = \nu f(\gamma)$. The critical value γ_c for $h_\nu(\gamma)$, defined by $h_\nu(\gamma_c) = \gamma_c^2(\sin(\alpha/2))^2$, now depends on ν, as well as α. Since we consider α to be fixed in this section, we write $\gamma_c = \gamma_c(\nu)$

LEMMA 4.1. *For* $0 < \alpha < \pi/4$,

$$\lim_{\nu \to 0+} \gamma_c(\nu) = 0 \qquad \text{and} \qquad \gamma_c(\nu) = \nu^{1/2}\left(\sin\frac{\alpha}{2}\right)^{-1} + O(1) \text{ as } \nu \to 0+.$$

Proof. From the definition of $\gamma_c(\nu)$, we have

$$\nu f(\gamma_c(\nu)) = \left(\gamma_c(\nu)\sin\frac{\alpha}{2}\right)^2. \tag{4.27}$$

Since $\nu f(\gamma) \to 0$ uniformly as $\nu \to 0+$, we see from (4.27) that $\gamma_c(\nu) \to 0$. Therefore, $\lim_{\nu \to 0+} f(\gamma_c(\nu)) = 1$, which implies

$$\lim_{\nu \to 0+} \frac{(\gamma_c(\nu)\sin(\alpha/2))^2}{\nu} = 1.$$

This completes the proof. ∎

COROLLARY 4.2. $h_\nu(\gamma_c(\nu)) \cong \nu$ *and* $h'_\nu(\gamma_c(\nu)) \cong \beta\nu$ *as* $\nu \to 0+$.

Now let Φ_ν denote the function Φ given by (3.22), with $h = h_\nu$.

PROPOSITION 4.3.

$$\Phi_\nu(\theta, \gamma_c(\nu), \alpha) = \nu^2 g(\theta, \alpha) + 0(\nu^{5/2})$$

as $\nu \to = +$, *where*

$$g(\theta, \alpha) = \cos(\theta - \alpha) + \cos\theta\cos\alpha + \sin(2\theta - \alpha)(\sin(\theta - \alpha) + \sin\theta\cos\alpha)/\sin^2\frac{\alpha}{2} + \sin(2\theta - \alpha)\sin(\theta - \alpha)\cos\alpha/\sin^4\frac{\alpha}{2}.$$

Proof. Using

$$h_\nu \approx \nu, \qquad h'_\nu \approx \beta\nu, \qquad \text{and} \qquad \gamma \approx \nu^{1/2}\left(\sin\frac{\alpha}{2}\right)^{-1},$$

in the expression for Φ in (3.24) we see that the 2nd term is of order $\nu^{5/2}$ while the other terms are of order ν^2. The proposition follows easily using Taylor's Theorem to expand $\Phi_\nu(\theta, \gamma_c(\nu), \alpha)$ around $\nu = 0$. ∎

Remark 4.4. We observe (using Maple V, for example) that the function $G(\alpha) = g(3\alpha/4, \alpha)$ is a monotonically increasing function for $0 < \alpha < \pi/3$ satisfying $\lim_{\alpha \to 0+} G(\alpha) = -\infty$ and $G = 0$ for $\alpha_0 \approx .27327\pi > \pi/4$. That is, $G(\alpha) < 0$ for $0 \le \alpha \le \pi/4$. (See Figs. 2(a) and 2(b), obtained with Maple V. In Fig. 2(a) we plot $G'(\alpha)$ and in Fig. 2(b) we plot $G'(\alpha)$ and $G(\alpha)$ for $2\pi/9 < \alpha < \pi/3$).

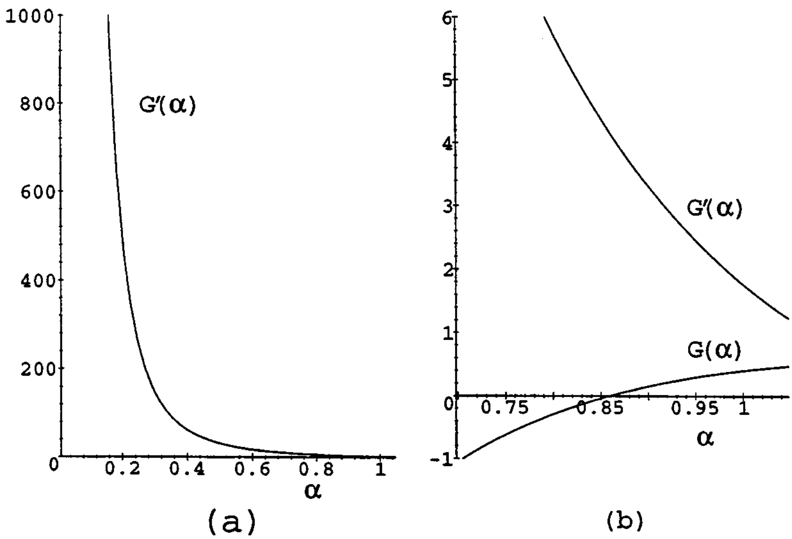

FIG. 2. Leading coefficient, $G(\alpha)$.

THEOREM 4.5. *The plastic rarefactions associated to system* (2.12) *are not always genuinely nonlinear. In particular, for* $h_\nu(\gamma) = \nu(1 - \gamma)$, *with* ν *small enough*, $\mathbf{a} \cdot \nabla a_1$ *is positive for some values of the stress, and negative for other values.*

Proof. Let $\alpha < \pi/6$ be fixed. Let $h_\nu(\gamma) = \nu(1 - \gamma)$. By Proposition 4.3

$$\Phi_\nu\left(\frac{3\alpha}{4}, \gamma_c(\nu), \alpha\right) = \nu^2 g\left(\frac{3\alpha}{4}, \alpha\right) + O(\nu^{5/2}).$$

Since $g(3\alpha/4, \alpha) < 0$, by Remark 3.2(a), $\Phi_\nu(3\alpha/4, \gamma_c, \alpha) < 0$ for all ν sufficiently small. For such values of ν, we have that $\mathbf{a} \cdot \nabla a_1 < 0$ in Regions I and II, but $\mathbf{a} \cdot \nabla a_1 > 0$ on the ray $\theta = 3\alpha/4$ and γ close to γ_c. This proves the theorem. ∎

5. NUMERICAL TEST

In this section, we describe a numerical test to check condition (3.23). The idea is to rewrite Φ as a polynomial and then study the roots.

We observe in Eq. (3.24) that the first term is homogeneous of degree 1 in the trigonometric pair $\{\cos\theta, \sin\theta\}$, and the remaining three terms are homogeneous of degree 3 in the same pair. This means that we will obtain a polynomial in $\tan\theta$ after performing the following operations on $\Phi/\cos^3\theta$:

(a) Expand all trigonometric functions involving θ in terms of $\sin\theta$ and $\cos\theta$ only.

(b) Replace $\sin\theta$ by $\tan\theta\cos\theta$. The resulting expression depends on θ only through $\tan\theta$ and $\cos\theta$.

(c) Replace $\sec^2\theta$ by $1 + \tan^2\theta$.

The resulting expression is a polynomial, Ψ_1, of degree 3 on $\tan\theta$,

$$\Psi_1(\theta, \gamma, \alpha) = a_2 \tan^3\theta + a_2 \tan^2\theta + a_1 \tan\theta + a_0,$$

where the coefficients a_i are functions of γ and α only.

The change of variables $z = (a_3)^{1/3}[\tan\theta - a_2/(3a_3)]$ transforms Ψ_1 into a generic polynomial $\Psi_2(z, \gamma, \alpha) = z^3 - a(\gamma, \alpha)z + b(\gamma, \alpha)$. Now condition (3.23), for $\alpha/2 < \theta < \alpha$ is equivalent to

$$\Psi_2(z, \gamma, \alpha) > 0, \qquad \text{for } z_0 < z < z_1, \tag{5.28}$$

where $z_0 = (a_3)^{1/3}[\tan(\alpha/2) - a_2/(3a_3)]$ and $z_1 = (a_3)^{1/3}[\tan\alpha - a_2/(3a_3)]$.

We notice that $\Psi_2(z)$, as a function of z only, has a relative minimum if and only if $a(\gamma, \alpha) > 0$. If a relative minimum exists, it would occur at $z_m = \sqrt{a/3}$ and take the value $b - 2(z_m)^3$.

We have the following theorem

THEOREM 5.1. *The characteristic fields associated to plastic rarefactions fails to be genuinely nonlinear* (*i.e., condition* (3.23) *is violated*) *if and only if the following conditions are satisfied at the same time*:

$$a > 0, \qquad \textit{and} \qquad b - 2(a/3)^{3/2} < 0.$$

Proof. Genuine nonlinearity of the plastic rarefactions is equivalent to condition (3.23) or, equivalently, to (5.28). From Propositions 3.3 and 3.5, we know that (3.23) is satisfied whenever $\theta \geq \alpha$ or $\theta \leq \alpha/2$. Thus if (3.23) is to be violated, it must happen in the region $\alpha/2 < \theta < \alpha$ or equivalently in $z_0 < z < z_1$. In particular $\Psi_2(z_0) \geq 0$ and $\Psi_2(z_1) \geq 0$ for all admissible values of γ and α.

It follows that $\Phi(\theta, \gamma, \alpha) < 0$ (for some value of θ, γ, α) if and only if $\Psi_2(z)$ has a *negative* local minimum in $z_0 < z < z_1$ for some values of γ and α. From the above remarks, $\Psi_2(z)$ can have a minimum if and only if $a > 0$, in which case the minimum takes the value $\Psi_2(z_m) = b - 2(a/3)^{3/2}$. This proves the theorem. ∎

For a given α, condition (5.28) can be tested numerically by checking whether the function $\phi(\gamma) = \Psi_2(z_m, \gamma, \alpha)$ is nonnegative for all values of $\gamma < \gamma_c$. We include an Appendix with the MapleV code for such test. In Fig. 3 we show two examples of the application of the test. In both cases $\alpha = \pi/6$ with different expressions for the hardening modulus: (a) $h(\gamma) = .27\sqrt{1-\gamma}$ and (b) $h(\gamma) = .02(1-\gamma)$. In Figs. 3(a) and 3(b) we plot the corresponding graphs of $\phi(\gamma)$. In the first case the test shows that condition (3.23) is always satisfied and therefore the plastic waves are genuinely nonlinear. In the second case the test shows that condition (3.23) is violated for γ close to γ_c. In this case the plastic waves are not genuinely nonlinear.

APPENDIX: MAPLE CODE

This is the maple file for the test of genuine nonlinearity. (Note: y stands for γ.)

```
> C := .27;      h := C*sqrt(1 - y);      # This line defines h(y).
>
> alpha := evalf(Pi/6);      # (chooses value of alpha)
> h' := diff(h, y);
```

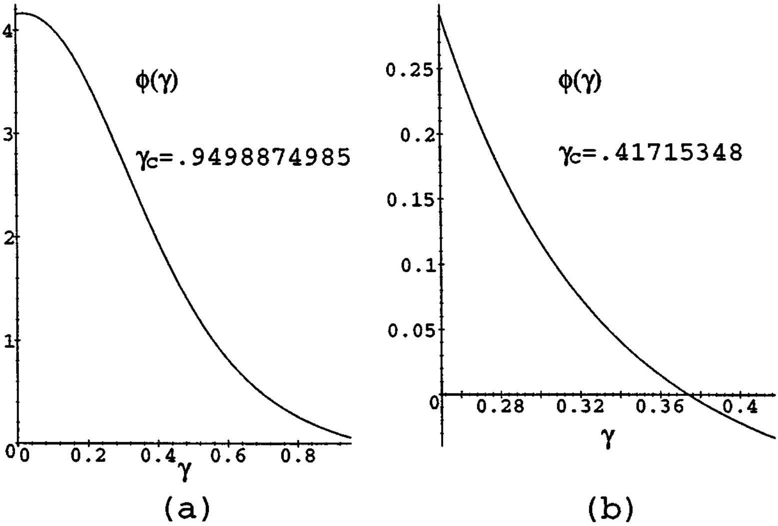

FIG. 3. (a) Genuinely nonlinear; (b) non-genuinely nonlinear.

```
> hc_eq := h = (y* sin(alpha/2))-^2:      y[c] := fsolve(hc_eq, y, 0..1):
> with(plots):
> Phi := h^2*(cos(theta-alpha) + cos(theta)*cos(alpha))
>            -h* h'*(cos(theta))^2*cos(theta-alpha)
>            +y^2* h* sin(2* theta-alpha)*(sin(theta-alpha)
>            +sin(theta)*cos(alpha))
>            +y^4* sin(2* theta-alpha)*sin(theta-alpha)*cos(alpha);
```

i.e.,

$$\Phi := h^2(\cos(\theta-\alpha)+\cos(\theta)\cos(\alpha)) - hh'\cos(\theta)^2\cos(\theta-\alpha)$$
$$+ y^2 h \sin(2\theta-\alpha)(\sin(\theta-\alpha)+\sin(\theta)\cos(\alpha))$$
$$+ y^4 \sin(2\theta-\alpha)\sin(\theta-\alpha)\cos(\alpha)$$

Next, we manipulate Φ to extract a polynomial in $tt = \tan(\theta)$:

```
> g0 := expand(Phi):
> g1 := g0/(cos(theta)^3):
> g2 := subs(sin(theta) = tt* cos(theta), g1):
> g3 := expand(simplify(g2)):
```

```
> g4 := expand(subs(1/(cos(theta))^2 = 1 + tt^2, g3)):
> Psi[1] := sort(g4, tt);
```

$$\Psi_1 := h^2 \sin(\alpha) tt^3 + y^4 \cos(\alpha)^2 \sin(\alpha) tt^3 + 2y^2 h \sin(\alpha)\cos(\alpha) tt^3 - y^2 h tt^2 + 2h^2 \cos(\alpha) tt^2 - y^4 \cos(\alpha) tt^2 + 3y^4 \cos(\alpha)^3 tt^2 + 5y^2 h \cos(\alpha)^2 tt^2 - hh' \sin(\alpha) tt - 4y^2 h \sin(\alpha)\cos(\alpha) tt + h^2 \sin(\alpha) tt - 3y^4 \cos(\alpha)^2 \sin(\alpha) tt + y^2 h - hh' \cos(\alpha) + y^4 \cos(\alpha) - y^2 h \cos(\alpha)^2 - y^4 \cos(\alpha)^3 + 2h^2 \cos(\alpha)$$

We write $\Psi_1 = a3 * tt^3 + a2 * tt^2 + a1 * tt + a0$, where

```
> a0 := coeff(Psi[1], tt, 0):a1 := coeff(Psi[1], tt, 1):
> a2 := coeff(Psi[1], tt, 2):a3 := coeff(Psi[1], tt, 3):
> g5 := sort(expand(Psi[1]/a3), tt): # normalize coefficient of cubic to 1.
```

Then $g5 = tt^3 + b2 * tt^2 + b1 * tt + b0$, where

```
> b0 := coeff(g5, tt, 0):b1 := coeff(g5, tt, 1):b2 := coeff(g5, tt, 2):
```

We now write the original $\Phi(\theta, \gamma, \alpha)$ as a polynomial Ψ_2 of z. Here $z = \tan\theta + b2/3$, where $b2$ is a function of y and α chosen to eliminate the z^2 term. The coefficients of Ψ_2 are aa and bb, both functions of y and α.

```
> g6 := subs(tt = z - b2/3, g5): g7 := sort(expand(g6), z):
> aa := -simplify(coeff(g7, z, 1)): bb := simplify(coeff(g7, z, 0)):
> Psi[2] := z^3 - aa*z + bb;
```

This gives the formula

$$\Phi_2 := z^3 - \tfrac{1}{3} \left(-3y^4 \cos(\alpha)^2 - y^4 - 8y^2 h \cos(\alpha) - 7h^2 \cos(\alpha)^2 + 3h^2 - 3hh' + 3hh' \cos(\alpha)^2\right) z \Big/ \left(-h^2 + h^2 \cos(\alpha)^2 - y^4 \cos(\alpha)^2 + y^4 \cos(\alpha)^4 - 2y^2 h \cos(\alpha) + 2y^2 \cos(\alpha)^3 h\right) - \tfrac{1}{27} \left(36h^2 y^2 + 18y^6 \cos(\alpha)^2 - 20h^3 \cos(\alpha)^3 + 9h^2 h' \cos(\alpha)^3 - 9y^2 hh' + 36h^3 \cos(\alpha) - 2y^6 + 12y^2 h^2 \cos(\alpha)^2 + 30y^4 h \cos(\alpha)\right.$$

$$+18y^4h\cos(\alpha)^3 - 9h^2h'\cos(\alpha) + 9y^2hh'\cos(\alpha)^2\Big)$$
$$\Big/\Big(\sin(\alpha)\big(-h^3 + h^3\cos(\alpha)^2 - 3hy^4\cos(\alpha)^2 + 3hy^4\cos(\alpha)^4$$
$$-3y^2h^2\cos(\alpha) + 3y^2h^2\cos(\alpha)^3 - y^6\cos(\alpha)^3 + y^6\cos(\alpha)^5\big)\Big)$$

> z0 := tan(alpha/2) + b2/3: z1 := tan(alpha) + b2/3: # define the range of z.

> zmin := simplify(sqrt(aa/3)): phi := simplify(bb-2*zmin^3): #min Psi[2] = phi

> at z = zmin

For a fixed value of α, we now have two functions of y labeled *aa* and *phi*. If for the same value of y, the following occurs,

$$aa > 0 \qquad \text{and} \qquad phi < 0,$$

then the system is not genuinely nonlinear for this choice of α. Such values can be observed from the following plots, in which $y[c]$ denotes γ_c.

> plot(aa, y = 0..y[c], title = 'aa-coefficient');

> plot(phi, y = 0..y, y = 0..y[c]);

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