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Modular invariant Frobenius algebras from ribbon Hopf algebra automorphisms

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ABSTRACT

For any finite-dimensional factorizable ribbon Hopf algebra H and any ribbon automorphism of H , we establish the existence of the following structure: an H -bimodule F_ω and a bimodule morphism Z_ω from Lyubashenko's Hopf algebra object K for the bimodule category to F_ω . This morphism is invariant under the natural action of the mapping class group of the one-punctured torus on the space of bimodule morphisms from K to F_ω . We further show that the bimodule F_ω can be endowed with a natural structure of a commutative symmetric Frobenius algebra in the monoidal category of H -bimodules, and that it is a special Frobenius algebra iff H is semisimple.

The bimodules K and F_ω can both be characterized as coends of suitable bifunctors. The morphism Z_ω is obtained by applying a monodromy operation to the coproduct of F_ω ; a similar construction for the product of F_ω exists as well.

Our results are motivated by the quest to understand the bulk state space and the bulk partition function in two-dimensional conformal field theories with chiral algebras that are not necessarily semisimple.

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1. Introduction

One remarkable feature of complex Hopf algebras is their intimate connection with low-dimensional topology, including invariants of knots, links and three-manifolds. These connections are par-

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ticularly well understood for semisimple Hopf algebras. The representation category of a semisimple factorizable finite-dimensional (weak) Hopf algebra is a modular tensor category [NTV] and thus allows one to construct a three-dimensional topological field theory. As a consequence, it provides finite-dimensional projective representations of mapping class groups of punctured surfaces.

It has been shown by Lyubashenko [Ly1, Ly3] that such representations of mapping class groups can be constructed for non-semisimple factorizable Hopf algebras H as well. This construction is in fact purely categorical, in the sense that it only uses the representation category as an abstract ribbon category with certain non-degeneracy properties. In the present paper we apply this construction not to the category of left H -modules, but rather to the category of H -bimodules. To this end we endow this category H -Bimod with the structure of a monoidal category using the coproduct of H (rather than by taking the tensor product \otimes_H over H as an associative algebra). With this tensor product, the category H -Bimod can be endowed with further structure such that it becomes a sovereign braided monoidal category.

For our present purposes we restrict to the case that the punctured surface in question is a one-punctured torus. Thus in the absence of punctures the mapping class group is the modular group $SL(2, \mathbb{Z})$; if punctures are present, then the mapping class group has additional generators given by Dehn twists around the punctures and by braiding homeomorphisms [Ly1, Sect. 4.3]. We denote the mapping class group of the one-punctured torus by $\Gamma_{1;1}$.

Specializing the results of [Ly1], we obtain a Hopf algebra object K in the monoidal category H -Bimod. For any H -bimodule X the vector space $\text{Hom}_{H|H}(K, X)$ of bimodule morphisms then carries a projective representation of $\Gamma_{1;1}$. The main result of this paper is the following assertion:

Theorem. *Let H be a (not necessarily semisimple) finite-dimensional factorizable ribbon Hopf algebra over an algebraically closed field of characteristic zero, and let $\omega : H \rightarrow H$ be an automorphism of H as a ribbon Hopf algebra. Then there is an object F_ω in the category H -Bimod and a morphism*

$$Z_\omega \in \text{Hom}_{H|H}(K, F_\omega)$$

that is invariant under the natural action [Ly1] of the mapping class group $\Gamma_{1;1}$ on $\text{Hom}_{H|H}(K, F_\omega)$.

The considerations leading to this result are inspired by structure one hopes to encounter in certain two-dimensional conformal field theories that are based on non-semisimple representation categories. More information about this motivation can be found in Appendix B; here it suffices to remark that F_ω is a candidate for what in conformal field theory is called the *algebra of bulk fields*, and that the morphism Z_ω is a candidate for a modular invariant *partition function*. Such a partition function should also enjoy integrality properties. As we will show elsewhere [FSS], for $\omega = id_H$ the relevant integers are closely related to the Cartan matrix of the algebra H .

To arrive at our result we show in fact first that the object F_ω actually carries a lot more natural structure: F_ω is a commutative symmetric Frobenius algebra in H -Bimod. Furthermore, the Frobenius algebra F_ω is a special¹ Frobenius algebra if and only if the Hopf algebra H is semisimple. A Frobenius algebra carries a natural coalgebra structure; the invariant morphism Z_ω is obtained by applying a monodromy operation to the coproduct of F_ω .

This paper is organized as follows. In Section 2 we introduce the relevant structure of a monoidal category on H -Bimod and construct, for the case $\omega = id_H$, the bimodule $F = F_{id_H}$ as a Frobenius algebra in H -Bimod. In Section 3 we endow the monoidal category H -Bimod with a natural braiding and show that with respect to this braiding the Frobenius algebra F is commutative. In Section 4 it is established that F is symmetric, has trivial twist, and is special iff H is semisimple. Modular invariance of Z_{id_H} is proven in Section 5. Section 6 is finally devoted to the case of a general ribbon Hopf algebra automorphism ω of H , which can actually be treated by modest modifications of the arguments of Sections 2–5. In Appendix A we gather some notions from category theory and explain

¹ A Frobenius is called special iff, up to non-zero scalars, the counit is a left inverse of the unit and the coproduct is a right inverse of the product, see Definition 4.6.

how the bimodules K and F_ω can be characterized as coends of suitable bifunctors. The latter shows that the objects in our constructions are canonically associated with the category of H -bimodules as an abstract category. Appendix B contains some motivation from (logarithmic) conformal field theory.

2. A Frobenius algebra in the bimodule category

2.1. Finite-dimensional ribbon Hopf algebras

In this section we collect some basic definitions and notation for Hopf algebras and recall that finite-dimensional Hopf algebras admit a canonical Frobenius algebra structure.

Throughout this paper, \mathbb{k} is an algebraically closed field of characteristic zero and, unless noted otherwise, H is a finite-dimensional factorizable ribbon Hopf algebra over \mathbb{k} . We denote by $m, \eta, \Delta, \varepsilon$ and s the product, unit, coproduct, counit and antipode of the Hopf algebra H .

There exist plenty of factorizable ribbon Hopf algebras (see e.g. [Bu]). For instance, the Drinfeld double of a finite-dimensional Hopf algebra K is factorizable ribbon provided that [KaR, Thm. 3] a certain condition for the square of the antipode of K is satisfied. Let us recall what it means that a Hopf algebra is factorizable ribbon.

Definition 2.1.

- (a) A Hopf algebra $H \equiv (H, m, \eta, \Delta, \varepsilon, s)$ is called *quasitriangular* iff it is endowed with an invertible element $R \in H \otimes H$ (called the *R-matrix*) that intertwines the coproduct and opposite coproduct, i.e. $\Delta^{\text{op}} = \text{ad}_R \circ \Delta$, and satisfies

$$(\Delta \otimes \text{id}_H) \circ R = R_{13} \cdot R_{23} \quad \text{and} \quad (\text{id}_H \otimes \Delta) \circ R = R_{13} \cdot R_{12}. \tag{2.1}$$

- (b) The *monodromy matrix* $Q \in H \otimes H$ of a quasitriangular Hopf algebra (H, R) is the invertible element

$$Q := R_{21} \cdot R \equiv (m \otimes m) \circ (\text{id}_H \otimes \tau_{H,H} \otimes \text{id}_H) \circ ((\tau_{H,H} \circ R) \otimes R). \tag{2.2}$$

- (c) A quasitriangular Hopf algebra (H, R) is called a *ribbon Hopf algebra* iff it is endowed with a central invertible element $v \in H$, called the *ribbon element*, that satisfies $s \circ v = v, \varepsilon \circ v = 1$ and $\Delta \circ v = (v \otimes v) \cdot Q^{-1}$.
- (d) A quasitriangular Hopf algebra (H, R) is called *factorizable* iff the monodromy matrix can be written as $Q = \sum_\ell h_\ell \otimes k_\ell$ with $\{h_\ell\}$ and $\{k_\ell\}$ two vector space bases of H .

Here and below, the symbol \otimes denotes the tensor product over \mathbb{k} , and for vector spaces V and W the linear map $\tau_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ is the flip map which exchanges the two tensor factors. Also, we canonically identify H with $\text{Hom}_{\mathbb{k}}(\mathbb{k}, H)$ and think of elements of (tensor products over \mathbb{k} of) H and $H^* = \text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ as (multi)linear maps. This has e.g. the advantage that many of our considerations still apply directly in the situation that H is a Hopf algebra, with adequate additional structure and properties, in an arbitrary \mathbb{k} -linear ribbon category instead of $\mathcal{Vect}_{\mathbb{k}}$. Various properties of the R-matrix and of the ribbon element, as well as of some further distinguished elements of H , will be recalled later on. Note that we do not assume the Hopf algebra H to be semisimple; in particular, the ribbon element does not need to be semisimple.

We also need a few further ingredients that are available for general finite-dimensional Hopf algebras, without assuming quasitriangularity, in particular the notions of (co)integrals and of a Frobenius structure for Hopf algebras.

Definition 2.2. A *left integral* of a Hopf algebra H is a morphism of left H -modules from the trivial H -module $(\mathbb{k}, \varepsilon)$ to the regular H -module (H, m) , i.e. an element $\Lambda \in H$ satisfying $m \circ (\text{id}_H \otimes \Lambda) = \Lambda \circ \varepsilon$.

A right cointegral of H is a morphism of right H -comodules from (\mathbb{k}, η) to (H, Δ) , i.e. an element $\lambda \in H^*$ satisfying $(\lambda \otimes id_H) \circ \Delta = \eta \circ \lambda$.

Right integrals and left cointegrals are defined analogously.

Recall [LS] that for a finite-dimensional Hopf \mathbb{k} -algebra the antipode is invertible and that H has, up to normalization, a unique non-zero left integral $\Lambda \in H$ and a unique non-zero right cointegral $\lambda \in H^*$. The number $\lambda \circ \Lambda \in \mathbb{k}$ is invertible. A factorizable ribbon Hopf algebra is unimodular [Ra3, Prop. 3(c)], i.e. the left integral Λ is also a right integral, implying that $s \circ \Lambda = \Lambda$.

The integral and the cointegral allow one to endow Hopf \mathbb{k} -algebras with more algebraic structure. The following characterization of Frobenius algebras will be convenient.

Definition 2.3. A Frobenius algebra A in $\mathcal{Vect}_{\mathbb{k}}$ is a vector space A together with (bi)linear maps m_A, η_A, Δ_A and ε_A such that (A, m_A, η_A) is an (associative, unital) algebra, $(A, \Delta_A, \varepsilon_A)$ is a (coassociative, counital) coalgebra and

$$(m_A \otimes id_A) \circ (id_A \otimes \Delta_A) = \Delta_A \otimes m_A = (id_A \otimes m_A) \circ (\Delta_A \otimes id_A), \tag{2.3}$$

i.e. the coproduct Δ_A is a morphism of A -bimodules.

We have

Lemma 2.4. A finite-dimensional Hopf \mathbb{k} -algebra $(H, m, \eta, \Delta, \varepsilon, s)$ carries a canonical structure of a Frobenius algebra A , with the same algebra structure on $A = H$, and with Frobenius coproduct and Frobenius counit given by

$$\Delta_A = (m \otimes s) \circ (id_A \otimes (\Delta \circ \Lambda)) \quad \text{and} \quad \varepsilon_A = (\lambda \circ \Lambda)^{-1} \lambda. \tag{2.4}$$

This actually holds more generally for finitely generated projective Hopf algebras over commutative rings (see e.g. [Pa,KaS]), as well as for any Hopf algebra in an additive ribbon category \mathcal{C} that has an invertible antipode and a left integral $\Lambda \in \text{Hom}(\mathbf{1}, H)$ and right cointegral $\lambda \in \text{Hom}(H, \mathbf{1})$ such that $\lambda \circ \Lambda \in \text{End}_{\mathcal{C}}(\mathbf{1})$ is invertible (see e.g. Appendix A.2 of [FSc]).

The Frobenius algebra structure given by (2.4) is unique up to rescaling the integral Λ by an invertible scalar. In the sequel, for a given choice of (non-zero) Λ , we choose the cointegral λ such that $\lambda \circ \Lambda = 1$.

2.2. H -Bimod as a monoidal category

Our focus in this paper is on natural structures on a distinguished H -bimodule, the coregular bimodule to be described below. To formulate these we need to endow the abelian category $H\text{-Bimod}$ of H -bimodules with the structure of a sovereign braided monoidal category.

The objects of the \mathbb{k} -linear abelian category $H\text{-Bimod}$ of bimodules over a Hopf \mathbb{k} -algebra H are triples (X, ρ, ϱ) such that (X, ρ) is a left H -module and (X, ϱ) is a right H -module and the left and right actions of H commute, $\rho \circ (id_H \otimes \varrho) = \varrho \circ (\rho \otimes id_H)$. Morphisms are \mathbb{k} -linear maps commuting with both actions. We denote the morphism spaces of $H\text{-Mod}$ and $H\text{-Bimod}$ by $\text{Hom}_H(-, -)$ and $\text{Hom}_{H|H}(-, -)$, respectively, while $\text{Hom}(-, -) \equiv \text{Hom}_{\mathbb{k}}(-, -)$ is reserved for \mathbb{k} -linear maps.

Just like the bimodules over any unital associative algebra, $H\text{-Bimod}$ carries a monoidal structure for which the tensor product is the one over H , for which the vector space underlying a tensor product bimodule $X \otimes_H Y$ is a non-trivial quotient of the vector space tensor product $X \otimes Y \equiv X \otimes_{\mathbb{k}} Y$. But for our purposes, we need instead a different monoidal structure on $H\text{-Bimod}$ for which also the coalgebra structure of H is relevant. This is obtained by pulling back the natural $H \otimes H$ -bimodule structure on $X \otimes Y$ along the coproduct to the structure of an H -bimodule. Thus if (X, ρ_X, ϱ_X) and (Y, ρ_Y, ϱ_Y) are H -bimodules, then their tensor product is $X \otimes Y$ together with the left and right actions

$$\begin{aligned} \rho_{X \otimes Y} &:= (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (\Delta \otimes id_X \otimes id_Y) \quad \text{and} \\ \varrho_{X \otimes Y} &:= (\varrho_X \otimes \varrho_Y) \circ (id_X \otimes \tau_{Y,H} \otimes id_H) \circ (id_X \otimes id_Y \otimes \Delta) \end{aligned} \tag{2.5}$$

of H . The monoidal unit for this tensor product is the one-dimensional vector space \mathbb{k} with both left and right H -action given by the counit, $\mathbf{1}_{H\text{-Bimod}} = (\mathbb{k}, \varepsilon, \varepsilon)$.

Obviously, (2.5) is just the standard tensor product

$$(X, \rho_X) \otimes^{H\text{-Mod}} (Y, \rho_Y) = (X \otimes Y, (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (\Delta \otimes id_X \otimes id_Y)) \tag{2.6}$$

of the category $H\text{-Mod}$ of left H -modules together with the corresponding tensor product of the category of right H -modules. For both monoidal structures the ground field \mathbb{k} , endowed with a left, respectively right, action via the counit, is the monoidal unit.

If H is a ribbon Hopf algebra, then (see e.g. Section XIV.6 of [Ka]) $H\text{-Mod}$ carries the structure of a ribbon category. Analogous further structure on $H\text{-Bimod}$ will become relevant later on, and we will introduce it in due course: a braiding on $H\text{-Bimod}$ in Section 3, and left and right dualities and a twist in Section 4.

2.3. The coregular bimodule

We now identify an object of the monoidal category $H\text{-Bimod}$ that is distinguished by the fact (see Appendix A.1) that it can be determined, up to unique isomorphism, by a universal property formulated in $H\text{-Bimod}$, and thus may be thought of as being canonically associated with $H\text{-Bimod}$ as a rigid monoidal category. Afterwards we will endow this object F with the structure of a Frobenius algebra in the monoidal category defined by the tensor product (2.5). As a vector space, F is the dual H^* of H .

Definition 2.5. The coregular bimodule $F \in H\text{-Bimod}$ is the vector space H^* endowed with the dual of the regular left and right actions of H on itself. Explicitly,

$$F = (H^*, \rho_F, \varrho_F), \tag{2.7}$$

with $\rho_F \in \text{Hom}(H \otimes H^*, H^*)$ and $\varrho_F \in \text{Hom}(H^* \otimes H, H^*)$ given by

$$\begin{aligned} \rho_F &:= (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes s \otimes b_H) \circ \tau_{H,H^*} \quad \text{and} \\ \varrho_F &:= (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes m \otimes id_{H^*}) \circ (id_{H^*} \otimes id_H \otimes \tau_{H^*,H}) \circ (id_{H^*} \otimes b_H \otimes s^{-1}). \end{aligned} \tag{2.8}$$

Expressions involving maps like ρ_F and ϱ_F tend to become unwieldy, at least for the present authors. It is therefore convenient to resort to a pictorial description. We depict the structure maps of the Hopf algebra H as²

$$m = \begin{array}{c} H \\ | \\ \text{---} \\ | \\ H \quad H \end{array} \quad \eta = \begin{array}{c} H \\ | \\ \circ \end{array} \quad \Delta = \begin{array}{c} H \quad H \\ \text{---} \\ | \\ H \end{array} \quad \varepsilon = \begin{array}{c} \circ \\ | \\ H \end{array} \quad s = \begin{array}{c} H \\ | \\ \circ \\ | \\ H \end{array} \quad s^{-1} = \begin{array}{c} H \\ | \\ \bullet \\ | \\ H \end{array} \tag{2.9}$$

² It is worth stressing that these pictures refer to the category $\text{Vect}_{\mathbb{k}}$ of finite-dimensional \mathbb{k} -vector spaces. Later on, we will occasionally also work with pictures for morphisms in more general monoidal categories \mathcal{C} ; to avoid confusion we will mark pictures of the latter type with the symbol $\boxed{\mathcal{C}}$.

the integral and cointegral as

$$\Lambda = \begin{array}{c} H \\ | \\ \text{[shaded box]} \end{array} \quad \lambda = \begin{array}{c} \text{[shaded box]} \\ | \\ H \end{array} \quad (2.10)$$

and the evaluation and coevaluation maps, dual maps, and flip maps of $\mathcal{Vect}_{\mathbb{k}}$ as

$$d_V = \begin{array}{c} \curvearrowright \\ V^* \quad V \end{array} \quad b_V = \begin{array}{c} V \quad V^* \\ \curvearrowleft \end{array} \quad f^V = \begin{array}{c} V^* \\ \curvearrowright \\ \text{[yellow box } f \text{]} \\ \curvearrowleft \\ W^* \end{array} \quad \tau_{V,W} = \begin{array}{c} W \quad V \\ \diagdown \quad \diagup \\ V \quad W \end{array} = \tau_{W,V}^{-1} = \begin{array}{c} W \quad V \\ \diagup \quad \diagdown \\ V \quad W \end{array} \quad (2.11)$$

The left-pointing arrows in the pictures for the evaluation and coevaluation indicate that they refer to the right duality of $\mathcal{Vect}_{\mathbb{k}}$. The evaluation and coevaluation \tilde{d} and \tilde{b} for the left duality of $\mathcal{Vect}_{\mathbb{k}}$ are analogously drawn with arrows pointing to the right. Also, for better readability we indicate the flip by either an over- or underbraiding, even though in the present context of the symmetric monoidal category $\mathcal{Vect}_{\mathbb{k}}$ both of them describe the same map.

In this graphical description the left and right actions (2.8) of H on F are given by

$$\begin{array}{c} H^* \\ \rho_F \\ \curvearrowleft \\ H \quad H^* \quad H \\ \alpha_F \end{array} = \begin{array}{c} H^* \\ \curvearrowright \\ \text{[red dots]} \\ \text{[red dots]} \\ \text{[red dots]} \\ \text{[red dots]} \\ s^{-1} \\ H \quad H^* \quad H \end{array} \quad (2.12)$$

Let us also mention that the Frobenius map $\Psi : H \rightarrow H^*$ and its inverse $\Psi^{-1} : H^* \rightarrow H$ are given by $\Psi(h) = \lambda \leftarrow s(h)$ and $\Psi^{-1}(p) = \Lambda \leftarrow p$, respectively (see e.g. [CW3]), i.e.

$$\Psi = \begin{array}{c} H^* \\ \text{[shaded box } \lambda \text{]} \\ \curvearrowright \\ \text{[red dot]} \\ s \\ H \end{array} \quad \text{and} \quad \Psi^{-1} = \begin{array}{c} H \\ \curvearrowleft \\ \text{[blue dot]} \\ \text{[shaded box } \Lambda \text{]} \\ H^* \end{array} \quad (2.13)$$

The statement that H is a Frobenius algebra (see Lemma 2.4) is equivalent to the invertibility of Ψ . That the two maps (2.13) are indeed each others' inverses means that

2.4. Morphisms for algebraic structure on the bimodule F

We now introduce the morphisms that endow the object F with the structure of a Frobenius algebra in the monoidal category $H\text{-Bimod}$. Very much like the coregular bimodule F itself, the algebra structure on F is a consequence of the universal properties of the coend of a functor $G_{\mathbb{k}}^H : H\text{-Mod}^{\text{op}} \times H\text{-Mod} \rightarrow H\text{-Bimod}$ (see Appendix A.1). Analogous coends with similar properties can be introduced for any rigid braided monoidal category, so that the Frobenius algebra F can be thought of as being canonically associated with the (abstract) monoidal category $H\text{-Mod}$ and the functor $G_{\mathbb{k}}^H$.

Definition 2.6. For H a finite-dimensional Hopf algebra, we introduce the following linear maps $m_F : H^* \otimes H^* \rightarrow H^*$, $\eta_F : \mathbb{k} \rightarrow H^*$, $\Delta_F : H^* \rightarrow H^* \otimes H^*$ and $\varepsilon_F : H^* \rightarrow \mathbb{k}$:

$$m_F := \Delta^*, \quad \eta_F := \varepsilon^*,$$

$$\Delta_F := [(id_H \otimes (\lambda \circ m)) \circ (id_H \otimes s \otimes id_H) \circ (\Delta \otimes id_H)]^*, \quad \varepsilon_F := \Lambda^*. \tag{2.15}$$

Again the graphical description appears to be convenient:

We would like to interpret the maps (2.16) as the structural morphisms of a Frobenius algebra in $H\text{-Bimod}$. To this end we must first show that these maps are actually morphisms of bimodules. We start with a few general observations.

Lemma 2.7. (i) For any Hopf algebra H we have

(2.17)

(ii) Further, if H is unimodular with integral Λ , we have

(2.18)

Proof. (i) The first equality holds by the defining properties of the antipode, unit and counit of H . The second equality follows by associativity and coassociativity, the third by the anti-coalgebra morphism property of the antipode, and the last by the connecting axiom for product and coproduct of the bialgebra underlying H .

(ii) The first equality in (2.18) follows by composing (2.17) with $id_H \otimes \Lambda$ and using that Λ is a left integral. The second equality in (2.18) follows by composing the left-right-mirrored version of (2.17) (which is proven in the same way as in (i)) with $\Lambda \otimes id_H$ and using that Λ is a right integral. \square

We will refer to the equality of the left- and right-hand sides of (2.17) as the *Hopf-Frobenius trick*.

Lemma 2.8. The map Δ_F introduced in (2.16) can alternatively be expressed as $\Delta_F = \Delta'_F$ with

(2.19)

Proof. Using the two equalities in (2.14) and coassociativity of Δ we obtain

(2.20)

Dualizing the expressions on the left- and right-hand sides of (2.20) establishes the claimed equality. \square

Proposition 2.9. When H -Bimod is endowed with the tensor product (2.5), \mathbb{k} is given the structure of the trivial H -bimodule $\mathbb{k}_\varepsilon = (\mathbb{k}, \varepsilon, \varepsilon)$ (the monoidal unit of H -Bimod) and H^* the H -bimodule structure (2.8), then the maps (2.15) are morphisms of H -bimodules.

Proof. (i) That m_F is a morphism of left H -modules is seen as

(2.21)

Here the first and last equalities just implement the definition (2.12) of the H -action, the second is the connecting axiom of H , and the third the anti-algebra morphism property of the antipode.

Similarly, that m_F is also a right module morphism follows as

(2.22)

(ii) That η_F is a left and right module morphism follows with the help of the properties $\varepsilon \circ m = \varepsilon \otimes \varepsilon$ and $\varepsilon \circ s = \varepsilon$ of the antipode. We have

(2.23)

and

(2.24)

respectively. This uses in particular the homomorphism property of the counit ε of H and the fact that $\varepsilon \circ s = \varepsilon$.

(iii) Next we apply the Hopf–Frobenius trick (2.17), which allows us to write

(2.25)

This tells us that Δ_F is a left module morphism.

(iv) For establishing the right module morphism property of Δ_F we recall from Lemma 2.8 that $\Delta_F = \Delta'_F$ with Δ'_F given by (2.19). The following chain of equalities shows that Δ'_F is a morphism of right H -modules:

(2.26)

Here the first equality combines the anti-coalgebra morphism property of the antipode and the connecting axiom, while the second equality uses that $\lambda \circ m = \lambda \circ \tau_{H,H} \circ (id_H \otimes S^2)$ and hence

$$\lambda \circ m \circ ((s \circ m) \otimes id_H) = \lambda \circ m \circ [s \otimes (m \circ \tau_{H,H} \circ (s^{-1} \otimes id_H))], \tag{2.27}$$

which can be shown (see [CW3, p. 4306]) by using that H is unimodular.

(v) Finally, the proof of the bimodule morphism property for ε_F is similar to the one for η_F . The sequence of equalities

(2.28)

shows that ε_F is a morphism of left modules. Here the second equality holds because Λ is a left integral. Using that Λ is also a right integral, one shows analogously that ε_F is a morphism of right modules. \square

2.5. The Frobenius algebra structure of F

Proposition 2.10. *The morphisms (2.15) endow the object $F = (H^*, \rho_F, \varrho_F)$ with the structure of a Frobenius algebra in H -Bimod (with tensor product (2.5)). That is, (F, m_F, η_F) is a (unital associative) algebra, $(F, \Delta_F, \varepsilon_F)$ is a (counital coassociative) coalgebra, and the two structures are connected by*

$$(id_{H^*} \otimes m_F) \circ (\Delta_F \otimes id_{H^*}) = \Delta_F \circ m_F = (m_F \otimes id_{H^*}) \circ (id_{H^*} \otimes \Delta_F). \tag{2.29}$$

Proof. (i) That $(F, m_F, \eta_F) = (H^*, \Delta^*, \varepsilon^*)$ is a unital associative algebra just follows from (and implies) the fact that (H, Δ, ε) is a counital coassociative coalgebra.

(ii) It follows directly from the coassociativity of Δ that

$$(id_{H^*} \otimes \Delta'_F) \circ \Delta_F = (\Delta_F \otimes id_{H^*}) \circ \Delta'_F. \tag{2.30}$$

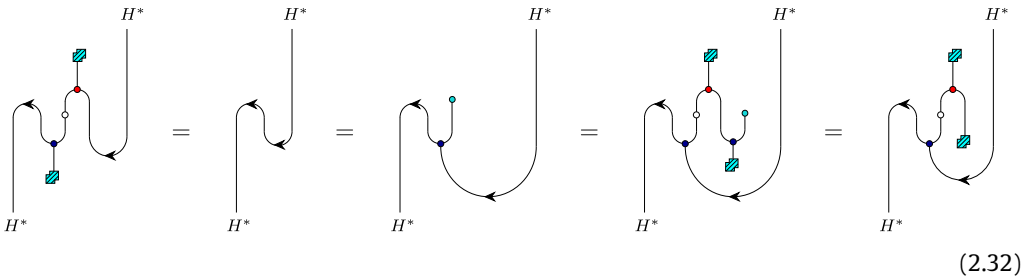
Since, as seen above, $\Delta'_F = \Delta_F$, this shows that Δ_F is a coassociative coproduct.

(iii) The coassociativity of Δ also implies directly the first of the Frobenius properties (2.29), as well as

$$(m_F \otimes id_{H^*}) \circ (id_{H^*} \otimes \Delta'_F) = \Delta'_F \circ m_F. \tag{2.31}$$

In view of $\Delta'_F = \Delta_F$, (2.31) is the second of the equalities (2.29).

(iv) That $\varepsilon_F = \Lambda^*$ is a counit for the coproduct Δ_F follows with the help of the invertibility (2.14) of the Frobenius map: we have



Here the left-hand side is $(id_{H^*} \otimes \varepsilon_F) \circ \Delta_F$, while the right-hand side is $(\varepsilon_F \otimes id_{H^*}) \circ \Delta_F$. \square

Remark 2.11. The spaces of left- and right-module morphisms, respectively, from F to $\mathbf{1}$ are given by $\mathbb{k}\Lambda_r$ and by $\mathbb{k}\Lambda_l$, respectively, with Λ_r and Λ_l non-zero left and right integrals of H . Thus a non-zero bimodule morphism from F to $\mathbf{1}$ exists iff H is unimodular, and in this case it is unique up to a non-zero scalar. In particular, up to a non-zero scalar the Frobenius counit ε_F is already completely determined by the requirement that it is a morphism of bimodules. In the situation at hand, the algebra F being a Frobenius algebra is thus a *property* rather than the choice of a structure.

3. Commutativity

The conventional tensor product (2.6) of bimodules generically does not admit a braiding. In contrast, the monoidal category H -Bimod, with tensor product as defined in (2.5), over a quasitriangular Hopf algebra admits braidings, and in fact can generically be endowed with several inequivalent ones. Among these inequivalent braidings, one is distinguished from the point of view of full local conformal field theory. We will select this particular braiding c and then show that with respect to this braiding c the algebra (F, m_F, η_F) is commutative.

The R-matrix $R \in H \otimes H$ is equivalent to a braiding $c^{H\text{-Mod}}$ on the category $H\text{-Mod}$ of left H -modules, consisting of a natural family of isomorphisms in $\text{Hom}_H(X \otimes Y, Y \otimes X)$ for each pair $(X, \rho_X), (Y, \rho_Y)$ of H -modules. These braiding isomorphisms are given by

$$c_{X,Y}^{H\text{-Mod}} = \tau_{X,Y} \circ (\rho_X \otimes \rho_Y) \circ (id_H \otimes \tau_{H,X} \otimes id_Y) \circ (R \otimes id_X \otimes id_Y), \tag{3.1}$$

where τ is the flip map. (When written in terms of elements $x \in X$ and $y \in Y$, this amounts to $x \otimes y \mapsto \sum_i s_i y \otimes r_i x$ for $R = \sum_i r_i \otimes s_i$, but recall that we largely refrain from working with elements.) The inverse braiding is given by a similar formula, with R replaced by $R_{21}^{-1} \equiv \tau_{H,H} \circ R^{-1}$. Besides R , also the inverse R_{21}^{-1} endows the category $H\text{-Mod}$ with the structures of a braided tensor category; the two braidings are inequivalent unless R_{21}^{-1} equals R , in which case the category is symmetric.

Likewise one can act with R and with R_{21}^{-1} from the right to obtain two different braidings on the category of right H -modules. As a consequence, with respect to the chosen tensor product on $H\text{-Bimod}$ there are two inequivalent natural braidings obtained by either using R both on the left and on the right, or else using (say) R_{21}^{-1} on the left and R on the right. For our present purposes (compare Lemma A.4(iii)) the second of these possibilities turns out to be the relevant braiding c . Pictorially, describing the R-matrix and its inverse by

$$R = \begin{array}{c} H \quad H \\ | \quad | \\ \text{grey box} \end{array} \quad \text{and} \quad R^{-1} = \begin{array}{c} H \quad H \\ | \quad | \\ \text{black box} \end{array} \tag{3.2}$$

the braiding on $H\text{-Bimod}$ looks as follows:

$$c_{X,Y} = \begin{array}{c} X \quad Y \\ | \quad | \\ \text{crossing} \\ | \quad | \\ X \quad Y \end{array} \begin{array}{l} \text{grey box } R \\ \text{black box } R^{-1} \end{array} \tag{3.3}$$

We are now in a position to state

Proposition 3.1. *The product m_F of the Frobenius algebra F in $H\text{-Bimod}$ is commutative with respect to the braiding (3.3):*

$$m_F \circ c_{F,F} = m_F. \tag{3.4}$$

Proof. We have

(3.5)

Here in the second equality the definition (2.8) of the H -actions on F is inserted. The third equality holds because the R -matrix satisfies

$$(s \otimes id_H) \circ R = R^{-1} = (id_H \otimes s^{-1}) \circ R, \tag{3.6}$$

which implies $(s \otimes s) \circ R^{-1} = R^{-1}$ as well as $(s^{-1} \otimes s^{-1}) \circ R = R$. The fourth equality follows by the defining property of R to intertwine the coproduct and opposite coproduct of H . \square

4. Symmetry, specialness and twist

By combining the dualities of $\mathcal{Vect}_{\mathbb{k}}$ with the antipode or its inverse, one obtains left and right dualities on the category $H\text{-Mod}$ of left modules over a finite-dimensional Hopf algebra H , and likewise for right H -modules. In the same way we can define left and right dualities on $H\text{-Bimod}$. Since the monoidal unit of $H\text{-Bimod}$ (with our choice of tensor product) is the ground field \mathbb{k} , we can actually take for the evaluation and coevaluation morphisms (and thus for the action of the functors on morphisms) just the evaluation and coevaluation maps (2.11) of $\mathcal{Vect}_{\mathbb{k}}$ and choose to define the action on objects $X = (X, \rho, \varrho) \in H\text{-Bimod}$ by

$$X^\vee := (X^*, \rho_\vee, \varrho_\vee) \quad \text{and} \quad {}^\vee X := (X^*, {}_\vee \rho, {}_\vee \varrho) \tag{4.1}$$

with

(4.2)

That the morphisms (4.2) are (left respectively right) H -actions follows from the fact that the antipode is an anti-algebra morphism, and that the evaluations and coevaluations are bimodule morphisms follows from the defining property $m \circ (s \otimes id_H) \circ \Delta = \eta \circ \varepsilon = m \circ (id_H \otimes s) \circ \Delta$ of the antipode. Note that with our definition of dualities³ we have

$$\vee(X^\vee) = X = (\vee X)^\vee \tag{4.3}$$

as equalities (not just isomorphisms) of H -bimodules.

The *canonical element* (also called *Drinfeld element*) $u \in H$ of a quasitriangular Hopf algebra (H, R) with invertible antipode is the element

$$u := m \circ (s \otimes id_H) \circ \tau_{H,H} \circ R. \tag{4.4}$$

u is invertible and satisfies $s^2 = ad_u$ [Ka, Prop. VIII.4.1]. We denote by $t \in H$ the inverse of the so-called special group-like element, i.e. the product

$$t := uv^{-1} \equiv m \circ (u \otimes v^{-1}) \tag{4.5}$$

of the Drinfeld element and the inverse of the ribbon element v . Since v is invertible and central, we have $ad_t = s^2$ and, as a consequence,

$$m \circ (s \otimes t) = m \circ (t \otimes s^{-1}) \quad \text{and} \quad m \circ (s^{-1} \otimes t^{-1}) = m \circ (t^{-1} \otimes s). \tag{4.6}$$

Also, since t is group-like we have $\varepsilon \circ t = 1$ and

$$s \circ t = t^{-1} = s^{-1} \circ t. \tag{4.7}$$

A *sovereign structure* on a category with left and right dualities is a choice of monoidal natural isomorphism π between the left and right duality functors [Dr, Def. 2.7].⁴ The category $H\text{-Mod}$ of left modules over a ribbon Hopf algebra H is sovereign iff the square of the antipode of H is inner [Bi,Dr]. Similarly, we have

³ The left and right duals of any object in a category with dualities are unique up to distinguished isomorphism. Our choice does not make use of the fact that H is a ribbon Hopf algebra. Another realization of the dualities on $H\text{-Mod}$ (and analogously on $H\text{-Bimod}$), which involves the special group-like element of H and hence does use the ribbon structure, is described e.g. in [Vi, Lemma 4.2].

⁴ Equivalently [Ye1, Prop. 2.11] one may require the existence of monoidal natural isomorphisms between the (left or right) double dual functors and the identity functor. The latter is called a balanced structure (see e.g. Section 1.7 of [Da1]), or sometimes also a pivotal structure (see e.g. Section 3 of [Sc]).

Lemma 4.1. For a ribbon Hopf algebra H with invertible antipode, the family π_X that is defined by

$$\pi_X := \text{[Diagram: A purple loop with two vertical lines labeled } X^* \text{ at the top and bottom. The loop has two crossings. Two small blue squares labeled } t \text{ are attached to the loop.]} \in \text{Hom}_{\mathbb{k}}(X^*, X^*) \tag{4.8}$$

for $X \in H\text{-Bimod}$ (with X^* the vector space dual to X) furnishes a sovereign structure on the category $H\text{-Bimod}$ of bimodules over H .

Proof. We must show that π_X is an invertible bimodule intertwiner from X^\vee to ${}^\vee X$ (i.e. the dual bimodules as defined in (4.1)), that the family $\{\pi_X\}$ is natural, and that it is monoidal, i.e. $\pi_{X \otimes Y} = \pi_X \otimes \pi_Y$.

(i) That π_X is a morphism in $\text{Hom}_{H|H}(X^\vee, {}^\vee X)$ is equivalent to

$$\text{[Diagram 1: } H \text{ and } X \text{ lines with a crossing and a blue square } t \text{]} = \text{[Diagram 2: } H \text{ and } X \text{ lines with a crossing and a blue square } t \text{]} \text{ and } \text{[Diagram 3: } X \text{ and } H \text{ lines with a crossing and a blue square } t \text{]} = \text{[Diagram 4: } X \text{ and } H \text{ lines with a crossing and a blue square } t \text{]} \tag{4.9}$$

This in turn follows directly by combining (4.6) and the (left, respectively right) representation properties.

(ii) With the help of the defining properties of the evaluation and coevaluation maps it is easily checked that

$$\pi_X^{-1} := \text{[Diagram: A purple loop with two vertical lines labeled } X^* \text{ at the top and bottom. The loop has two crossings. Two small blue squares labeled } t^{-1} \text{ are attached to the loop.]} \in \text{Hom}_{\mathbb{k}}(X^*, X^*) \tag{4.10}$$

is a linear two-sided inverse of π_X . That π_X^{-1} is a bimodule morphism is then automatic.

(iii) That the family $\{\pi_X\}$ of isomorphisms furnishes a natural transformation from the right to the left duality functor means that for any morphism $f \in \text{Hom}_{H|H}(X, Y)$ one has

$${}^\vee f \circ \pi_Y = \pi_X \circ f^\vee \tag{4.11}$$

as morphisms in $\text{Hom}_{H|H}(Y^\vee, {}^\vee X)$. Now by sovereignty of $\mathcal{V}ect_{\mathbb{k}}$ we know that ${}^\vee f = f^\vee$ as linear maps from Y^* to X^* , and as a consequence (4.11) is equivalent to

$$f \circ \varphi_X = \varphi_Y \circ f \tag{4.12}$$

as morphisms in $\text{Hom}_{H|H}(X, Y)$, where φ_X is the left action on X with $t \in H$ composed with the right action on X with t . (4.12), in turn, is a direct consequence of the fact that f is a bimodule morphism.

(iv) That π_X is monoidal follows from the fact that t is group-like. \square

Definition 4.2.

- (a) An *invariant pairing* on an algebra $A = (A, m, \eta)$ in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is a morphism $\kappa \in \text{Hom}_{\mathcal{C}}(A \otimes A, \mathbf{1})$ satisfying $\kappa \circ (m \otimes id_A) = \kappa \circ (id_A \otimes m)$.
- (b) A *symmetric algebra* (A, κ) in a sovereign category \mathcal{C} is an algebra A in \mathcal{C} together with an invariant pairing κ that is symmetric, i.e. satisfies

$$\text{Diagrammatic equation (4.13)} \tag{4.13}$$

Remark 4.3. (i) Unlike the pictures used so far (and most of the pictures below), which describe morphisms in $\mathcal{V}ect$, (4.13) refers to morphisms in the category \mathcal{C} rather than in $\mathcal{V}ect$; to emphasize this we have added the box \mathcal{C} to the picture. Also note that the morphisms (4.13) involve the left and right dualities of \mathcal{C} , but do not assume a braiding. Thus the natural setting for the notion of symmetry of an algebra is the one of sovereign categories \mathcal{C} ; a braiding on \mathcal{C} is not needed.

(ii) An algebra with an invariant pairing κ is Frobenius iff κ is non-degenerate, see e.g. [FSt, Sect. 3].

(iii) The two equalities in (4.13) actually imply each other.

In the case of the category $H\text{-Bimod}$ with sovereign structure π as defined in (4.8), the equalities (4.13) read

$$\text{Diagrammatic equation (4.14)} \tag{4.14}$$

Theorem 4.4. For any unimodular finite-dimensional ribbon Hopf algebra H the pair (F, κ_F) with F the coregular bimodule (with Frobenius algebra structure as defined above) and

$$\kappa_F := \varepsilon_F \circ m_F = (\Delta \circ \Lambda)^* \tag{4.15}$$

is a symmetric Frobenius algebra in $H\text{-Bimod}$.

Proof. That the pairing κ_F is invariant follows directly from the coassociativity of Δ . To establish that κ_F is symmetric, consider the following equalities:

(4.16)

The first equality is Theorem 3(d) of [Ra2], and involves the right modular element (also known as distinguished group-like element) g of H , which by definition satisfies $g \circ \lambda = (id_H \otimes \lambda) \circ \Delta$. The second equality uses that $g = t^2$ (which holds by Theorem 2(a) and Corollary 1 of [Ra1], specialized to unimodular H) and $s^2 = ad_t$.

Using also the identity $\rho_F \circ (t^{-1} \otimes id_{H^*}) = (m \circ (t \otimes id_H))^*$ (which, in turn, uses (4.7)), it follows that the equality of the left and right sides of (4.16) is nothing but the dualized version of the first of the equalities (4.14) for the case $A = F$ and $\kappa = \kappa_F$. \square

Next we observe:

Lemma 4.5. *The morphisms (2.15) satisfy*

$$\varepsilon_F \circ \eta_F = \varepsilon \circ \Lambda \quad \text{and} \quad m_F \circ \Delta_F = (\lambda \circ \varepsilon) id_{H^*}. \tag{4.17}$$

Proof. We have

(4.18)

Here the last equality uses the defining property of the antipode s of H . \square

Definition 4.6. A Frobenius algebra $(A, m_A, \eta_A, \Delta_A, \varepsilon_A)$ in a \mathbb{k} -linear monoidal category is called *special* [FRS,EP] (or strongly separable [Mü]) iff $\varepsilon_A \circ \eta_A = \xi id_{\mathbf{1}}$ and $m_A \circ \Delta_A = \zeta id_A$ with $\xi, \zeta \in \mathbb{k}^\times$.

It is known that a finite-dimensional Hopf algebra H is semisimple iff the Maschke number $\varepsilon \circ \Lambda \in \mathbb{k}$ is non-zero, and it is cosemisimple iff $\lambda \circ \varepsilon \in \mathbb{k}$ is non-zero [LS]; also, in characteristic zero cosemisimplicity is implied by semisimplicity [LR, Thm. 3.3]. Thus we have:

Corollary 4.7. *The Frobenius algebra F in $H\text{-Bimod}$ is special iff H is semisimple.*

As already pointed out we do not, however, assume that H is semisimple. We now note a consequence of the fact that F is commutative and symmetric, irrespective of whether H is semisimple or not. We first observe:

Lemma 4.8. *The braided monoidal category $H\text{-Bimod}$ of bimodules over a finite-dimensional ribbon Hopf \mathbb{k} -algebra H is balanced. The twist endomorphisms are given by*

$$\theta_X = \text{diagram} \tag{4.19}$$

with v the ribbon element of H .

Proof. We have seen that $H\text{-Bimod}$ is braided, and according to Lemma 4.1 it is sovereign. Now a braided monoidal category with a (left or right) duality is sovereign iff it is balanced, see e.g. Proposition 2.11 of [Ye1].

For any sovereign braided monoidal category \mathcal{C} the twist endomorphisms θ_X can be obtained by combining the braiding, dualities and sovereign structure according to

$$\theta_X = \text{diagram} \tag{4.20}$$

With the explicit form (3.3) of the braiding and (4.8) of the sovereign structure, this results in

$$\theta_X = \text{diagram} \tag{4.21}$$

Using the relations $t = v^{-1}u$ and $s^2 = ad_t$, the fact that v is central and the relation (4.4) between the canonical element u and the R -matrix then gives the formula (4.19). \square

Remark 4.9. (i) By using that $v \in H$ is central and satisfies $s \circ v = v$, it follows immediately from (4.19) that the Frobenius algebra F has trivial twist,

$$\theta_F = id_{H^*}. \tag{4.22}$$

(ii) In fact, a commutative symmetric Frobenius algebra in any sovereign braided category has trivial twist. This was shown in Proposition 2.25(i) of [FFRS] for the case that the category is strictly sovereign (i.e. that the sovereign structure is trivial in the sense that $\pi_X = id_X$ for all X), and the proof easily carries over to general sovereign categories. Conversely, the fact that F is a symmetric algebra can be derived by combining the triviality (4.22) of the twist with commutativity.

(iii) That F is commutative and symmetric implies [FFRS, Prop. 2.25(iii)] that it is cocommutative as well.

5. Modular invariance

Our focus so far has been on a natural object in the sovereign braided finite tensor category $H\text{-Bimod}$, the symmetric Frobenius algebra F . But in any such category there exists another natural object K , which has been studied by Lyubashenko. K is a Hopf algebra, and it plays a crucial role in the construction of mapping class group actions. The construction of these mapping class group actions relies on the presence of several distinguished endomorphisms of K . The existence of these endomorphisms is a consequence of universal properties characterizing the Hopf algebra object K . More precisely, apart from the antipode s_K of K , Lyubashenko obtains invertible endomorphisms $S_K, T_K \in \text{End}_{\mathcal{C}}(K)$ that obey the relations [Ly1, Thm 2.1.9]

$$(S_K T_K)^3 = \lambda S_K^2 \quad \text{and} \quad S_K^2 = s_K^{-1} \tag{5.1}$$

with some scalar λ that depends on the category \mathcal{C} in question.

These endomorphisms are the central ingredient for the construction of representations of mapping class groups of punctured Riemann surfaces on morphism spaces of \mathcal{C} . In particular, Lyubashenko [Ly1, Sect. 4.3] constructed a projective representation of the mapping class group $\Gamma_{1;m}$ of the m -punctured torus on morphism spaces of the form $\text{Hom}_{\mathcal{C}}(K, X_1 \otimes X_2 \otimes \dots \otimes X_m)$, for (X_1, X_2, \dots, X_m) any m -tuple of objects of \mathcal{C} . Specializing to the case of one puncture, $m = 1$, there is, for any object X of \mathcal{C} , a projective $\Gamma_{1;1}$ -action $\Gamma_{1;1} \times \text{Hom}_{\mathcal{C}}(K, X) \rightarrow \text{Hom}_{\mathcal{C}}(K, X)$. The mapping class group $\Gamma_{1;1}$ is generated by three generators S, T and D , where D is the Dehn twist around the puncture. The representation of $\Gamma_{1;1}$ satisfies

$$(S, f) \mapsto f \circ S_K^{-1}, \quad (T, f) \mapsto f \circ T_K^{-1} \quad \text{and} \quad (D, f) \mapsto \theta_X \circ f \tag{5.2}$$

with θ the twist of \mathcal{C} .

These general constructions apply in particular to the finite tensor category $H\text{-Bimod}$. The main goal in this section is to use the coproduct of the Frobenius algebra F to construct an element in $\text{Hom}_{H|H}(K, F)$ that is invariant under the action of $\Gamma_{1;1}$. A similar construction allows one to derive an invariant element in $\text{Hom}_{H|H}(K \otimes F, \mathbf{1})$ from the product of F . For the motivation to detect such elements and for possible applications in full local conformal field theory we refer to Appendix B.

5.1. Distinguished endomorphisms of coends

A finite tensor category [EO] is a \mathbb{k} -linear abelian rigid monoidal category with enough projectives and with finitely many simple objects up to isomorphism, with simple tensor unit, and with every object having a composition series of finite length. Both $H\text{-Mod}$ and $H\text{-Bimod}$, for H a finite-dimensional unimodular ribbon Hopf algebra, belong to this class of categories, see [LM, Ly1].

Let \mathcal{C} be a sovereign braided finite tensor category. As shown in [Ly1,Ke] (compare also [Vi] or, as a review, Sections 4.3 and 4.5 of [FSc]), there exists an object K in \mathcal{C} that carries a natural structure of a Hopf algebra in \mathcal{C} . Moreover, there is a two-sided integral as well as a Hopf pairing for this Hopf algebra K . Combining the integral of K and other structure of the category, one constructs [LM,Ly1, Ly2] distinguished morphisms S_K and T_K satisfying (5.1) in $\text{End}_{\mathcal{C}}(K)$.

The Hopf algebra K can be characterized as the *coend*⁵

$$K = \int^X F(X, X) = \int^X X^\vee \otimes X \tag{5.3}$$

of the functor F that acts on objects as $(X, Y) \mapsto X^\vee \otimes Y$. As described in some detail in Appendix A.3, in the case $\mathcal{C} = H\text{-Bimod}$ the object K is the *coadjoint bimodule* $\mathbb{H}H^*$. That is, the underlying vector space is the tensor product $H^* \otimes_{\mathbb{k}} H^*$, and this space is endowed with a left H -action by the coadjoint left action (A.26) on the first factor, and with a right H -action by the coadjoint right action on the second factor.

In the case of a general braided finite tensor category \mathcal{C} the morphisms S_K and T_K in $\text{End}_{\mathcal{C}}(K)$ are defined with the help of the braiding c and the twist θ of \mathcal{C} , respectively [LM,Ly1,Ly2]. T_K is given by the dinatural family

$$\tag{5.4}$$

Here it is used that a morphism f with domain the coend K is uniquely determined by the dinatural family $\{f \circ \iota_X^K\}$ of morphisms. For S one defines

$$S_K := (\varepsilon_K \otimes id_K) \circ Q_{K,K} \circ (id_K \otimes \Lambda_K), \tag{5.5}$$

where ε_K and Λ_K are the counit and the two-sided integral of the Hopf algebra K , respectively, while the morphism $Q_{K,K} \in \text{End}_{\mathcal{C}}(K \otimes K)$ is determined through monodromies $c_{Y^\vee, X} \circ c_{X, Y^\vee}$ according to

$$\tag{5.6}$$

⁵ The definition of the coend K of a functor $G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$, including the associated dinatural family ι_X^K of morphisms $\iota_X^K \in \text{Hom}_{\mathcal{C}}(F(X, X), K)$, will be recalled at the beginning of Appendix A.1.

We also note that the Hopf algebra K is endowed with a Hopf pairing ω_K , given by [Ly1]

$$\omega_K = (\varepsilon_K \otimes \varepsilon_K) \circ \mathcal{Q}_{K,K}. \tag{5.7}$$

5.2. The Drinfeld map

Let us now specialize the latter formulas to the case of our interest, i.e. $\mathcal{C} = H\text{-Bimod}$. Then the coend is $K = \mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*$, with dinatural family $\iota^K = i^{\triangleright\triangleleft}$ given by (A.29), while the twist is given by (4.19) and the braiding by (3.3). Further, the structure morphisms of the categorical Hopf algebra $\mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*$ can be expressed through those of the algebraic Hopf algebra H ; in particular, the counit and integral are $\varepsilon_{\triangleright\triangleleft} = \eta^\vee \otimes \eta^\vee$ (see (A.32)) and $\Lambda_{\triangleright\triangleleft} = \lambda^\vee \otimes \lambda^\vee$ (see (A.36)). The monodromy morphism $\mathcal{Q}_{\mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*, \mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*} \in \text{End}_{H|H}(\mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^* \otimes \mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*)$ that was introduced in (5.6) then reads

$$\mathcal{Q}_{\mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*, \mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*} = \tag{5.8}$$

while the general formulas for T_K and S_K specialize to the morphisms

$$T_{\triangleright\triangleleft} = \tag{5.9}$$

in $\text{End}_{H|H}(\mathbf{H}\mathbf{H}_{\triangleright\triangleleft}^*)$. Note that the morphism $S_{\triangleright\triangleleft}$ is composed of (variants of) the Frobenius map (2.13) and the Drinfeld map

$$f_Q := (d_H \otimes id_H) \circ (id_{H^*} \otimes Q) \in \text{Hom}(H^*, H). \tag{5.10}$$

In order that $S_{\triangleright\triangleleft}$ is invertible, which is necessary for having projective mapping class group representations, it is necessary and sufficient that the Drinfeld map f_Q is invertible.

Remark 5.1. By the results for general \mathcal{C} , $S_{\triangleright\triangleleft}$ is indeed a morphism in H -Bimod. But this is also easily checked directly: One just has to use that the Drinfeld map intertwines the left coadjoint action ρ_{\triangleright} (see (A.26)) of H on H^* and the left adjoint action ρ_{ad} of H on itself [CW1, Prop. 2.5(5)], i.e. that

$$f_Q \in \text{Hom}_H(H_{\triangleright}^*, H_{\text{ad}}), \tag{5.11}$$

together with the fact that the cointegral λ satisfies (since H is unimodular) [Ra2, Thm. 3]

$$\lambda \circ m = \lambda \circ m \circ \tau_{H,H} \circ (id_H \otimes S^2). \tag{5.12}$$

Remark 5.2. The Drinfeld map f_Q of a finite-dimensional quasitriangular Hopf algebra H is invertible iff H is factorizable. In the semisimple case, factorizability is the essential ingredient for a Hopf algebra to be modular [NTV, Lemma 8.2]. Here, without any assumption of semisimplicity, we see again a direct link between invertibility of S and factorizability.

Remark 5.3. The coadjoint H -module H_{\triangleright}^* is actually the coend (5.3) for the case that \mathcal{C} is the category H -Mod of left H -modules. In this case the endomorphism (5.5) is precisely the composition

$$S_{\triangleright} = \Psi \circ f_Q \tag{5.13}$$

of the Drinfeld and Frobenius maps (also called the quantum Fourier transform), see e.g. [LM,FGST]. Further, the Drinfeld map f_Q is related to the Hopf pairing $\omega_{H_{\triangleright}^*}$ from (5.7) for the coend H_{\triangleright}^* in H -Mod by

$$f_Q = (S \otimes \omega_{H_{\triangleright}^*}) \circ (id_H \otimes \tau_{H^*,H^*}) \circ (b_H \otimes id_{H^*}). \tag{5.14}$$

In particular, since the antipode is invertible, the Drinfeld map of H is invertible iff the Hopf pairing of H_{\triangleright}^* is non-degenerate.

Remark 5.4. For factorizable H , the Drinfeld map f_Q maps any non-zero cointegral λ of H to a non-zero integral Λ . Thus we may (and do) choose λ and Λ (uniquely, up to a common factor ± 1) such that besides $\lambda \circ \Lambda = 1$ we also have

$$f_Q(\lambda) = \Lambda \tag{5.15}$$

(see [GW, Thm. 2.3.2] and [CW2, Rem. 2.4]). Together with $s \circ \Lambda = \Lambda$ and the property (5.12) of the cointegral it then also follows that

$$f_{Q^{-1}}(\lambda) = \Lambda \quad \text{as well as} \quad f_{Q^{-1}} \in \text{Hom}_H(H_{\triangleright}^*, H_{\text{ad}}), \tag{5.16}$$

where $f_{Q^{-1}}$ is the morphism (5.10) with the monodromy matrix Q replaced by its inverse Q^{-1} . Further, one has [CW2, Lemma 2.5]

$$f_Q \circ \Psi \circ f_{Q^{-1}} \circ \Psi = id_H, \tag{5.17}$$

which in turn by comparison with (5.13) shows that $\Psi \circ f_{Q^{-1}} = S_{\triangleright}^{-1}$. Further, the latter identity and (5.13) are equivalent to the relations

(5.18)

Remark 5.5. According to the first of the formulas (5.9) we have $T_{\triangleright\triangleleft} = T_{\triangleright}^{-1} \otimes T_{\triangleright}$. Using (5.12) and (5.13) it follows that $S_{\triangleright\triangleleft} = S_{\triangleright}^{-1} \otimes S_{\triangleright}$ as well. As a consequence, the first of the relations (5.1) is realized with $\lambda = 1$. Thus in the case of the category H -Bimod of our interest, with coend K , the projective representation (5.2) of the mapping class group $\Gamma_{1;1}$ of the one-punctured torus is actually a *genuine* $\Gamma_{1;1}$ -representation.

5.3. Action of $\Gamma_{1;1}$ on morphism spaces

Consider now the representation $\rho_{K,Y}$ of $\Gamma_{1;1}$ on the spaces $\text{Hom}_{\mathcal{C}}(K, Y)$ just mentioned. The group $\Gamma_{1;1}$ is generated by three elements S, T and D , where D is the Dehn twist around the puncture, while S and T are modular transformations that act on the surface in the same way as in the absence of the puncture. The generators are subject to the relations $(ST)^3 = S^2$ (like for the modular group) and $S^4 = D$. Of particular interest to us are *invariants* of the $\Gamma_{1;1}$ -action on the spaces $\text{Hom}_{\mathcal{C}}(K, Y)$, i.e. morphisms g satisfying

$$g \circ \rho_{K,Y}(\gamma) = g \tag{5.19}$$

for all $\gamma \in \Gamma_{1;1}$.

Morphisms in $\text{Hom}_{\mathcal{C}}(K, Y)$ can in particular be obtained by defining a linear map t_Q from $\text{Hom}_{\mathcal{C}}(X, Y \otimes X)$ to $\text{Hom}_{\mathcal{C}}(K, Y)$ as a universal partial trace, to which we refer as the *partial monodromy trace*. Thus for $f \in \text{Hom}_{\mathcal{C}}(X, Y \otimes X)$ we set

$$t_Q(f) := \text{Diagram} \in \text{Hom}_{\mathcal{C}}(K, Y) \tag{5.20}$$

where $Q_{\mathcal{H}, Y}^1 \in \text{End}_{\mathcal{C}}(K \otimes Y)$ is defined by

$$Q_{K,Y}^1 := \text{[Diagram]} \tag{5.21}$$

Note that, by the naturality of the braiding, the morphisms (5.21) are natural in Y .

Remark 5.6. (i) It follows from elementary properties of the braiding that for any object X of \mathcal{C} the morphism $(\varepsilon_K \otimes id_X) \circ Q_{K,X}^1$ endows X with the structure of a K -module internal to \mathcal{C} .

(ii) If \mathcal{C} is a (semisimple, strictly sovereign) modular tensor category, then the invariance property (5.19) for $\gamma = S$ and $g = t_Q(f)$ is equivalent to the definition of S -invariance of morphisms in $\text{Hom}_{\mathcal{C}}(Y \otimes X, X)$ that is given in [KoR, Def. 3.1(i)].

Specializing now to $\mathcal{C} = H\text{-Bimod}$ and $X = Y$ being the Frobenius algebra F in $H\text{-Bimod}$, we can state one of the main results of this paper:

Theorem 5.7. *The partial monodromy trace of the coproduct Δ_F is $\Gamma_{1;1}$ -invariant.*

We will prove this statement by establishing, in Lemma 5.8 and Lemma 5.9 below, separately invariance under the two generators T and S of $\Gamma_{1;1}$. Note that this implies in particular invariance under the Dehn twist D ; the latter can also be directly deduced from the fact that $D_{\triangleright \triangleleft} = \theta_F$ together with the result (4.22) that F has trivial twist. Before investigating the action of S and T , let us first present the partial monodromy trace $t_Q(\Delta_F)$ in a convenient form. To this end we first note that, invoking the explicit form (3.3) of the braiding and (4.8) of the sovereign structure of $H\text{-Bimod}$, we have

$$Q_{K,F}^1 = \text{[Diagram]} \text{ and hence}$$

$$t_Q(f) = \text{Diagram} \tag{5.22}$$

for any $f \in \text{Hom}_{H|H}(F, F \otimes F)$. Specializing (5.22) to the partial monodromy trace of the coproduct Δ_F , i.e. inserting Δ_F from (2.19), yields

$$t_Q(\Delta_F) = \text{Diagram} \tag{5.23}$$

This can be rewritten as follows:

$$t_Q(\Delta_F) = \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} \tag{5.24}$$

Here in the first step it is used that $S^2 = \text{ad}_t$ and that $\lambda \circ m \circ (g \otimes \text{id}_H) = \lambda \circ s$ (which is a *left cointegral*), while the second equality follows by the fact that the antipode is an anti-algebra morphism and by associativity of m . We can now use the fact (see (5.16)) that the morphism $f_{Q^{-1}}$ intertwines the coadjoint and adjoint actions; we then have

$$t_Q(\Delta_F) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \quad (5.25)$$

The diagram shows three stages of a morphism transformation. The first stage shows a complex structure with strands labeled H^* , a box labeled λ , and boxes labeled Q^{-1} and Q . The second stage is identical to the first. The third stage shows a simplified structure where the Q^{-1} and Q boxes are replaced by a single box labeled Λ .

where the last equality uses the first of the identities (5.18) together with the fact that the antipode is an anti-coalgebra morphism and that $s \circ \Lambda = \Lambda$.

Lemma 5.8. *The morphism $t_Q(\Delta_F)$ is T -invariant, i.e. satisfies $t_Q(\Delta_F) \circ T_{\triangleright\triangleleft} = t_Q(\Delta_F)$.*

Proof. Invoking the expressions for $T_{\triangleright\triangleleft}$ given in (5.9) and for $t_Q(\Delta_F)$ given on the right-hand side of (5.25), and using the centrality of the ribbon element $v \in H$, we have

$$t_Q(\Delta_F) \circ T_{\triangleright\triangleleft} = \begin{array}{c} \text{Diagram} \\ \text{Diagram} \end{array} \quad (5.26)$$

The diagram shows the composition of $t_Q(\Delta_F)$ and $T_{\triangleright\triangleleft}$. It features strands labeled H^* , a box labeled Λ , and boxes labeled v and v^{-1} .

Recalling now the identity (2.18), the central elements v and v^{-1} cancel each other, hence (5.26) equals $t_Q(\Delta_F)$. \square

Lemma 5.9. *The morphism $t_Q(\Delta_F)$ is S -invariant, i.e. satisfies $t_Q(\Delta_F) \circ S_{\triangleright\triangleleft} = t_Q(\Delta_F)$.*

Proof. We will show that $t_Q(\Delta_F)$ is invariant under S^{-1} . Applying definition (5.9), we can use the identities (5.18) to obtain

$$t_Q(\Delta_F) \circ S_{\triangleright\triangleleft}^{-1} = \text{Diagram 1} = \text{Diagram 2} \tag{5.27}$$

Further we note that owing to the identities (5.12), $s^{-2} \circ \Lambda = \Lambda$ and (2.14) we can write

$$\text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5} = \text{Diagram 6} = \text{Diagram 7} \tag{5.28}$$

It follows that

$$t_Q(\Delta_F) \circ S_{\triangleright\triangleleft}^{-1} = \text{Diagram 8} = \text{Diagram 9} \tag{5.29}$$

This coincides with the right-hand side of (5.25) and thus with $t_Q(\Delta_F)$. \square

To neatly summarize the results above we state

Definition 5.10. A coalgebra $(C, \Delta_C, \varepsilon_C)$ in $H\text{-Bimod}$ is called *modular invariant* iff the morphism $t_Q(\Delta_C) \in \text{Hom}_{H|H}(K, C)$ is $\Gamma_{1;1}$ -invariant.

Thus what we have shown can be rephrased as

Corollary 5.11. The Frobenius algebra $F \in H\text{-Bimod}$ introduced in (2.15) is modular invariant.

Proof. Invariance of $t_Q(\Delta_C)$ under the action of the generators T and S of $\Gamma_{1;1}$ has been shown in Lemma 5.8 and Lemma 5.9. Invariance under the action of the generator D of $\Gamma_{1;1}$ follows immediately from the fact that F has trivial twist. \square

Remark 5.12. (i) The morphism $t_Q(\Delta_F)$ is non-zero. Indeed one can show that $\varepsilon_F \circ t_Q(\Delta_F)$ can be expanded as a bilinear form in the simple characters of Lyubashenko’s Hopf algebra $\mathcal{H} \in H\text{-Mod}$, with coefficients given by the Cartan matrix of H , i.e. by the matrix that describes the composition series of indecomposable projectives, and these \mathcal{H} -characters are non-zero. In conformal field theory terms, this means that the Cartan matrix – which is a quantity directly associated to the category – is the right substitute of the charge conjugation matrix in the non-semisimple case. However, establishing this result requires methods different from those on which our focus is in this paper.

(ii) In the same way as the partial monodromy trace (5.20) associates a morphism in $\text{Hom}_{\mathcal{C}}(K, Y)$ to a morphism in $\text{Hom}_{\mathcal{C}}(X, Y \otimes X)$, one may introduce another partial monodromy trace t'_Q that maps morphisms in $\text{Hom}_{\mathcal{C}}(X \otimes Y, X)$ linearly to morphism in $\text{Hom}_{\mathcal{C}}(K \otimes Y, \mathbf{1})$, say as

$$t'_Q(f) := \begin{array}{c} \text{Diagram: A vertical line on the left labeled } K \text{ has a blue dot at the top. A red box labeled } Q_{K,Y}^1 \text{ is on this line. A vertical line on the right labeled } Y \text{ has a green box labeled } F \text{ on it. A green loop starts from the } Y \text{ line, goes up, left, down, and right, crossing the } K \text{ line. A small green box labeled } \pi_X^{-1} \text{ is on the } K \text{ line where the loop crosses.} \end{array} \quad (5.30)$$

It is not difficult to check that the morphism $t'_Q(m_F)$ obtained this way from the product of the Frobenius algebra F is modular invariant in the sense that $t'_Q(m_F) \circ \rho_{K \otimes F, \mathbf{1}}(\gamma) = t'_Q(m_F)$ for all $\gamma \in \Gamma_{1;1}$. Indeed, $t'_Q(m_F)$ is related to $t_Q(\Delta_F)$ by

$$t'_Q(m_F) = \begin{array}{c} \text{Diagram: Three vertical lines labeled } H^* \text{ at the bottom. The middle line has a pink box labeled } t_Q(\Delta_F) \text{ and a higher pink box labeled } \Psi^{-1}. \text{ A loop starts from the middle line, goes up, left, down, and right, crossing the middle line.} \end{array} \quad (5.31)$$

with Ψ the Frobenius map, and as a consequence (using that $(s^{-2})^* \otimes id_{H^*}$ commutes with the action of $\Gamma_{1;1}$ and that $\tau_{H^*, H^*} \circ S_{\triangleright \triangleleft} = S_{\triangleright \triangleleft}^{-1}$ and $\tau_{H^*, H^*} \circ T_{\triangleright \triangleleft} = T_{\triangleright \triangleleft}^{-1}$) modular invariance of $t'_Q(m_F)$ is equivalent to modular invariance of $t_Q(\Delta_F)$. Accordingly, from the perspective of $H\text{-Bimod}$ alone we could

as well have referred to algebras rather than coalgebras in Definition 5.10. Indeed, this is the option that was chosen for the semisimple case in [KoR, Def. 3.1(ii)]. Our preference for coalgebras derives from the fact that, as described in Appendix B, the morphism space $\text{Hom}_{\mathcal{C}}(K, F)$ plays a more direct role than $\text{Hom}_{\mathcal{C}}(K \otimes F, \mathbf{1})$ in the motivating context of modular functors and conformal field theory.

(iii) More generally, for any non-negative integers m and n , the mapping class group $\Gamma_{1;m+n}$ of the $(m+n)$ -punctured torus acts on the morphism space $\text{Hom}_{\mathcal{C}}(K, F^{\otimes m+n})$ [Ly1, Sect. 4.3] and thus, using the canonical isomorphism of the Frobenius algebra F with its dual, also on $\text{Hom}_{\mathcal{C}}(K \otimes F^{\otimes m}, F^{\otimes n})$. By suitably composing the morphisms (5.23) and (5.30) with product and coproduct morphisms of F , one easily constructs a morphism in $\text{Hom}_{\mathcal{C}}(K \otimes F^{\otimes m}, F^{\otimes n})$ that, owing to commutativity and cocommutativity of F and to the triviality of its twist, is invariant under this action of $\Gamma_{1;m+n}$.

6. The case of non-trivial Hopf algebra automorphisms

A Hopf algebra automorphism of a Hopf algebra H is a linear map from H to H that is both an algebra and a coalgebra automorphism and commutes with the antipode. For H a ribbon Hopf algebra with R -matrix R and ribbon element v , an automorphism ω of H is said to be a ribbon Hopf algebra automorphism iff $(\omega \otimes \omega)(R) = R$ and $\omega(v) = v$. For any H -bimodule (X, ρ, ϱ) and any pair of algebra automorphisms ω, ω' of H there is a corresponding (ω, ω') -twisted bimodule ${}^{\omega}X^{\omega'} = (X, \rho \circ (\omega \otimes id_X), \varrho \circ (id_X \otimes \omega'))$. If ω and ω' are Hopf algebra automorphisms, then the twisting is compatible with the monoidal structure of H -Bimod, and if they are even ribbon Hopf algebra automorphisms, then it is compatible with the ribbon structure of H -Bimod.

In this section we observe that to any finite-dimensional factorizable ribbon Hopf algebra H and any ribbon Hopf algebra automorphism of H there is again associated a Frobenius algebra in H -Bimod, which moreover shares all the properties, in particular modular invariance, of the Frobenius algebra F that we obtained in the previous sections. The arguments needed to establish this result are simple modifications of those used previously. Accordingly we will be quite brief.

Proposition 6.1. (i) For H a finite-dimensional factorizable ribbon Hopf algebra over \mathbb{k} and ω a Hopf algebra automorphism of H , the bimodule $F_{\omega} := {}^{id_H}(F)^{\omega}$ carries the structure of a Frobenius algebra. The structure morphisms of F_{ω} as a Frobenius algebra are given by the formulas (2.15) (thus as linear maps they are the same as for $F \equiv F_{id_H}$).

(ii) F_{ω} is commutative and symmetric, and it is special iff H is semisimple.

(iii) If ω is a ribbon Hopf algebra automorphism, then F_{ω} is modular invariant.

Proof. The proofs of all statements are completely parallel to those in the case $\omega = id_H$. The only difference is that the various morphisms one deals with, albeit coinciding as linear maps with those encountered before, are now morphisms between different H -bimodules than previously. That they do intertwine the relevant bimodule structures follows by combining the simple facts that (since ω is compatible with the ribbon structure of H -Bimod) $(F \otimes F)^{\omega} = F_{\omega} \otimes F_{\omega}$ as a bimodule and that a linear map $f \in \text{Hom}(X, Y)$ for $X, Y \in H$ -Bimod lies in the subspace $\text{Hom}_{H|H}(X, Y)$ iff it lies in the subspace $\text{Hom}_{H|H}(X^{\omega}, Y^{\omega})$. \square

Furthermore, again the Frobenius algebra F_{ω} is canonically associated with H -Bimod as an abstract category. Indeed, analogously as in Proposition A.3, one sees that F_{ω} can be constructed as a coend, namely the one of the functor $G_{\otimes_{\mathbb{k}}}^{H;\omega} : H\text{-Mod}^{\text{op}} \times H\text{-Mod} \rightarrow H\text{-Bimod}$ that acts on morphisms as $f \times g \mapsto f^{\vee} \otimes_{\mathbb{k}} g$ and on objects by mapping $(X, \rho_X) \times (Y, \rho_Y)$ to

$$(X^* \otimes_{\mathbb{k}} Y, [\rho_{X^{\vee}} \circ (\omega^{-1} \otimes id_{X^*})] \otimes id_Y, id_{X^*} \otimes (\rho_Y \circ \tau_{Y,H} \circ (id_Y \otimes S^{-1}))) \tag{6.1}$$

(or, in other words, $G_{\otimes_{\mathbb{k}}}^{H;\omega} = (?^{\omega^{-1}} \times Id) \circ G_{\otimes_{\mathbb{k}}}^H$ with the functor $G_{\otimes_{\mathbb{k}}}^H$ (whose coend is F) given by (A.2)):

Proposition 6.2. The H -bimodule F_ω together with the dinatural family of morphisms

$$i_X^{F_\omega} := (\omega^{-1})^* \circ i_X^F, \tag{6.2}$$

with i_X^F as defined in (A.5), is the coend of the functor $G_{\otimes_{\mathbb{k}}}^{H;\omega}$.

Proof. Again the proof is parallel to the one for the case $\omega = id_H$, the difference being that the automorphisms $\omega^{\pm 1}$ need to be inserted at appropriate places. For instance, the equalities

(6.3)

and

(6.4)

which generalize the relations (A.8) and (A.9), respectively, demonstrate that the linear maps $i_X^{F_\omega} \in \text{Hom}(X^* \otimes_{\mathbb{k}} X, H^*)$ are indeed bimodule morphisms in $\text{Hom}_{H|H}(G_{\otimes_{\mathbb{k}}}^{H;\omega}(X, X), F_\omega)$. \square

Remark 6.3. (i) When discussing twists of F we can restrict to the case that only the, say, right module structure is twisted, because the bimodule ${}^\omega H^{\omega'}$ is isomorphic to $id_H H^{\omega^{-1} \circ \omega'}$.

(ii) It follows from the automorphism property of ω that together with Λ also $\omega(\Lambda)$ is a non-zero two-sided integral of H . As a consequence, just like in the case $\omega = id_H$ considered in Remark 2.11, the counit ε_F of F_ω is uniquely determined up to a non-zero scalar.

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Appendix A. Coend constructions

A.1. The coregular bimodule as a coend

A dinatural transformation $F \rightrightarrows B$ from a functor $F : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$, to an object $B \in \mathcal{D}$ is a family of morphisms $\varphi = \{\varphi_X : F(X, X) \rightarrow B\}_{X \in \mathcal{C}}$ such that the diagram

$$\begin{array}{ccc}
 F(Y, X) & \xrightarrow{F(id_Y, f)} & F(Y, Y) \\
 \downarrow F(f, id_X) & & \downarrow \varphi_Y \\
 F(X, X) & \xrightarrow{\varphi_X} & B
 \end{array} \tag{A.1}$$

commutes for all $f \in \text{Hom}(X, Y)$. For instance, the family $\{d_X\}$ of evaluation morphisms of a rigid monoidal category \mathcal{C} forms a dinatural transformation from the functor that acts as $X \times Y \mapsto X^\vee \otimes Y$ to the monoidal unit $\mathbf{1} \in \mathcal{C}$.

Dinatural transformations from a given functor F to an object of \mathcal{D} form a category, with the morphisms from $(F \rightrightarrows B, \varphi)$ to $(F \rightrightarrows B', \varphi')$ being given by morphisms $f \in \text{Hom}_{\mathcal{D}}(B, B')$ satisfying $f \circ \varphi_X = \varphi'_X$ for all $X \in \mathcal{C}$. A coend (A, ι) for the functor F is an initial object in this category. If the coend of F exists, then it is unique up to unique isomorphism; one denotes it by $\int^X F(X, X)$. A morphism with domain $\int^X F(X, X)$ and codomain Y is equivalent to a family $\{f_X\}_{X \in \mathcal{C}}$ of morphisms from $F(X, X)$ to Y such that (Y, f) is a dinatural transformation.

For H a finite-dimensional Hopf algebra over \mathbb{k} , endow the categories $H\text{-Mod}$ and $H\text{-Bimod}$ of left H -modules and of H -bimodules, respectively, with the tensor products (2.6) and (2.5) and with the dualities described at the beginning of Section 4. Consider the tensor product (bi)functor

$$G_{\otimes_{\mathbb{k}}}^H : H\text{-Mod}^{op} \times H\text{-Mod} \rightarrow H\text{-Bimod} \tag{A.2}$$

that acts on objects as

$$(X, \rho_X) \times (Y, \rho_Y) \xrightarrow{G_{\otimes_{\mathbb{k}}}^H} (X^* \otimes_{\mathbb{k}} Y, \rho_{X^\vee} \otimes id_Y, id_{X^*} \otimes (\rho_Y \circ \tau_{Y, H} \circ (id_Y \otimes s^{-1}))) \tag{A.3}$$

and on morphisms as $f \times g \mapsto f^\vee \otimes_{\mathbb{k}} g$. Pictorially, the action on objects is

(A.4)

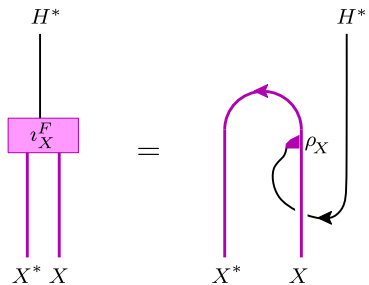
Remark A.1. The category $H\text{-Mod}^{\text{op}} \times H\text{-Mod}$ is naturally endowed with a tensor product, acting on objects as $(X \times Y) \times (X' \times Y') \mapsto (X \otimes^{H\text{-Mod}} X') \times (Y' \otimes^{H\text{-Mod}} Y)$. With respect to this tensor product and the tensor product (2.5) on $H\text{-Bimod}$, $G_{\otimes_{\mathbb{k}}}^H$ together with the associativity constraints from $\mathcal{Vect}_{\mathbb{k}}$ is a monoidal functor.

In this appendix we show that the coregular H -bimodule F introduced in Definition 2.8 is the coend of the functor $G_{\otimes_{\mathbb{k}}}^H$. We first present the appropriate dinatural family.

Lemma A.2. The family (ι_X^F) of morphisms

$$\iota_X^F := (d_X \otimes id_{H^*}) \circ [id_{X^*} \otimes (\rho_X \circ \tau_{X,H}) \otimes id_{H^*}] \circ (id_{X^*} \otimes id_X \otimes b_H) \tag{A.5}$$

in $H\text{-Bimod}$, pictorially given by

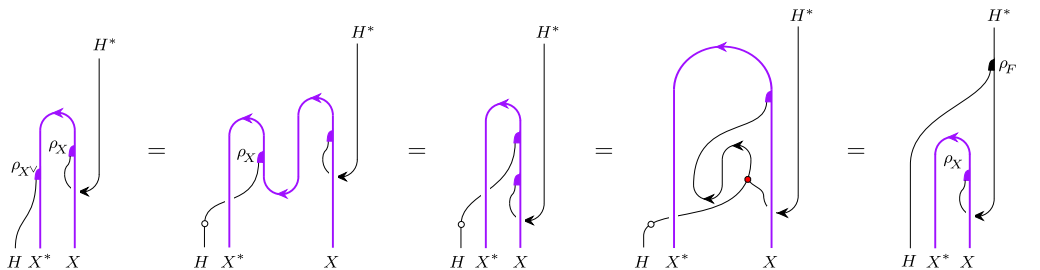

(A.6)

is dinatural for the functor $G_{\otimes_{\mathbb{k}}}^H$, i.e.

$$\iota_Y^F \circ G_{\otimes_{\mathbb{k}}}^H(id_Y, f) = \iota_X^F \circ G_{\otimes_{\mathbb{k}}}^H(f, id_X) \tag{A.7}$$

for any $f \in \text{Hom}_H(X, Y)$.

Proof. (i) First note that the maps (A.5) are a priori just linear maps in $\text{Hom}_{\mathbb{k}}(X^* \otimes_{\mathbb{k}} X, H^*)$. However, when H^* is endowed with the H -bimodule structure (2.8) and $X^* \otimes_{\mathbb{k}} X$ with the one implied by (A.2), we have the chain of equalities


(A.8)

showing that (A.6) intertwines the left action of H , and

(A.9)

showing that it also intertwines the right action.

(ii) The dinaturalness property amounts to the equality of the left- and right-hand sides of

(A.10)

for any module morphism f from X to Y . Now the first equality in (A.10) holds by definition of f^* , and the second equality holds because f is a module morphism. \square

Proposition A.3. *The H -bimodule F together with the dinatural family (ι_X^F) given by (A.5) is the coend of the functor $G_{\otimes_k}^H$,*

$$(F, \iota^F) = \int^X G_{\otimes_k}^H(X, X). \tag{A.11}$$

Proof. We have to show that (F, ι^F) is an initial object in the category of dinatural transformations from $G_{\otimes_k}^H$ to a constant.

(i) Let j^Z be a dinatural transformation from $G_{\otimes_k}^H$ to $Z \in H\text{-Bimod}$. Given any $X \in H\text{-Mod}$ and any $x_0 \in \text{Hom}_k(k, X)$ (i.e. element of X), applying the dinaturalness property of j^Z to the morphism

$f_{x_0} := \rho_X \circ (id_H \otimes x_0) \in \text{Hom}_H(H, X)$ (with H regarded as an H -module via the regular left action) yields $j_X^Z \circ (id_{X^*} \otimes x_0) = j_H^Z \circ (i_X^F \otimes \eta) \circ (id_{X^*} \otimes x_0)$. Namely, we have

$$\text{Diagram 1} \equiv \text{Diagram 2} = \text{Diagram 3} \equiv \text{Diagram 4} \tag{A.12}$$

and thus, after composition with $id_{X^*} \otimes \eta$,

$$j_X^Z \circ (id_{X^*} \otimes x_0) = \text{Diagram 1} = \text{Diagram 2} = j_H^Z \circ (i_X^F \otimes \eta) \circ (id_{X^*} \otimes x_0) \tag{A.13}$$

with i_X^F from (A.5). Since $x_0 \in \text{Hom}_{\mathbb{k}}(\mathbb{k}, X)$ is arbitrary, we actually have

$$j_X^Z = j_H^Z \circ (i_X^F \otimes \eta) \tag{A.14}$$

for any bimodule Z and dinatural transformation j^Z from $G_{\mathbb{k}}^H$ to Z .

(ii) Now consider the linear map

$$\kappa^Z := j_H^Z \circ (id_{H^*} \otimes \eta) \tag{A.15}$$

from H^* to Z . This is in fact a bimodule morphism from F to Z : Compatibility with the left H -action follows directly from the fact that j_H^Z is a morphism of bimodules, and thus in particular of left modules, while compatibility with the right H -action is seen as follows:

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5} \tag{A.16}$$

Here the element $h \in \text{Hom}_{\mathbb{k}}(\mathbb{k}, H)$ is arbitrary; the second equality invokes the dinaturalness of j^Z for the map $m \circ (id_H \otimes (s^{-1} \circ h)) \in \text{End}_H(H)$.

(iii) In terms of the morphism κ^Z , (A.14) amounts to

$$j_X^Z = \kappa^Z \circ i_X^F. \tag{A.17}$$

This establishes existence of the morphism from F to Z that is required for the universal property of the coend.

(iv) It remains to show that κ^Z is uniquely determined. This just follows by specializing (A.17) to the case $X = H$ and observing that i_H^F has a right-inverse. The latter property holds because of $i_H^F \circ (id_{H^*} \otimes \eta) = (d_H \otimes id_{H^*}) \circ (id_{H^*} \otimes b_H) = id_{H^*}$. \square

A.2. Some equivalences of braided monoidal categories

We note the following equivalences, where as usual H^{op} is H with opposite product $m \circ \tau_{H,H}$ (and with the same coproduct), and H^{coop} is H with opposite coproduct $\tau_{H,H} \circ \Delta$ (and with the same product).

Lemma A.4. (i) For any Hopf algebra H there are equivalences

$$H\text{-Bimod} \simeq (H \otimes_{\mathbb{k}} H)\text{-Mod} \simeq (H \otimes_{\mathbb{k}} H^{op})\text{-Mod} \tag{A.18}$$

as abelian categories.

(ii) The equivalences (A.18) extend to equivalences

$$H\text{-Bimod} \simeq (H \otimes_{\mathbb{k}} H^{coop})\text{-Mod} \simeq (H \otimes_{\mathbb{k}} H^{op})\text{-Mod} \tag{A.19}$$

as monoidal categories, with respect to the tensor products (2.6) on $H\text{-Mod}$ and (2.5) on $H\text{-Bimod}$. The constraint morphisms for the tensor functor structures of the equivalence functors are all identities.

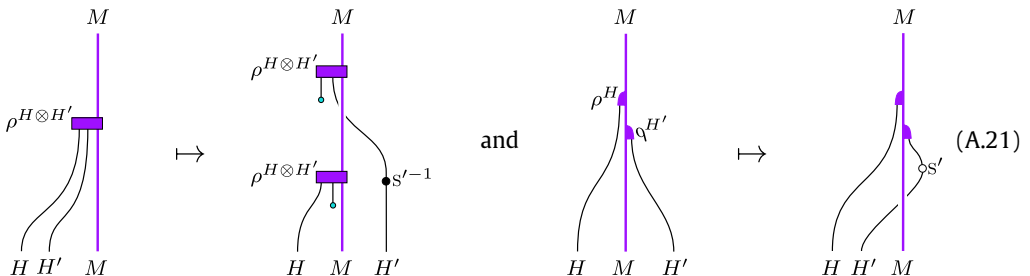
(iii) If the Hopf algebra H is quasitriangular with R -matrix R , then the equivalences (A.19) extend to equivalences

$$H\text{-Bimod} \simeq (\bar{H} \otimes_{\mathbb{k}} H^{coop})\text{-Mod} \simeq (\bar{H} \otimes_{\mathbb{k}} H^{op})\text{-Mod} \tag{A.20}$$

as braided monoidal categories, where $H\text{-Bimod}$ is endowed with the braiding (3.3) and \bar{H} is H with R -matrix R_{21}^{-1} . (Also, H^{op} is endowed with the natural quasitriangular structure inherited from H , i.e. has R -matrix R_{21} .)

Proof. (i) We derive each of the equivalences in a somewhat more general context.

For any two Hopf algebras H and H' there is an equivalence $H\text{-}H'\text{-Bimod} \simeq (H \otimes_{\mathbb{k}} H')\text{-Mod}$ as abelian categories. The equivalence is furnished by the two functors which on morphisms are the identity and which map objects according to



respectively.

Similarly, an equivalence $H\text{-}H'\text{-Bimod} \simeq (H \otimes_{\mathbb{k}} H'^{\text{op}})\text{-Mod}$ as abelian categories is furnished by functors that differ from those in (A.21) by just omitting the (inverse) antipode (compare e.g. [FRS, Prop. 4.6]).

(ii) For the first equivalence, compatibility with the tensor product follows for the second functor in (A.21) as

(A.22)

and analogously for the first functor, as well as for the second equivalence.

(iii) The Hopf algebras $\bar{H} \otimes_{\mathbb{k}} H^{\text{coop}}$ and $\bar{H} \otimes_{\mathbb{k}} H^{\text{op}}$ have natural quasitriangular structures, with R-matrices given by $(id_H \otimes c_{H,H} \otimes id_H) \circ (R_{21}^{-1} \otimes R^{-1})$. By direct calculation one checks that the two functors given in (A.21) (with $H' = H$), respectively the ones with the occurrences of the antipode removed, not only furnish an equivalence between $H\text{-Bimod}$ and $(\bar{H} \otimes_{\mathbb{k}} H^{\text{coop}})\text{-Mod}$, respectively $(\bar{H} \otimes_{\mathbb{k}} H^{\text{op}})\text{-Mod}$, as abelian monoidal categories, but map the braidings of these categories to each other as well.

Also note that the R-matrix furnishes an equivalence between $H^{\text{coop}}\text{-Mod}$ and $H\text{-Mod}$ as monoidal categories, so that in the equivalences (A.20) we could as well use H instead of H^{coop} . \square

Remark A.5. In view of Lemma A.4, Proposition A.3 is implied by Theorem 7.4.13 of [KL].

The significance of the coend (A.11) and of the equivalences in Lemma A.4 actually transcends the framework of the (bi)module categories considered in this paper. Namely, one can consider the situation that $H\text{-Mod}$ is replaced by a more general ribbon category \mathcal{C} , while the role of $H\text{-Bimod}$ is taken over by the Deligne product of \mathcal{C} with itself. Recall [De, Sect. 5] that the Deligne tensor product of two \mathbb{k} -linear abelian categories \mathcal{C} and \mathcal{D} that are locally finite, i.e. all morphism spaces of which are finite-dimensional and all objects of which have finite length, is a category $\mathcal{C} \boxtimes \mathcal{D}$ together with a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$ that is right exact and \mathbb{k} -linear in both variables and has the following universal property: for any bifunctor G from $\mathcal{C} \times \mathcal{D}$ to a \mathbb{k} -linear abelian category \mathcal{E} being right exact and \mathbb{k} -linear in both variables there exists a unique right exact \mathbb{k} -linear functor $G_{\boxtimes} : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$ such that $G \cong G_{\boxtimes} \circ \boxtimes$. In short, bifunctors from $\mathcal{C} \times \mathcal{D}$ become functors from $\mathcal{C} \boxtimes \mathcal{D}$. The category $\mathcal{C} \boxtimes \mathcal{D}$ is again \mathbb{k} -linear abelian and locally finite.

By the universal property of the Deligne product, there is a unique functor

$$G_{\boxtimes}^H : H\text{-Mod} \boxtimes H\text{-Mod} \longrightarrow H\text{-Bimod} \tag{A.23}$$

such that the bifunctor (A.2) can be written as the composition $G_{\boxtimes}^H = G_{\boxtimes}^H \circ (?^{\vee} \boxtimes Id)$, with the functor $?^{\vee} \boxtimes Id = \boxtimes \circ (?^{\vee} \times Id)$ acting as $X \times Y \mapsto X^{\vee} \boxtimes Y$ and $f \times g \mapsto f^{\vee} \boxtimes g$. On objects of $\mathcal{C} \boxtimes \mathcal{D}$ that are of the form $U \boxtimes V$ with $U \in \mathcal{C}$ and $V \in \mathcal{D}$, the functor G_{\boxtimes}^H acts as

$$(X, \rho_X) \boxtimes (Y, \rho_Y) \xrightarrow{G_{\boxtimes}^H} (X \otimes Y, \rho_X \otimes id_Y, id_X \otimes (\rho_Y \circ \tau_{Y,H} \circ (id_Y \otimes s^{-1}))). \tag{A.24}$$

Now by combining Proposition 5.3 of [De] with the first equivalence in (A.18), one sees (compare also e.g. [Fr, Ex. 7.10]) that the functor G_{\square}^H is an equivalence of abelian categories. Further, $H\text{-Mod} \boxtimes H\text{-Mod}$ has a natural monoidal structure [De, Prop. 5.17] as well as a braiding (which on objects of the form $U \boxtimes V$ acts as $(c_{U',U}^{H\text{-Mod}})^{-1} \otimes_{\mathbb{k}} c_{V,V'}^{H\text{-Mod}}$). With respect to these the equivalence (A.23) can be endowed with the structure of an equivalence of braided monoidal categories.

Observations analogous to those made here for the category $C_H = H\text{-Mod}$ in fact apply to any locally finite \mathbb{k} -linear abelian ribbon category \mathcal{C} . Hereby the Frobenius algebra F in $H\text{-Bimod}$ can be understood as a particular case of the coend

$$F_{\mathcal{C}} := \int^X X^{\vee} \boxtimes X \tag{A.25}$$

of the functor $?^{\vee} \boxtimes Id : C^{op} \times C \rightarrow \bar{C} \boxtimes C$ (where \bar{C} is C with opposite braiding), which exists for any such category C . This coend $F_{\mathcal{C}}$ has already been considered in [Ke] and [KL, Sect. 5.1.3]. It is natural to expect that also in this general setting the coend $F_{\mathcal{C}}$ still carries a natural Frobenius algebra structure. However, so far we only know that $F_{\mathcal{C}}$ is naturally a unital associative algebra in $\bar{C} \boxtimes C$.

For $C = H\text{-Mod}$, the category $\bar{C} \boxtimes C$ is also equivalent, as a ribbon category, to the center of C , and thus to the category of Yetter–Drinfeld modules over H . Hence instead of with H -bimodules we could equivalently work with Yetter–Drinfeld modules over H . In particular, the Frobenius algebra F can be recognized as the so-called [FFRS, Da2] full center of the tensor unit of $H\text{-Mod}$; in the Yetter–Drinfeld setting, this is described in Example 5.5 of [Da3].

A.3. The coend $\mathbf{HH}_{\triangleright\triangleleft}^*$ in $H\text{-Bimod}$

The coadjoint left and right actions $\rho_{\triangleright} \in \text{Hom}(H \otimes H^*, H^*)$ and $\rho_{\triangleleft} \in \text{Hom}(H^* \otimes H, H^*)$ of H on its dual H^* are by definition the morphisms

and

$$\rho_{\triangleright} = \text{ [Diagram 1] } \quad \text{and} \quad \rho_{\triangleleft} = \text{ [Diagram 2] } \tag{A.26}$$

We call the H -bimodule that consists of the vector space $H^* \otimes_{\mathbb{k}} H^*$, endowed with the coadjoint left H -action on the first tensor factor and with the coadjoint right H -action on the second factor, the *coadjoint bimodule* and denote it by $\mathbf{HH}_{\triangleright\triangleleft}^*$. That is,

$$\mathbf{HH}_{\triangleright\triangleleft}^* = (H^* \otimes_{\mathbb{k}} H^*, \rho_{\triangleright} \otimes id_{H^*}, id_{H^*} \otimes \rho_{\triangleleft}). \tag{A.27}$$

We will now show that this bimodule arises as the coend of the functor

$$\otimes \circ (?^{\vee} \times Id) : H\text{-Bimod}^{op} \times H\text{-Bimod} \rightarrow H\text{-Bimod}, \tag{A.28}$$

where \otimes and $?^{\vee}$ are the tensor product (2.5) and right duality (4.1) of $H\text{-Bimod}$. A crucial input is the braided monoidal equivalence described in Lemma A.4(iii).

Proposition A.6. Let H be a finite-dimensional ribbon Hopf \mathbb{k} -algebra. Then the H -bimodule $\mathbf{HH}_{\triangleright\triangleleft}^*$ together with the family $i_X^{\triangleright\triangleleft}$ of morphisms

(A.29)

from $X^\vee \otimes X$ to $\mathbf{HH}_{\triangleright\triangleleft}^*$, for $X = (X, \rho_X, \varrho_X) \in H\text{-Bimod}$, is the coend for the functor (A.28):

$$(\mathbf{HH}_{\triangleright\triangleleft}^*, i^{\triangleright\triangleleft}) = \int^X X^\vee \otimes X. \tag{A.30}$$

Proof. The statement follows from the results of [Ly1, Sect. 1.2] and [Vi, Sect. 4.5] for the coend of the functor $\otimes \circ (?^\vee \times Id)$ from $H'\text{-Mod}^{\text{op}} \times H'\text{-Mod}$ to $H'\text{-Mod}$, with the Hopf algebra $H' = \overline{H} \otimes H^{\text{op}}$, by transporting them via the equivalence (A.20) to $H\text{-Bimod}$.

We omit the details, but find it instructive to compare a few aspects of a direct proof to the corresponding parts of the proof of Lemma A.2 and of Proposition A.3:

First, dinaturalness follows by an argument completely parallel to the one used in (A.10) to show dinaturalness of the family (A.5). Second, the role of the morphism f_{x_\circ} (that is, left action of H on an element x_\circ of X) that appears in formula (A.12) is taken over by the map

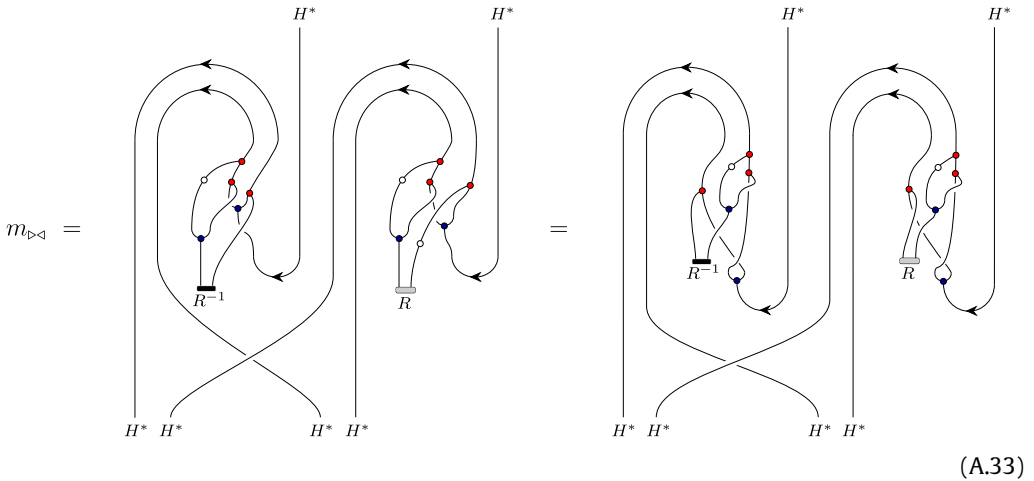
(A.31)

This map is a bimodule morphism from $H \otimes_{\mathbb{k}} H$ – regarded as an H -bimodule $(H \otimes_{\mathbb{k}} H)_{\text{reg}}$ via the regular left and right actions on the second and first factor, respectively – to X . Analogously as in (A.13) one shows that for any dinatural transformation j^Z from the functor (A.28) to $Z \in H\text{-Bimod}$ one has $j_X^Z \circ (id_{X^*} \otimes x_\circ) = \kappa^Z \circ i_X^{\triangleright\triangleleft} \circ (id_{X^*} \otimes x_\circ)$, with the map κ^Z defined by $\kappa^Z := j_{(H \otimes_{\mathbb{k}} H)_{\text{reg}}}^Z \circ (id_{H^*} \otimes id_{H^*} \otimes \eta \otimes \eta)$. And again κ^Z is a bimodule morphism, so that the existence part of the universal property of the coend is established. Uniqueness follows by specializing to the case $X = (H \otimes_{\mathbb{k}} H)_{\text{reg}}$ and observing that $i_{(H \otimes_{\mathbb{k}} H)_{\text{reg}}}^{\triangleright\triangleleft}$ is an epimorphism (as is e.g. seen from $i_{(H \otimes_{\mathbb{k}} H)_{\text{reg}}}^{\triangleright\triangleleft} \circ (id_{H^*} \otimes id_{H^*} \otimes \eta \otimes \eta) = id_{H^*} \otimes id_{H^*}$). \square

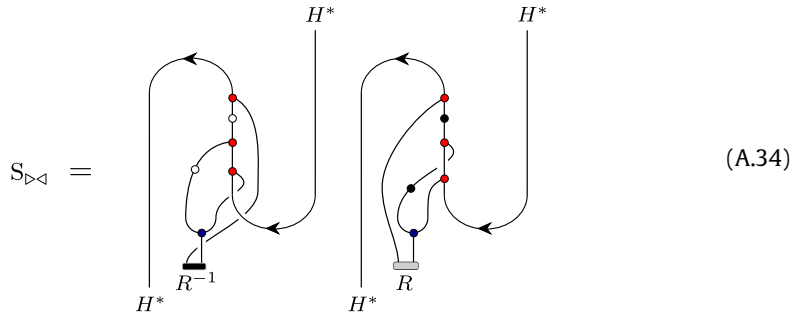
Corollary A.7. *The H -bimodule $\mathbf{HH}_{\triangleright\triangleleft}^*$ carries the structure of a Hopf algebra, with structure morphisms given as follows. The unit, counit and coproduct are*

$$\begin{aligned} \eta_{\triangleright\triangleleft} &= \varepsilon^\vee \otimes \varepsilon^\vee, & \varepsilon_{\triangleright\triangleleft} &= \eta^\vee \otimes \eta^\vee, \\ \Delta_{\triangleright\triangleleft} &= (\text{id}_{H^*} \otimes \tau_{H^*, H^*} \otimes \text{id}_{H^*}) \circ ((m^{\text{op}})^\vee \otimes m^\vee), \end{aligned} \tag{A.32}$$

the product is



and the antipode is



Proof. We just have to specialize the general results of [Ly2], which apply to the coend of the functor $\otimes \circ (?^\vee \otimes \text{Id}) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ in any \mathbb{k} -linear abelian ribbon category \mathcal{C} , to the case $\mathcal{C} = H\text{-Bimod}$. The calculations are straightforward, and except for the multiplication and the antipode they are very short.

Let us just mention that the first equality in (A.33) follows from the general results (see [Ly2, Prop. 2,3], as well as [Vi, Sect. 1.6] or [FSc, Sect. 4.3]) together with (4.2) and the defining relation (2.1) of the R -matrix. The second equality in (A.33) follows with the help of standard manipulations from the fact that the R -matrix intertwines the coproduct and the opposite coproduct. \square

Proposition A.8. (i) *If Λ is a two-sided integral of H , then*

$$\lambda_{\triangleright\triangleleft} := \Lambda^\vee \otimes \Lambda^\vee \tag{A.35}$$

is two-sided cointegral of the Hopf algebra $(\mathbf{HH}_{\triangleright\triangleleft}^*, m_{\triangleright\triangleleft}, \eta_{\triangleright\triangleleft}, \Delta_{\triangleright\triangleleft}, \varepsilon_{\triangleright\triangleleft}, S_{\triangleright\triangleleft})$.

(ii) If λ is a right cointegral of H , then

$$\Lambda_{\triangleright\triangleleft} := \lambda^\vee \otimes \lambda^\vee \tag{A.36}$$

is a two-sided integral of $(\mathbf{H}H_{\triangleright\triangleleft}^*, m_{\triangleright\triangleleft}, \eta_{\triangleright\triangleleft}, \Delta_{\triangleright\triangleleft}, \varepsilon_{\triangleright\triangleleft}, S_{\triangleright\triangleleft})$.

Proof. (i) Inserting the definitions one has

$$\begin{aligned} (\lambda_{\triangleright\triangleleft} \otimes \text{id}_{H^* \otimes H^*}) \circ \Delta_{\triangleright\triangleleft} &= (m \circ (\Lambda \otimes \text{id}_{H^*}))^* \otimes (m \circ (\text{id}_{H^*} \otimes \Lambda))^* \quad \text{and} \\ (\text{id}_{H^* \otimes H^*} \otimes \lambda_{\triangleright\triangleleft}) \circ \Delta_{\triangleright\triangleleft} &= (m \circ (\text{id}_{H^*} \otimes \Lambda))^* \otimes (m \circ (\Lambda \otimes \text{id}_{H^*}))^*. \end{aligned} \tag{A.37}$$

Since Λ is a two-sided integral of H , both of these expressions are equal to $\eta_{\triangleright\triangleleft} \circ \lambda_{\triangleright\triangleleft}$.

(ii) That $\Lambda_{\triangleright\triangleleft}$ is a left integral readily follows from the first expression for the product in (A.33) together with the fact that λ is a right cointegral and that it satisfies (5.12). That $\Lambda_{\triangleright\triangleleft}$ is also a right cointegral follows in the same way by using instead the second expression in (A.33) for the product. \square

Appendix B. Motivation from conformal field theory

A major motivation for the mathematical results of this paper comes from structures that originate in full local two-dimensional conformal field theory. In this appendix we briefly describe some of these structures.

In representation theoretic approaches to conformal field theory the starting point is a chiral symmetry algebra together with its category \mathcal{C} of representations. For any mathematical structure that formalizes the physical concept of chiral symmetry algebra, the category \mathcal{C} can be endowed with a lot of additional structure. In particular, in many cases it leads to a so-called modular functor. A modular functor actually consists of a collection of functors. Namely, to any compact Riemann surface $\Sigma_{g,n}$ of genus g and with a finite number n of marked points it assigns a functor

$$F_{\Sigma_{g,n}} : \mathcal{C}^{\boxtimes n} \rightarrow \mathcal{Vect} \tag{B.1}$$

from $\mathcal{C}^{\boxtimes n}$ to the category \mathcal{Vect} of finite-dimensional complex vector spaces. This collection of functors is required to obey a system of compatibility conditions, which in particular expresses factorization constraints and accommodates actions of mapping class groups of surfaces. Thus, selecting for a genus- g surface $\Sigma_{g,n}$ with n marked points any n -tuple (V_1, V_2, \dots, V_n) of objects of the category \mathcal{C} , we obtain a finite-dimensional complex vector space $F_{\Sigma_{g,n}}(V_1, V_2, \dots, V_n)$ which carries an action of the mapping class group of $\Sigma_{g,n}$. In chiral conformal field theory, this space plays the role of the space of conformal blocks with chiral insertion of type V_i at the i th marked point of $\Sigma_{g,n}$.

In the particular case that the category \mathcal{C} is finitely semisimple, the structure of a modular functor is reasonably well understood. Specifically, precise conditions are known under which the representation category of a vertex algebra \mathcal{V} is a modular tensor category. In this case the Reshetikhin–Turaev construction allows one to obtain a modular functor just on the basis of \mathcal{C} as an abstract category. In a remarkable development, Lyubashenko and others (see [KL] and references cited there) have extended many aspects of this story to a larger class of monoidal categories that are not necessarily semisimple any longer. In particular, given an abstract monoidal category with adequate additional properties, one can still construct representations of mapping class groups.

Representation categories that are not semisimple are of considerable physical interest; they arise in particular in various systems of statistical mechanics. The corresponding models of conformal field theory have been termed “logarithmic” conformal field theories. A complete characterization of this class of models has not been achieved yet, but a necessary requirement ensuring tractability is that the category \mathcal{C} , while being non-semisimple, still possesses certain finiteness properties, e.g. each object should have a composition series of finite length.

In the present paper we consider an even more restricted, but non-empty, subclass, namely the one for which the monoidal category \mathcal{C} is equivalent to the representation category of a finite-dimensional factorizable ribbon Hopf algebra. Finite-dimensional Hopf algebras H_{KL} have indeed been associated, via a Kazhdan–Lusztig correspondence, to certain classes of logarithmic conformal field theories. These Hopf algebras H_{KL} do not have an R-matrix, albeit they do have a monodromy matrix that is even factorizable [FGST] (so that in particular the partial monodromy traces which we introduced in Section 5 can still be defined). Accordingly our results do not perfectly match the presently available conformal field theoretic proposals. On the other hand, it is apparent that the Hopf algebras H_{KL} are not quite the appropriate algebraic structures: their representation categories, albeit being equivalent to the representation categories of the relevant vertex algebras as abelian categories, are not equivalent to them as monoidal categories.⁶

The Riemann surface of interest to us is $\Sigma_{1,1}$, a one-punctured torus. This surface is distinguished by the fact [Ye2] that it carries a natural Hopf algebra structure in the category of three-cobordisms. For general reasons, the functor $F_{\Sigma_{1,1}}$ is representable:

$$F_{\Sigma_{1,1}} \cong \text{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, -). \quad (\text{B.2})$$

In this way we obtain for the category \mathcal{C} a distinguished object, and this object is actually a Hopf algebra in \mathcal{C} . The construction in [Ly1] turns the logic around: it starts with a Hopf algebra object $K_{\mathcal{C}} \in \mathcal{C}$ canonically associated to the braided category and constructs the functor as $F_{\Sigma_{1,1}}(V) = \text{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, V)$ for any object $V \in \mathcal{C}$.

From the point of view of two-dimensional conformal field theory, the one-punctured torus is the surface relevant for partition functions. We are interested in this paper in a candidate for the partition function of the space of bulk fields and thus in one-point functions of bulk fields on the torus. The space $\mathcal{H}_{\text{bulk}}$ of bulk fields carries the structure of a bimodule over the chiral symmetry algebra \mathcal{V} . In the case that the category \mathcal{C} is semisimple, a particularly simple solution is given by the bulk state space $\bigoplus_i S_i^{\vee} \otimes_{\mathcal{C}} S_i$, where the (finite) summation is over all isomorphism classes $[S_i]$ of simple \mathcal{V} -modules. The corresponding partition function is the so-called *charge conjugation* modular invariant. It has been conjectured [QS,GR] that this type of bulk state space exists in the non-semisimple case as well, and that as a left \mathcal{V} -module it decomposes as

$$\mathcal{H}_{\text{bulk}} \cong \bigoplus_i P_i^{\vee} \otimes_{\mathcal{C}} S_i, \quad (\text{B.3})$$

with P_i the projective cover of the simple \mathcal{V} -module S_i .

According to the principle of holomorphic factorization, a correlation function for a conformal real surface is an element in the space of conformal blocks associated to the oriented double of the surface. Thus a one-point function on the torus is a specific element in the space of conformal blocks associated to the double of the torus (as a real surface), that is, of the disconnected sum of two copies of $\Sigma_{1,1}$ with opposite orientation. For any selection of a pair (V_1, V_2) of objects of \mathcal{C} at the two points on the double cover that lie over the one insertion point on the torus, this space of conformal blocks is the tensor product $\text{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, V_1)^* \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{C}}(K_{\mathcal{C}}, V_2)$. More compactly, this space can be written as a morphism space of another braided tensor category $\mathcal{D} := \bar{\mathcal{C}} \boxtimes \mathcal{C}$, which has its own canonical Hopf algebra object $K_{\mathcal{D}}$. As we have noted in Section A.2, if \mathcal{C} is the category $H\text{-Mod}$ of left modules over a finite-dimensional factorizable ribbon Hopf algebra H , then \mathcal{D} can be identified with the category of bimodules over H , with a tensor product derived from the coproduct on H .

Compatibility of (B.3) with short exact sequences implies that the character of the projective cover P_i is a linear combination of simple characters, with coefficients given by the entries of the Cartan matrix of the category. Thus in the charge conjugation case the Cartan matrix provides the coefficients

⁶ Also, constructing algebras with the help of the Kazhdan–Lusztig correspondence involves some arbitrariness. It has been suggested [ST] that one should better work with Hopf algebras in a category of Yetter–Drinfeld modules built from braided vector spaces, rather than Hopf algebras in \mathcal{Vect} .

in the bilinear combination of characters that, owing to holomorphic factorization, describes the bulk partition function. Indeed, as mentioned in Remark 5.12(iii), the same structure is seen when expressing the morphism $\varepsilon_F \circ t_Q(\Delta_F)$, which (as follows from Remark 5.6(i)) is nothing but the character of F for the algebra $K_{\mathcal{D}}$, as a bilinear combination of characters for the algebra $K_{\mathcal{C}}$.

Correlation functions in conformal field theory should be invariant under the relevant mapping class group. For the one-point correlation function on the torus we are thus interested in finding an object $F \in \mathcal{D}$ corresponding to the space of bulk fields as well as a vector

$$Z_F \in \text{Hom}_{\mathcal{D}}(K_{\mathcal{D}}, F) \quad (\text{B.4})$$

that is invariant under the action of the mapping class group $\Gamma_{1;1}$ of the one-punctured torus. Moreover, comparison with the semisimple situation, in which \mathcal{C} is a modular tensor category, indicates that the object F should possess a structure of a commutative symmetric Frobenius algebra in \mathcal{D} . The partition function, given by $\varepsilon_F \circ Z_F$, is then invariant under the modular group $\text{SL}(2, \mathbb{Z})$.

This is precisely what the present paper achieves for the case $\mathcal{C} \simeq H\text{-Mod}$: given a ribbon Hopf algebra automorphism ω of H , we obtain a commutative symmetric Frobenius algebra F_{ω} in the category $H\text{-Bimod}$. As an object, F_{ω} is the twisted coregular bimodule ${}^{id_H}(F)_{\omega}$, so that e.g. its decomposition as a left H -module precisely reproduces the decomposition (B.3) above. (The conjecture (B.3) has only been made for the case corresponding to trivial automorphism $\omega = id_H$, though.) Also note that according to Remark 6.3(ii) the counit of F_{ω} is unique up to a non-zero scalar; in the conformal field theory context this amounts to uniqueness of the vacuum state. The partial monodromy trace (5.23) of the coproduct $\Delta : F_{\omega} \rightarrow F_{\omega} \otimes F_{\omega}$ furnishes a $\Gamma_{1;1}$ -invariant morphism $Z_{\omega} \in \text{Hom}(K_{\mathcal{D}}, F_{\omega})$. The morphism $\varepsilon_{F_{\omega}} \circ Z_{\omega}$, associated to H and ω , is a natural candidate for a modular invariant partition function on the torus.

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