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Clique separator decomposition of hole-free and diamond-free graphs and algorithmic consequences

Andreas Brandstädt[a,*], Vassilis Giakoumakis[b], Frédéric Maffray[c]

[a] *Fachbereich Informatik, Universität Rostock, A.-Einstein-Str. 21, D-18051 Rostock, Germany*

[b] *MIS (Modélisation, Information & Systèmes), Université de Picardie Jules Verne, Amiens, France*

[c] *C.N.R.S., Laboratoire G-SCOP, Grenoble-INP, Université Joseph Fourier, Grenoble Cedex, France*

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ABSTRACT

Clique separator decomposition, introduced by Whitesides and Tarjan, is one of the most important graph decompositions. A *hole* is a chordless cycle with at least five vertices. A *paraglider* is a graph with five vertices a, b, c, d, e and edges $ab, ac, bc, bd, cd, ae, de$. We show that every (hole, paraglider)-free graph admits a clique separator decomposition into graphs of three very specific types. This yields efficient algorithms for various optimization problems in this class of graphs.

1. Introduction, motivation and related work

Graph decompositions play an important role in structural and algorithmic aspects of graph theory. A *clique separator* (or *clique cutset*) of a graph G is a clique K in G such that $G \setminus K$ has more connected components than G. An *atom* is a graph without clique separator. An *atom of a graph* G is any induced subgraph of G that is an atom. Whitesides [38] proved that a clique separator decomposition of a graph can be determined in polynomial time; Tarjan [36] improved that result and showed that the decomposition can be applied to various optimization problems such as minimum fill-in, maximum weight independent set (MWIS), maximum weight clique, and coloring: if the problem is solvable in polynomial time on the atoms of a hereditary graph class $\mathcal{C}$, then it is solvable in polynomial time on class $\mathcal{C}$. In this paper, we are going to analyze the structure of atoms in two subclasses of hole-free graphs.

A *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complementary graph of a hole. A graph is *hole-free* (*antihole free*, respectively) if it contains no induced subgraph which is isomorphic to a hole (an antihole, respectively). The words odd and even, when applied to a hole or antihole, refer to the number of its vertices. For any integer $n \geq 1$, let K_n denote a complete graph with n vertices and P_n denote a chordless path with n vertices. For $n \geq 3$, let C_n denote a chordless cycle with n vertices. So any C_n with $n \geq 5$ is a hole. Note that, in our terminology, C_4 is not a hole. The graph $K_4 \setminus e$ (i.e., a clique on four vertices minus one edge) is called *diamond*. A *paraglider* is a graph with five vertices a, b, c, d, e and seven edges $ab, ac, bc, bd, cd, ae, de$ (see Fig. 1). Note that a paraglider contains a diamond. Here we will study the class of (hole, paraglider)-free graphs (HP-free graphs for short). Some of the results also apply to the subclass of (hole, diamond)-free graphs (HD-free graphs).

* Corresponding author.
E-mail addresses: ab@informatik.uni-rostock.de (A. Brandstädt), vassilis.giakoumakis@u-picardie.fr (V. Giakoumakis), frederic.maffray@inpg.fr (F. Maffray).

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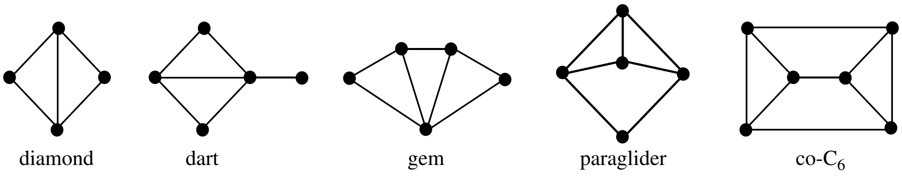

Fig. 1. Diamond, dart, gem, paraglider, and co-C_6.

Recall that a graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the maximum clique size in H. The celebrated *Strong Perfect Graph Theorem* proved by Chudnovsky et al. [19] states (as conjectured by Berge [2]) that *a graph is perfect if and only if it is odd-hole free and odd-antihole free*.

Cycle properties of graphs and their algorithmic aspects play a fundamental role in combinatorial optimization, discrete mathematics, and computer science. Various graph classes are characterized in terms of cycle properties — among them are the classes of chordal graphs, weakly chordal graphs, and perfect graphs, which are of great importance for algorithmic graph theory and various applications. A graph is *chordal* (also called *triangulated*) if it contains no chordless cycle on at least four vertices. See, for example, [13,23,30] for the many facets of chordal graphs. A famous theorem of Dirac [21] states that *every chordal graph either is a clique or has a clique cutset*. It follows that a graph is chordal if and only if it is decomposable by clique separator decomposition into atoms that are cliques. HP-free graphs obviously generalize chordal graphs.

Recently there has been much work on related classes such as even-hole-free (forbidding also C_4) and diamond-free graphs [28] (see also [37]) and [22] dealing with the structure and recognition of (C_4, diamond)-free graphs. The classes of *weakly chordal* graphs and *chordal bipartite* graphs are also of importance here. A graph is *weakly chordal* (or *weakly triangulated*) if it is hole-free and antihole free. The classes of weakly chordal graphs and HP-free graphs are incomparable, as shown by the examples of the paraglider (which is weakly chordal but not HP-free) and $\overline{C_6}$ (which is HP-free but not weakly chordal). A graph is *bipartite* if it contains no cycle of odd length, and *chordal bipartite* if it is bipartite and contains no hole. Chordal bipartite graphs were introduced in [24]. HD-free graphs generalize the class of chordal bipartite graphs; moreover, diamond-free chordal graphs are the well-known block graphs — see [13] for various characterizations and the importance of chordal bipartite graphs as well as of block graphs. In [11,17], various characterizations of (dart, gem)-free chordal graphs are given; among others, it is shown that a graph is (dart, gem)-free chordal if and only if it results from substituting cliques into the vertices of a block graph.

Since every hole C_k with $k \geq 7$ contains the disjoint union of P_2 and P_3, and the paraglider is the complementary graph of $P_2 \cup P_3$, it follows that HP-free graphs contain no odd hole and no odd antihole. Thus, by the Strong Perfect Graph Theorem, HP-free graphs are perfect. Our structural results for atoms of HP-free graphs, however, will give a more direct way to show perfection of HP-free graphs. It is well known [3,27] that a graph is perfect if and only if its atoms are perfect; and it turns out (as we will show below) that the atoms of HP-free graphs belong to simple classes of perfect graphs.

A *matched co-bipartite graph* is a graph H that consists of two disjoint cliques of size k, with $k \geq 3$, such that the edges between these two cliques form a matching with k edges. Note that $\overline{C_6}$ is a matched co-bipartite graph.

A *complete multipartite graph* is a graph whose vertex set can be partitioned into parts $S_1, \ldots, S_k$ such that any two vertices are adjacent if and only if they belong to distinct parts.

Our main result is the following theorem.

Theorem 1. *A graph G is (hole, paraglider) free if and only if every atom of G is either*

- *a complete multipartite graph, or*
- *the join of a chordal bipartite graph and a (possibly empty) clique, or*
- *the join of a matched co-bipartite graph and a (possibly empty) clique.*

The proof of Theorem 1 is given in Section 2. By Tarjan [36], Theorem 1 has various algorithmic consequences; in Section 3, we describe these and others.

We finish this section by recalling some definitions and notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* $N(x)$ of a vertex x in G is the set $N(x) = \{u \in V(G) \mid ux \in E\}$. The neighborhood $N(X)$ of a subset $X \subseteq V$ is the set $\{u \in V(G) \mid u$ is adjacent to a vertex of $X\}$. Given a subgraph H of G, let $N_H(x)$ denote the set $N(x) \cap V(H)$, and let $N_H(X)$ denote the set $N(X) \cap V(H)$. Given a set $S \subset V(G)$ and a vertex x, we say that x is *complete* to S if it is adjacent to every vertex of S, and *anticomplete* to S if it is not adjacent to any vertex of S.

The *complementary graph* of G is the graph $\overline{G}$ whose vertex set is $V(G)$ and edge set is $\{xy \mid x \neq y$ and $xy \notin E(G)\}$.

A set $U \subseteq V(G)$ is *independent* if its vertices are pairwise nonadjacent. A set $U \subseteq V(G)$ is a *clique* if its vertices are pairwise adjacent.

For any subgraph H of G, we let $G \setminus H$ denote the subgraph induced by the set of vertices $V(G) \setminus V(H)$.

Let $\mathcal{F}$ be a set of graphs. A graph G is $\mathcal{F}$ *free* if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$. As already mentioned, G is *hole-free* (is *antihole free*, respectively) if no induced subgraph of G is isomorphic to a hole (an antihole, respectively).

2. Structure of (hole, paraglider)-free and (hole, diamond)-free atoms

Let $\mathcal{G}_1$ be the class of complete multipartite graphs, $\mathcal{G}_2$ be the class of graphs that are the join of a chordal bipartite graph and a clique, and $\mathcal{G}_3$ be the class of graphs that are the join of a matched co-bipartite graph and a clique. Let us refer to these three classes as *basic*. In view of Theorem 1, we want to show that every HP-free atom is in one of the three basic classes. Note that every atom is connected (we consider the empty set as a clique, so a disconnected graph has a clique cutset).

The following theorem describes the structure of HP-free atoms that contain a $\overline{C_6}$.

Theorem 2. *Let G be an HP-free atom that contains an induced $\overline{C_6}$. Then G is the join of a matched co-bipartite graph and a (possibly empty) clique.*

Proof. Let G be an HP-free atom. Suppose that G contains a $\overline{C_6}$. Let H be a maximal matched co-bipartite graph that extends a $\overline{C_6}$ in G. Let $V(H)$ be partitioned into two cliques $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, with $k \geq 3$, where $a_1b_1, \ldots, a_kb_k$ are the edges between A and B. We claim the following.

$$\text{For every vertex } x \text{ of } G \setminus H,\ \text{ either } N_H(x) \text{ is a clique or } N_H(x) = V(H). \quad \square \tag{1}$$

Proof of (1). Suppose that $N_H(x)$ is not a clique, so, up to relabeling, x is adjacent to a_1 and b_2. Then x is adjacent to one of a_3 and b_3, for otherwise $\{x, a_1, a_3, b_3, b_2\}$ induces a C_5. Assume, up to symmetry, that x is adjacent to a_3. Then x is adjacent to a_2, for otherwise $\{x, a_1, a_2, a_3, b_2\}$ induces a paraglider. Moreover, if $k \geq 4$, then x is adjacent to each a_i with $4 \leq i \leq k$, for otherwise either $\{x, a_1, a_i, b_i, b_2\}$ induces a C_5 (if x is not adjacent to b_i) or $\{x, a_1, a_2, a_i, b_i\}$ induces a paraglider (if x is adjacent to b_i). Then x is adjacent to each b_j with $1 \leq j \leq k$, for otherwise $\{x, a_2, b_2, a_j, b_j\}$ induces a paraglider. Thus $N_H(x) = V(H)$. $\square$

Let R be the set of vertices that are complete to $V(H)$. Then

$$R \text{ is a clique.} \tag{2}$$

Proof of (2). If R contains non-adjacent vertices u and v, then $\{u, v, a_1, b_2, b_3\}$ induces a paraglider. $\square$

$$\text{Let } F \text{ be any component of } G \setminus (V(H) \cup R). \text{ Then } N_H(F) \text{ is a clique.} \tag{3}$$

Proof of (3). Suppose that there are non-adjacent vertices x and y in $N_H(F)$. Let u be a neighbor of x in F and v be a neighbor of y in F. Note that $u \neq v$ by (1), and since u and v are not in R. There is a chordless path P between u and v in F. We choose x, y, u, v and P such that P is as short as possible. Up to relabeling, let $x = a_1$ and $y = b_2$. Since u and v are not in R, (1) implies that ub_3 and va_3 are not edges. Any interior vertex w of P is not adjacent to a_1 or a_3, for otherwise the subpath of P between w and v contradicts the choice of P; and similarly, w is not adjacent to b_2 or b_3. Since $V(P) \cup \{a_1, a_3, b_2, b_3\}$ cannot contain a hole, it must be that ua_3 and vb_3 are edges and $P = uv$. Now, since u is adjacent to a_1 and a_3, by (1), we have $N_H(u) \subseteq A$, and similarly, $N_H(v) \subseteq B$. Then u is adjacent to a_2, for otherwise $\{u, a_1, a_2, b_2, v\}$ induces a C_5; and if $k \geq 4$, u is adjacent to each a_i with $4 \leq i \leq k$, for otherwise there is a C_6 or C_5 (depending on the adjacency of v and b_i) induced among u, a_1, a_i, b_i, b_2, v. So $N_H(u) = A$, and similarly, $N_H(v) = B$. But then $V(H) \cup \{u, v\}$ induces a matched co-bipartite graph, which contradicts the maximality of H. $\square$

In conclusion, if $G \setminus (V(H) \cup R)$ has a component F, then, by (2) and (3), $N_H(F) \cup R$ is a clique cutset (that separates F from $H \setminus N_H(F)$), a contradiction to the fact that G is an atom. Therefore we have $V(G) = V(H) \cup R$, and so G is the join of a matched co-bipartite graph and a clique; that is, G is in class $\mathcal{G}_3$. This finishes the proof of Theorem 2. $\square$

Note that in a (hole, diamond)-free graph G that contains a $\overline{C_6}$, say H, no vertex can be complete to $V(H)$.

Corollary 1. *If G is a (hole, diamond)-free atom containing an induced $\overline{C_6}$ then G is a matched co-bipartite graph.*

Now we examine the case when there is no $\overline{C_6}$. We first need an easy lemma.

Lemma 1. *In a chordal bipartite graph H, let P be a chordless even path and v be a vertex adjacent to the two endvertices of P. Then v is adjacent to every second vertex of P and not adjacent to the other vertices of P.*

Proof. Let $P = p_0 - \cdots - p_k$ ($k \geq 2$). If the lemma does not hold, there are consecutive vertices p_i and p_{i+1} of P that are either (a) both adjacent to v or (b) both not adjacent to v. In case (a), H contains a triangle. In case (b), let h be the largest integer with $0 \leq h < i$ and j be the smallest integer with $i + 1 < j \leq k$ such that v is adjacent to p_h and p_j. Then $\{v, p_h, \ldots, p_j\}$ induces a hole, a contradiction to H being chordal bipartite. $\square$

Let $K_{p,q}$ denote the complete bipartite graph with parts of size p and q respectively, and let $K_{3,3} \setminus e$ be obtained from a $K_{3,3}$ by removing one edge.

Theorem 3. *Let G be an HP-free atom that contains no $\overline{C_6}$. Suppose that G contains a $K_{3,3} \setminus e$. Then G is the join of a chordal bipartite graph and a (possibly empty) clique.*

Proof. Let H be a bipartite subgraph of G such that H contains a $K_{3,3} \setminus e$, H has no clique cutset, and $V(H)$ is maximal with this property (G has such a subgraph because $K_{3,3} \setminus e$ itself has no clique cutset). If $G = H$, there is nothing to prove, so let $G \neq H$. Recall that G and H are connected. Let x be any vertex of $G \setminus H$. Our aim is to prove claim (6) below, and for that purpose we need two intermediate steps.

$$\text{Consider any 6-tuple } \{v_1, \dots, v_6\} \subset V(H) \text{ with edges } v_iv_{i+1} \ (\text{mod } 6) \text{ and } v_1v_4, \text{ where optionally each of } v_2v_5 \text{ and } v_3v_6 \text{ may also exist. If } \{v_1, \dots, v_4\} \subset N(x), \text{ then also } \{v_5, v_6\} \subset N(x). \quad \square \tag{4}$$

Proof of (4). Since $\{v_1, v_4\}$ is not a cutset of H, there is a shortest path P from $\{v_2, v_3\}$ to $\{v_5, v_6\}$ in $H \setminus \{v_1, v_4\}$. We prove the claim by induction on the length of P. Let $P = p_0 - \cdots - p_k$, with $k \geq 1$.

First, suppose that P is odd. So, up to symmetry, let $p_0 = v_2$, $p_k = v_5$, and $V(P) \cap \{v_1, v_3, v_4, v_6\} = \emptyset$. If $k = 1$ (i.e., v_2v_5 is an edge), then x must be adjacent to v_5, for otherwise $\{x, v_2, v_3, v_4, v_5\}$ induces a paraglider; and to v_6, for otherwise $\{x, v_1, v_4, v_5, v_6\}$ induces a paraglider. Now let $k \geq 3$; that is, v_2v_5 and v_3v_6 are not edges. Since H is bipartite, there is no edge v_6p_i with i even. If there is an edge v_6p_i with i odd and $i < k$, then the path $p_0 - \cdots - p_i - v_6$ is shorter than P, a contradiction. So there is no such edge, and $p_0 - \cdots - p_k - v_6$ is a chordless path P'. By Lemma 1 applied to P' and v_1, vertex v_1 is adjacent to $p_2, p_4, \dots, p_{k-1}$ (and not adjacent to $p_1, p_3, \dots, p_k$). Likewise, $v_3 - p_0 - \cdots - p_k$ is a chordless path, and v_4 is adjacent to $p_1, p_3, \dots, p_{k-2}$ (and not adjacent to $p_2, p_4, \dots, p_{k-1}$). Then x is adjacent to p_1, for otherwise $\{x, v_1, v_4, p_0, p_1\}$ induces a paraglider; and to p_2, for otherwise $\{x, v_1, v_4, p_1, p_2\}$ induces a paraglider. Then, by the induction hypothesis, applied to the 6-tuple $\{p_1, p_2, v_1, v_4, v_5, v_6\}$ with path $p_2 - \cdots - p_k$, vertex x is adjacent to v_5 and v_6.

Now suppose that P is even, so, up to symmetry, let $p_0 = v_2$, $p_k = v_6$, $k \geq 2$, and $V(P) \cap \{v_1, v_3, v_4, v_5\} = \emptyset$. By Lemma 1 applied to P and v_1, vertex v_1 is adjacent to $p_2, p_4, \dots, p_{k-2}$ (and not adjacent to $p_1, p_3, \dots, p_{k-1}$). Since H is bipartite, there is no edge v_3p_i with i odd. If there is an edge v_3p_i with i even and $i > 0$, then the path $v_3 - p_i - \cdots - p_k$ is shorter than P, a contradiction. So there is no such edge, and $v_3 - p_0 - \cdots - p_k - v_5$ is a chordless path P'. By Lemma 1 applied to P' and v_4, vertex v_4 is adjacent to $p_1, p_3, \dots, p_{k-1}$ (and not adjacent to $p_2, p_4, \dots, p_k$). Then x is adjacent to p_1, for otherwise $\{x, v_1, v_4, p_0, p_1\}$ induces a paraglider; and to p_2, for otherwise $\{x, v_1, v_4, p_1, p_2\}$ induces a paraglider. Then, by the induction hypothesis, applied to the 6-tuple $\{p_1, p_2, v_1, v_4, v_5, v_6\}$ with path $p_2 - \cdots - p_k$, vertex x is adjacent to v_5 and v_6. Thus (4) holds. $\square$

$$\text{If there is a } P_3 \text{ in } H \text{ whose three vertices are in } N(x), \text{ then } V(H) \subseteq N(x). \tag{5}$$

Proof of (5). Let $W = \{a, b, c\}$ be the vertex set of a P_3 in H, with edges ab and bc, such that $W \subseteq N(x)$, and let z be any vertex in $V(H) \setminus W$. Call a *W-link* any path in H from z to W that contains exactly one vertex from W. Since $\{b, c\}$ is not a clique cutset of H, there is a W-link from z to a, and we let p be the length of a shortest such path. Likewise, there is a W-link from z to c, and we let q be the length of a shortest such path. Note that p and q have the same parity, since H is bipartite. We define $\ell_W(z) = \min\{p, q\}$ and $L_W(z) = \max\{p, q\}$. We prove that x is adjacent to z by induction on $\ell_W(z)$, and also, when $\ell_W(z) = 1$, by induction on $L_W(z)$. We may assume that $p \leq q$, so $\ell_W(z) = p$. Let $P = u_0 - \cdots - u_p$ be a W-link from z to a of length p, with $u_0 = a$ and $u_p = z$, and let $Q = v_0 - \cdots - v_q$ be a W-link from z to c of length q, with $v_0 = c$ and $v_q = z$.

First, suppose that $p = 1$; that is, z is adjacent to a. It follows that q is odd. Let j be the smallest integer such that there exists an edge av_j ($j \leq q$). Then $a - v_j - v_{j-1} - \cdots - v_0$ is a chordless path R, of length $j + 1$, and j is odd, since H is bipartite. By Lemma 1 applied to R and b, vertex b is adjacent to every second vertex of R (i.e., to $v_2, v_4, \dots, v_{j-1}$). Suppose that $j \geq 3$. If x has no neighbor in $\{v_1, v_2\}$, then $V(R) \cup \{x\}$ contains a hole (that contains x, c, v_1, v_2, v_3), a contradiction. So x is adjacent to one of v_1 and v_2; and it must be adjacent to both, for otherwise $\{x, b, c, v_1, v_2\}$ induces a paraglider. For each even h with $h < j-1$, this argument can be repeated with v_h instead of c and $\{v_{h+1}, v_{h+2}\}$ instead of $\{v_1, v_2\}$; thus we obtain by induction on h that x is adjacent to every vertex of R. Then we set $W' = \{a, b, v_{j-1}\}$ and observe that $\ell_{W'}(z) = 1$ and $v_{j-1} - v_j - \cdots - v_q$ is a W'-link, so $L_{W'}(z) < L_W(z)$, and, by the induction hypothesis, x is adjacent to z. Therefore $j = 1$. Then x is adjacent to v_1, for otherwise $\{x, a, b, c, v_1\}$ induces a paraglider. If $q = 1$, we are done; therefore let $q \geq 3$. By Lemma 1 applied to Q and a, vertex a is adjacent to $v_3, v_5, \dots, v_{q-2}$ (and not to $v_2, \dots, v_{q-1}$). For each odd h with $3 \leq h \leq q-2$, we have $\ell_W(v_h) = 1$ and $L_W(v_h) < q$, so x is adjacent to v_h; moreover, x is adjacent to v_{h-1}, for otherwise $\{x, a, v_{h-2}, v_{h-1}, v_h\}$ induces a paraglider. By (4) applied to the 6-tuple $\{a, v_{q-4}, \dots, v_q\}$, we obtain that x is adjacent to z.

Now suppose that $p \geq 2$. By the induction hypothesis, x is adjacent to u_1, because $\ell_W(u_1) < p$. Set $W' = \{u_1, a, b\}$, and observe that $u_p - \cdots - u_1$ is a W'-link from z to u_1, so $\ell_{W'}(z) < \ell_W(z)$, and, by the induction hypothesis, x is adjacent to z. Thus (5) holds. $\square$

$$\text{If } x \text{ is any vertex of } G \setminus H, \text{ then either } N_H(x) \text{ is a (possibly empty) clique or } N_H(x) = V(H). \tag{6}$$

Proof of (6). Suppose that x has two non-adjacent neighbors u and v in H. Let H_x be the subgraph induced by $V(H) \cup \{x\}$. Suppose that H_x has a clique cutset K. If $x \in K$, then $K \setminus \{x\}$ is a clique cutset of H, a contradiction. So $x \notin K$. Let C be the component of $H_x \setminus K$ that contains x, and let D be another component of $H_x \setminus K$. Since u and v are not adjacent, at least one of them, say u, is not in K; so $u \in C$. But then K is a clique cutset of H (that separates u from D), a contradiction. Thus H_x has no clique cutset. The maximality of $V(H)$ implies that H_x is not bipartite; and so H_x contains a triangle, which contains x. Let a and b be two neighbors of x in H that are adjacent. One of u and v, say u, is not in $\{a, b\}$. Let $T = \{a, b, u\}$, and let T^* be the vertex set of a connected subgraph of H that contains T. Choose T such that T^* is as small as possible. If $T^* = T$

then T induces a P_3, and (5) implies that $N_H(x) = V(H)$. Now let us assume that $T^* \neq T$; i.e., u is not adjacent to any of a and b. Since $\{a\}$ is not a cutset of H, there is a shortest path P from b to u in $H \setminus \{a\}$. Let $P = p_0 - \cdots - p_k$, with $p_0 = b$ and $p_k = u$. Then x is not adjacent to p_1, for otherwise we could take $T = \{a, b, p_1\}$; and x is adjacent to p_2, for otherwise $V(P) \cup \{x\}$ contains a hole (that contains x, b, p_1, p_2, p_3). Then we can assume that $u = p_2$ and $T^* = \{a, b, p_1, p_2\}$. Since $\{b, p_1\}$ is not a cutset of H, there is a shortest path Q from a to u in $H \setminus \{b, p_1\}$. Let $Q = q_0 - \cdots - q_\ell$, with $q_0 = a$ and $q_\ell = u$. The bipartiteness of H implies that ℓ is odd, b is not adjacent to any q_i with i odd, and p_1 is not adjacent to any q_j with j even. Then x is not adjacent to q_1, for otherwise we could take $T = \{a, b, q_1\}$; and x is adjacent to q_2, for otherwise $V(Q) \cup \{x\}$ contains a hole (that contains x, a, q_1, q_2, q_3). Then b is not adjacent to q_2, for otherwise $\{x, a, b, q_1, q_2\}$ induces a paraglider; and b is not adjacent to any q_j with j even ($j \geq 4$), for otherwise $\{b, a, q_1, \ldots, q_j\}$ induces a hole. Then p_1 is not adjacent to q_1, for otherwise $\{x, a, q_1, p_1, u\}$ induces a hole. But then, letting i be the largest integer such that p_1 is adjacent to q_i ($i \leq \ell$), we see that $\{p_1, b, a, q_1, q_2, \ldots, q_i\}$ induces a hole, a contradiction. Thus (6) holds. □

Let R be the set of vertices that are complete to $V(H)$. Then

R is a clique. (7)

Proof of (7). Since H contains a $K_{3,3} \setminus e$, there are three vertices a, b, c in H that induce a subgraph with exactly one edge. If R contains two non-adjacent vertices x and y, then $\{a, b, c, x, y\}$ induces a paraglider. Thus (7) holds. □

If F is any component of $G \setminus (V(H) \cup R)$, then $N_H(F)$ is a clique. (8)

Proof of (8). Suppose to the contrary that there are non-adjacent vertices u and v in $N_H(F)$. Let x be a neighbor of u in F and y be a neighbor of v in F. There is a chordless path P between x and y in F. We choose u, v, x, y and P such that P is as short as possible. By (6), P has length at least 1. Since H has no clique cutset, it is 2-connected, and by Menger's theorem [31,33] there are two paths Q and Q' between u and v in H such that $V(Q) \cap V(Q') = \{u, v\}$. The choice of P implies that u has no neighbor in $P \setminus \{x\}$ and v has no neighbor in $P \setminus \{y\}$. It must be that some interior vertex s of Q has a neighbor z in P, for otherwise $V(P) \cup V(Q)$ induces a hole.

Suppose that Q has length at least 3. Then, up to symmetry, s is not adjacent to v. If $z \neq x$, then the subpath $P[z, y]$ contradicts the choice of P. So $z = x$. By (6), and since $x \notin R$, $\{u, s\}$ is a clique; i.e., s is the neighbor of u on Q. Then it must be that some interior vertex t of $Q \setminus \{u\}$ has a neighbor in P, for otherwise $V(P) \cup V(Q) \setminus \{u\}$ induces a hole. As above (with s), we obtain that the only neighbor of t in P is y, and consequently $\{t, v\}$ is a clique; i.e., t is the neighbor of v on Q. Now $V(P) \cup V(Q) \setminus \{u, v\}$ induces a chordless cycle, so it must have length 4, so st and xy are edges of H. Thus $Q = u - s - t - v$. Since Q' has length at least 3, we also have $Q' = u - s' - t' - v$, where s' is adjacent to x and not to y, and t' is adjacent to y and not to x. Note that ss' and tt' are not edges, because H is bipartite. Then st' is not an edge, for otherwise $\{x, u, s, s', t'\}$ induces a paraglider; and similarly $s't$ is not an edge. But then $\{u, s, t, v, t', s'\}$ induces a hole in H, a contradiction. Therefore Q and Q' have length 2. Thus $Q = u - s - v$ and $Q' = u - s' - v$, where we know already that s has a neighbor in P, and, similarly, s' has a neighbor in P. Note that ss' is not an edge, since H is bipartite. There is a subpath P' of P whose endvertices are adjacent to s and s' respectively, and the choice of P implies that $P' = P$; i.e., up to symmetry, s is adjacent to x, s' is adjacent to y, and there is no other edge between P and $\{s, s'\}$. If P has length at least 2, then $V(P) \cup \{u, s'\}$ induces a hole, a contradiction. So P has length 1. But then $\{x, y, u, v, s, s'\}$ induces a $\overline{C}_6$, a contradiction. Thus (8) holds. □

In conclusion, if $G \setminus (V(H) \cup R)$ has a component F, then, by (7) and (8), $N_H(F) \cup R$ is a clique cutset (that separates F from $H \setminus N_H(F)$), a contradiction to the fact that G is an atom. Therefore we have $V(G) = V(H) \cup R$, and so G is the join of a chordal bipartite graph and a clique; i.e., G is in class $\mathcal{G}_2$. This finishes the proof of Theorem 3. □

Theorem 4. *Let G be an HP-free atom that contains no $\overline{C_6}$ and no $K_{3,3} \setminus e$. Suppose that G contains a C_4. Then G is a complete multipartite graph.*

Proof. Let J be an induced subgraph of G that is the complete join of k non-empty stable sets $S_1, \ldots, S_k$, with $k \geq 2$, such that at least two of these stable sets have size at least 2. Note that a C_4 is such a graph. We assume also that J is such that $V(J)$ is maximal with this property. If $G = J$, then G is a complete multipartite graph, so let us assume that $G \neq J$. Let F be any component of $G \setminus J$. We claim that

$N_J(F)$ is a clique. □ (9)

Proof of (9). Suppose that there are non-adjacent vertices x and y in $N_J(F)$. Let u be a neighbor of x in F and v be a neighbor of y in F. There is a chordless path P between u and v in F. We choose x, y, u, v and P such that P is as short as possible. Up to relabeling, let $x, y \in S_1$. By the definition of J, there are non-adjacent vertices a and b in $J \setminus S_1$.

First, suppose that $u = v$. For any two distinct integers $i, j \in \{2, \ldots, k\}$, vertex u must be complete to S_i or to S_j, for otherwise there are non-neighbors s, t of u with $s \in S_i$ and $t \in S_j$, and $\{u, x, y, s, t\}$ induces a paraglider. Therefore we may assume that u is complete to $S_2 \cup \cdots \cup S_{k-1}$. Suppose that u is complete to S_k. If u is also complete to S_1, then the subgraph induced by $V(J) \cup \{u\}$ is the join of $k+1$ stable sets $S_1, \ldots, S_k, \{u\}$, which contradicts the choice of J. So u has a non-neighbor z in S_1. Then $\{a, b, u, x, z\}$ induces a paraglider. Therefore u is not complete to S_k. Moreover, if u has a neighbor s and a non-neighbor t with $s, t \in S_k$, then $\{u, x, y, s, t\}$ induces a paraglider. So u is anticomplete to S_k. If u is complete to S_1, then the subgraph induced by $V(J) \cup \{u\}$ is the join of k stable sets $S_1, \ldots, S_{k-1}, S_k \cup \{u\}$, which contradicts the choice of J. So u has a non-neighbor z in S_1. If $a, b \in S_j$ with $j < k$, then $\{a, b, u, x, z\}$ induces a paraglider. So $a, b \in S_k$. But then $\{a, b, u, x, y, z\}$ induces a $K_{3,3} \setminus e$, a contradiction.

Now suppose that $u \neq v$. The choice of P implies that x has no neighbor in $P \setminus \{u\}$ and y has no neighbor in $P \setminus \{v\}$. Then a must have a neighbor in P, for otherwise $V(P) \cup \{a, x, y\}$ induces a hole; and similarly b has a neighbor in P. So there is a subpath P' of P whose endvertices are adjacent to a and b respectively, and the choice of P implies that $P' = P$. Thus, up to symmetry, a is adjacent to u, b is adjacent to v, and there is no other edge between P and $\{a, b\}$. If P has length at least 2, then $V(P) \cup \{a, y\}$ induces a hole. So P has length 1, but then $\{u, v, x, y, a, b\}$ induces a $\overline{C}_6$. Thus (9) holds. □

In conclusion, if $G \setminus J$ has a component F, then, by (9), $N_J(F)$ is a clique cutset (that separates F from $J \setminus N_J(F)$), a contradiction to the fact that G is an atom. Therefore we have $G = J$, so G is a complete multipartite graph; i.e. $G \in \mathcal{G}_1$. This finishes the proof of Theorem 4. □

Proof of Theorem 1. First, suppose that G has an induced subgraph H that is either a hole or a paraglider. Since H has no clique cutset, H must be an induced subgraph of some atom of G. So if every atom is HP-free, then G is HP-free.

Conversely, suppose that G is an HP-free graph, and let A be any atom of G. If A contains no C_4, then A is chordal, so Dirac's theorem [21] implies that A is a clique (which is a complete multipartite graph). If A contains a C_4 but no $\overline{C_6}$ and no $K_{3,3} \setminus e$, then Theorem 4 implies that A is a complete multipartite graph. If A contains a $K_{3,3} \setminus e$ but no $\overline{C_6}$, then Theorem 3 implies that A is the join of a chordal bipartite graph and a clique. If A contains a $\overline{C_6}$, then Theorem 2 implies that A is the join of a matched co-bipartite graph and a clique. This finishes the proof of Theorem 1. □

3. Algorithmic consequences

It was shown in [36] that, for various optimization problems such as minimum fill-in, maximum independent set, maximum clique, and coloring, whenever these problems are efficiently solvable on the atoms of a graph class, they are efficiently solvable on all graphs of the class. More precisely, given a graph G with n vertices and m edges, the algorithm from [36] finds in time $\mathcal{O}(nm)$ a clique separator decomposition of G with at most n atoms. For each optimization problem, the problem is solved on the atoms and the optimal solutions are combined to produce an optimal solution for G (the way they are combined depends on the problem). If the complexity of solving the problem for any atom of G is $\mathcal{O}(f(n, m))$, then the total complexity for G is $\mathcal{O}(nm + nf(n, m))$.

For perfect graphs, maximum independent set, maximum clique, and coloring are known to be solvable in polynomial time [25,26] using the ellipsoid method (but from a practical point of view, this is not an efficient solution of the problems). (Hole, paraglider)-free graphs are perfect because of the Strong Perfect Graph Theorem (a simpler and more direct way to prove this uses Theorem 1 and the fact that a graph is perfect if its atoms are perfect). The clique separator approach gives direct combinatorial algorithms for the problems mentioned above in the case of HP-free graphs, as we show now.

Recognition. Chordal bipartite graphs can be recognized in time $\mathcal{O}(\min\{m \log n, n^2\})$ [29,32,35]. The recognition of complete multipartite graphs and of matched co-bipartite graphs can be easily done in linear time. We can decide if a graph G is HP-free as follows. Use the method of [36] to find a clique separator decomposition of G into at most n atoms. For each atom A, check whether A belongs to one of the three basic classes $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$. If not, then the input graph is not (hole, paraglider) free. The total complexity is $\mathcal{O}(\min\{nm \log n, n^3\})$.

Maximum weight independent set. For matched co-bipartite graphs and complete multipartite graphs, MWIS is trivial. For bipartite graphs, MWIS can be solved in time $\mathcal{O}(n^3)$ [33]. Thus, the time bound for MWIS on HP-free graphs is $\mathcal{O}(n^4)$.

Maximum clique and coloring. On each of the three basic classes the two problems are very simple and can be solved in time $\mathcal{O}(n + m)$. Here combining the solutions obtained on the atoms means simply taking the largest of them; this operation does not add a factor in the complexity. So these problems can be solved in time $\mathcal{O}(n + m)$ on HP-free graphs if a clique separator decomposition of the input graph is given. For maximum clique on diamond-free graphs, however, there is an even simpler way to solve the problem efficiently. Since the neighborhood of every vertex is P_3-free, it consists of pairwise disjoint cliques. So there are at most m maximal cliques, and we can list them explicitly and find an optimal clique.

Minimum fill-in. On graphs in classes $\mathcal{G}_1$ and $\mathcal{G}_3$ Minimum fill-in is very simple and can be solved in time $\mathcal{O}(n + m)$. For chordal bipartite graphs, a $\mathcal{O}(n^4)$ algorithm is given in [34]. Here, combining the solutions obtained on the atoms means simply taking the union of the fill-in sets; this operation does not add a factor in the complexity. So minimum fill-in can be solved in time $\mathcal{O}(n^4)$ on HP-free graphs.

Maximum weight induced matching (MWIM). A set M of edges is an *induced matching* in G if the pairwise distance of the edges in M is at least two in G. The MWIM problem asks for an induced matching of maximum weight. In [15], it is shown that, for a hereditary class $\mathcal{C}$ of graphs, MWIM is solvable in polynomial time if MWIM is solvable in polynomial time on the atoms of $\mathcal{C}$. For chordal bipartite graphs, a polynomial-time solution is given in [18]. In a complete multipartite graph there is no induced matching of size 2, and in a matched co-bipartite graph there is no induced matching of size 3, so in either class the search for a maximum weight induced matching is trivial.

4. Conclusion

We have described here the structure of (hole, paraglider)-free atoms and some algorithmic consequences. In a forthcoming paper [4] we will analyze the structure of (hole, diamond)-free graphs and its algorithmic consequences in more detail.

There are various other aspects and papers which are related to our work as described below.

4.1. Related results for subclasses of P_5-free graphs

In [1], Alekseev showed that (P_5, paraglider)-free atoms are $3K_2$-free, which leads to a polynomial-time algorithm for the MWIS problem, since $3K_2$-free graphs contain at most $\mathcal{O}(n^4)$ inclusion-maximal independent sets. In [12], we improved this result by generalizing the forbidden paraglider subgraph. In [7], we give a more detailed structural analysis of (P_5, paraglider)-free atoms. In [16], we describe the structure of prime (P_5, co-chair)-free graphs and give algorithmic applications. The complexity of the MWIS problem for P_5-free graphs is an open problem. It is also open for (P_5, C_5)-free graphs; such graphs are hole-free. Thus, it is interesting to study subclasses of P_5-free graphs (subclasses of (P_5, C_5)-free graphs, respectively).

4.2. Clique-width

In [6], we describe the simple structure of (P_5, diamond)-free graphs; such graphs can contain C_5, and thus (P_5, diamond)-free graphs are in general not perfect and their class is incomparable with the class of (hole, diamond)-free graphs. (P_5, diamond)-free graphs have bounded clique-width — see, e.g., [20] for the notion and algorithmic implications of bounded clique-width, which has tremendous consequences for efficiently solving hard problems on such graph classes. For the more general class of (P_5, gem)-free graphs, the situation is similar: by the Strong Perfect Graph Theorem, (hole, gem)-free graphs are perfect, since antiholes with at least seven vertices contain a gem. The structure of (P_5, gem)-free graphs and some algorithmic applications were described in [5,10]. In [9], it was shown that (P_5, gem)-free graphs have bounded clique-width.

The clique-width of (hole, diamond)-free graphs, however, is unbounded, since, for example, the subclass of chordal bipartite graphs (which are the (hole, triangle)-free graphs) has unbounded clique-width [14]. This illustrates that corresponding subclasses of hole-free graphs are more interesting than those of P_5-free graphs.

4.3. Open problems

It would be interesting to describe the structure of (hole, gem)-free graphs. In particular, how can one avoid to use the Strong Perfect Graph Theorem for showing that (hole, gem)-free graphs are perfect?

In [8], we give a polynomial-time algorithm for the MWIS problem on (hole, co-chair)-free graphs. It would be interesting to obtain better structural results on these graphs.

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