A stable numerical method for Volterra integral equations with discontinuous kernel

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Abstract

Numerical methods for Volterra integral equations with discontinuous kernel need to be tuned to their peculiar form. Here we propose a version of the trapezoidal direct quadrature method adapted to such a type of equations. In order to delineate its stability properties, we first investigate about the behavior of the solution of a suitable (basic) test equation and then we find out under which hypotheses the trapezoidal direct quadrature method provides numerical solutions which inherit the properties of the continuous problem.

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1. Introduction

In this paper we consider Volterra integral equations (VIEs) with two constant delays \( \tau_2 > \tau_1 > 0 \)

\[
y(t) = f(t) + \int_{t-\tau_1}^{t} k_1(t-s)g(y(s)) \, ds + \int_{t-\tau_2}^{t-\tau_1} k_2(t-s)g(y(s)) \, ds + \int_{0}^{t-\tau_2} k_3(t-s)g(y(s)) \, ds,
\]

\( t \in I := [\tau_2, T], \)

(1.1)

with \( y(t) = u(t), \ t \in [0, \tau_2], \) where \( u(t) \) is a known function such that

\[
u(\tau_2) = f(\tau_2) + \int_{\tau_2-\tau_1}^{\tau_2} k_1(\tau_2-s)g(u(s)) \, ds + \int_{0}^{\tau_2-\tau_1} k_2(\tau_2-s)g(u(s)) \, ds.
\]

(1.2)
Equations of type (1.1) often arise in many physical and biological models (see, for example [1,2,6,8,14,16] and references therein) and in general in any convolution VIE of Hammerstein type [9]

\[ y(t) = f(t) + \int_0^t k(t-s)g(y(s))\,ds, \quad (1.3) \]

where the kernel \( k \) has jump discontinuities defined as follows:

\[ k(t) = \begin{cases} 
  k_1(t) & 0 \leq t < \tau_1, \\
  k_2(t) & \tau_1 \leq t < \tau_2, \\
  k_3(t) & \tau_2 \leq t \leq T.
\end{cases} \]

In the following we assume that the given real-valued functions \( f = f(t) \) and \( k_1, k_2, k_3 \) are at least continuous on \([0,T]\) and on \([0,\tau_1], [\tau_1, \tau_2] \) and \([\tau_2,T]\) respectively and \( g \) satisfies the Lipschitz condition with respect to \( y \). Existence and uniqueness results for (1.1) can be easily proved by comparison with the theory for VIEs (see, for example [13]). In fact (1.1) can be recast in the form of a classical VIE by proceeding recursively on the intervals \([\tau_2, \tau_2+\tau_1], [\tau_2+\tau_1, \tau_2+2\tau_1] \) and so on (method of steps).

Provided that \( k, g, f \) and \( u \) in (1.1) are sufficiently smooth, condition (1.2) assures the continuity of \( y(t) \) for \( t \geq 0 \) and \( y^{(l)}(t) \), \( l = 1, 2, \ldots \), presents some points of primary discontinuities and it is continuous for \( t > l\tau_2 \).

The objective of this paper is a crucial understanding of the behavior of numerical simulations of (1.1) by means of the Trapezoidal Direct Quadrature (TDQ) method. To be more specific, we are interested to describe the sensitivity of the solution of (1.1) under a perturbation on the datum \([\tau_2, \tau_2+\tau_1], [\tau_2+\tau_1, \tau_2+2\tau_1] \) and on two different points of view on this subject. First of all, Eq. (1.4) can be considered as the linearized equation of (1.1) under a perturbation on the datum \( f(t) \) and then to determine under which conditions an approximate solution exhibits the same sensitivity. In this paper we study the stability by using the classical approach based on test equations (see, for example [1,2,7,8,12,16]). We consider the test equation

\[ y(t) = 1 + \lambda_1 \int_{t-\tau_1}^{t} y(s)\,ds + \lambda_2 \int_{t-\tau_2}^{t-\tau_1} y(s)\,ds + \lambda_3 \int_{0}^{t-\tau_2} y(s)\,ds, \quad (1.4) \]

with \( t \in [\tau_2, T] \) and \( y(t) = u(t) \) known in \([0, \tau_2]\), and we refer to [2], [5, Chapter 7] and [13, Sections 7.4, 7.5] for a more general discussion on the use of test equations in the study of stability for VIEs. Here we confine our discussion on two different points of view on this subject. First of all, Eq. (1.4) can be considered as the linearized equation of the error with respect to a constant perturbation on the forcing function \( f(t) \) and a bounded perturbation on the known function \( u(t) \), \( t \in [0, \tau_2] \) ([2], [5, p. 411, Definition 7.1.4]). In other words, let \( \tilde{y} \) be the solution of (1.1), subject to a constant perturbation \( \epsilon \) on \( f \) and to a bounded perturbation \( \psi(t) \) on \( u(t) \) such that

\[ \psi(\tau_2) = f(\tau_2) + \int_{\tau_2-\tau_1}^{\tau_2} k_1(\tau_2-s)g(\psi(s))\,ds + \int_{0}^{\tau_2-\tau_1} k_2(\tau_2-s)g(\psi(s))\,ds, \]

and let \( e(t) = \tilde{y}(t) - y(t) \), then \( e(t) \) satisfies an equation which can be approximated by (1.4) provided the following simplified assumptions:

\[ k_i(t-s)g'(y(s)) \approx k_i(0)g'(y(0)) = \lambda_i, \quad i = 1, 2, 3. \]

On the other hand, (1.4) can be viewed as a very simple equation for which some conditions for the asymptotic stability of the solution are known (see [11]). Taking into account these two interpretations of the test equation, we say that a method is stable with respect to (1.4), when the numerical solution of the test equation mimics the qualitative properties of the analytical one. This will be made more precise in Sections 3 and 4. Of course, many different stability definitions can be given for the continuous problem (1.1). The utility of test equations like (1.4) is also evident in the case of the corresponding discrete equations. As a matter of fact, the reason for defining stability with respect to the simple test equation (1.4), is that it allows a full analysis of the difference equation resulting from the discretization by a numerical method.

Moreover, we note that Eq. (1.4) is equivalent to the delay differential equation (DDE)

\[ y'(t) = \lambda_1 y(t) + (\lambda_2 - \lambda_1) y(t - \tau_1) + (\lambda_3 - \lambda_2) y(t - \tau_2), \quad (1.5) \]
for which the stability analysis is well known. By using the sufficient condition stated in [12], we can assert that the solution of Eq. (1.4) is asymptotically stable if

\[ \lambda_1 + |\lambda_2 - \lambda_1| + |\lambda_3 - \lambda_2| < 0. \]  

(1.6)

Stability conditions for (1.4) in the particular case \( \lambda_2 = \lambda_3 = 0 \) can be found in [2], however they cannot be derived from (1.6). Moreover, we observe that in the case \( \lambda_1 = \lambda_2 = a, \lambda_3 = a + b, (1.4) \) corresponds to Eq. (1.5a) in [16] and the stability conditions there reported can be obtained directly from (1.6). The case \( \lambda_1 = \lambda_2 = 0 \) can be found in [8].

In Section 2 we introduce the adapted TDQ method when applied to Eq. (1.1) and we give some convergence results, while in Section 3 we study the nature of the constraints for \( \lambda_1, \lambda_2, \lambda_3 \) and for the stepsize \( h \) that lead to an asymptotically stable numerical solution whenever the solution \( y(t) \) of (1.4) is asymptotically stable.

An important class of problems of the kind (1.1) consists of the Volterra equations where the kernels \( k_1 \) and \( k_3 \) are identically zero, that is

\[ y(t) = f(t) + \int_{t_1}^{t} k(t-s)g(y(s))ds, \quad t > t_2. \]  

(1.7)

An example of application in age structured population dynamic derives from the integral formulation of the problem introduced in [4, Section 5]. For this reason, Eq. (1.7) will be referred to as the model problem. For the corresponding test equation (1.4) (with \( \lambda_1 = \lambda_3 = 0 \)), the classical stability analysis of [12] and [1] and the results in [2] and [16] cannot be applied. Therefore, the most of this paper is concerned with the study of the test equation related to (1.7).

In Section 4 we rely on our investigations on the behavior of the analytical solution of this test equation and we find conditions for the TDQ method to reproduce this behavior. In particular, in Section 4.1, Theorem 4.2 gives some sufficient conditions on the parameters \( \lambda, \tau_1 \) and \( \tau_2 \) of the test equation and on the known function \( u(t) \) for the positiveness and the boundedness of its analytical solution \( y(t) \). Under the same hypotheses of Theorem 4.2, we explicitly give, in Theorem 4.3, the limiting value of \( y(t) \) as \( t \) tends to infinity. What is more, in Lemma 4.1, we state a sufficient condition on \( \lambda \) for the oscillatory behavior of \( y(t) \). In Section 4.2 we prove the discrete analogues of the above mentioned results, more specifically, Theorems 4.4 and 4.5 state that, under the same hypotheses of Theorems 4.2 and 4.3, the numerical solution of the test equation generated by the TDQ method is positive, bounded and its limiting value at infinity coincides with that of \( y(t) \).

Some numerical experiments are shown in Section 5. Finally, in Section 6 our concluding remarks are reported.

2. The method: formulation and convergence

Let \( \Pi_N = \{t_0: 0 = t_0 < t_1 < \cdots < t_N = T \} \) be a partition of the time interval \([0, T]\) with constant stepsize \( h = t_{n+1} - t_n, n = 0, \ldots, N - 1 \) and assume that

\[ h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}, \]  

(2.1)

with \( r_1, r_2 \) positive and integer. The discretization of problem (1.1) by the adapted TDQ method yields

\[
y_n = f(t_n) + \frac{h}{2} \left[ k_1(r_1 h)g(y_{n-r_1}) + 2 \sum_{j=n-r_1+1}^{n-1} k_1((n-j)h)g(y_j) + k_1(0)g(y_n) \right]
\]

\[ + \frac{h}{2} \left[ k_2(r_2 h)g(y_{n-r_2}) + 2 \sum_{j=n-r_2+1}^{n-r_1-1} k_2((n-j)h)g(y_j) + k_2(r_1 h)g(y_{n-r_1}) \right]
\]

\[ + \frac{h}{2} \left[ k_3(nh)g(y_0) + 2 \sum_{j=1}^{n-r_2-1} k_3((n-j)h)g(y_j) + k_3(r_2 h)g(y_{n-r_2}) \right], \quad n > r_2, \]  

(2.2)

where \( y_n \) represents an approximation to the exact solution \( y \) of (1.1) at the point \( t_n \) and \( y_l = u(lh), l = 0, 1, \ldots, r_2 \), where \( u(t) \) is the known function introduced in the previous section.
The method (2.2) coincides with the classical trapezoidal method [13, p. 96] applied to (1.3) only in the case of continuous kernels.

The consistency of (2.2) can be easily proved by using the standard techniques for VIEs described, for example, in [5] and [13]. By applying the classical theory on convergence (based on the discrete Gronwall-type inequalities [5, p. 40]), the method of steps and the fact that the discontinuity points $\tau_2$ for the first derivative and $\tau_2, \tau_1 + \tau_2, 2\tau_2$ for the second derivative of $y$ belong to the mesh $\Pi_N$, we obtain the following result:

**Theorem 2.1.** Let $y_n$ be the numerical solution of (1.1) obtained by the TDQ method (2.2). Assume that the sufficient conditions on $k_1, k_2, k_3$ and $g$ for the existence and uniqueness of the solution $y(t)$ of (1.1) are satisfied. What is more, let us suppose that $k_1, k_2, k_3 \in C^2[0, \tau_1], C^2[\tau_1, \tau_2]$ and $C^2[\tau_2, T]$ respectively, and that $u \in C^2[0, \tau_2]$ and $f \in C^2[0, T]$. If the stepsize $h$ satisfies the condition (2.1), then

$$\max_{1 \leq n \leq N} \| y(t_n) - y_n \| = O(h^2).$$

Hence, the order of the adapted trapezoidal method (2.2) is not influenced by the discontinuities in the kernel.

3. Stability

Let $y_n$ be the approximation to the solution of (1.1) given by (2.2), we are now interested in determine delay independent criteria for the stability of $y_n$. For this reason, we apply the TDQ method to the test equation (1.4)

$$y_n = 1 + \lambda_1 \frac{h}{2} \left[ y_{n-r_1} + 2 \sum_{j=n-r_1+1}^{n-1} y_j + y_n \right] + \lambda_2 \frac{h}{2} \left[ y_{n-r_2} + 2 \sum_{j=n-r_2+1}^{n-r_1-1} y_j + y_{n-r_1} \right] + \lambda_3 \frac{h}{2} \left[ y_0 + 2 \sum_{j=1}^{n-r_2-1} y_j + y_{n-r_2} \right]$$

and we find the conditions on $\lambda_1, \lambda_2, \lambda_3$ such that $y_n$ asymptotically vanishes for $h$ satisfying (2.1), whenever (1.6) holds. By writing down (3.1) for $y_n$ and $y_{n+1}$ and combining the two expressions, the following relation comes out:

$$y_{n+1} = \left( 1 - \lambda_1 \frac{h}{2} \right)^{-1} \left[ 1 + \lambda_1 \frac{h}{2} \right] y_n + \frac{h}{2} (\lambda_2 - \lambda_1) (y_{n+1-r_1} + y_{n-r_1}) + \frac{h}{2} (\lambda_3 - \lambda_2) (y_{n+1-r_2} + y_{n-r_2}).$$

In order to obtain the stability conditions for the scalar discrete Volterra equation (3.2) we can apply the results in [10, Theorem 3.1] obtained by the Liapunov approach, then the TDQ method turns to be asymptotically stable if the condition

$$\left| 1 + \lambda_1 \frac{h}{2} \right| + h|\lambda_2 - \lambda_1| + h|\lambda_3 - \lambda_2| < \left| 1 - \lambda_1 \frac{h}{2} \right|$$

is satisfied. In particular we can state the result below.

**Lemma 3.1.** The condition (3.3) is satisfied if and only if

$$\begin{align*}
(i) & \quad h|\lambda_2 - \lambda_1| + |\lambda_3 - \lambda_2| < 2, \\
(ii) & \quad h\lambda_1 + h|\lambda_2 - \lambda_1| + |\lambda_3 - \lambda_2| < 0.
\end{align*}$$

Since (ii) coincides with (1.6), we get the following sufficient condition for the stability of the numerical solution.

**Theorem 3.2.** Assume that Eq. (1.4) is asymptotically stable (that is $\lambda_1, \lambda_2, \lambda_3$ satisfy (1.6)), then $y_n$ asymptotically vanishes if

$$h|\lambda_2 - \lambda_1| + |\lambda_3 - \lambda_2| < 2.$$
Stability of the TDQ method when applied to the particular cases of (1.4) \((\lambda_2 = \lambda_3 = 0 \text{ and } \lambda_1 = \lambda_2 = a, \lambda_3 = a + b)\) can be derived from [2,16], respectively. As we could expect, our lemma cannot be applied to the case \(\lambda_2 = \lambda_3 = 0\) [2], while in the case of [16] the condition we get is more restrictive. Finally we observe that in the trivial case \(\lambda_1 = \lambda_2 = \lambda_3\) (1.4) coincides with the well known basic test equation [5, Chapter 7], [13, Section 7.5] and Theorem 3.2 is equivalent to the classical \(A\)-stability result for the TDQ method [13, Section 7.5].

4. Stability for the model problem

As mentioned above, interesting applications lead to test equations of the kind (1.4) where \(\lambda_1 = \lambda_3 = 0\). We are in the particular case

\[
y(t) = 1 + \lambda \int_{t-\tau_1}^{t-\tau_2} y(s) \, ds, \quad t > \tau_2
\]

for which the stability condition (1.6) cannot be applied. Here we suppose that \(y(t) = u(t), \; t \in [0, \tau_2]\), where \(u(t)\) is a known function such that

\[
u(\tau_2) = 1 + \lambda \int_0^{\tau_2-\tau_1} u(s) \, ds.
\]

Notice that Eq. (4.1) is equivalent to the DDE

\[
y'(t) = \lambda (y(t - \tau_1) - y(t - \tau_2)).
\]

Some conditions assuring the boundedness of the solution of a DDE strictly related to (4.2) can be found in [15]. The classical theory on the stability of (4.1)–(4.2) involves the analysis of the roots of the characteristic equation [3, p. 190]. However, to our knowledge, there are no explicit results in our particular case. Therefore, we look at the stability from the first point of view introduced in Section 1, and we consider (4.1) as the linearized equation of the error with respect to a constant perturbation on the forcing function. In this perspective we first delineate the behavior of the solution of (4.1)–(4.2) and then study the conditions for which the numerical solution behaves like the continuous one.

4.1. Qualitative behavior of the continuous solution

First of all, we will prove the following lemma.

**Lemma 4.1.** Let \(y(t)\) be the solution of (4.1), if \(\lambda \leq 0\) then \(y(t)\) exhibits oscillation (i.e., slope changes).

**Proof.** Assume that \(y'(t) > 0\), for \(t > \tilde{t}\), then there exists \(\tilde{t}\) such that \(y(t - \tau_1) > y(t - \tau_2)\) for all \(t > \tilde{t}\). Since \(\lambda\) is negative, this last relation implies that \(\lambda (y(t - \tau_1) - y(t - \tau_2)) < 0\) and thus, by Eq. (4.2), \(y'(t) < 0, \; t > \tilde{t}\) which is in contradiction with our assumption. The same procedure can be applied if we assume that \(y'(t) < 0\). \(\square\)

This lemma states that \(y(t)\) is not ultimately monotone, however this course does not exclude that it converges as \(t\) goes to infinity. The theorem below gives additional information about the asymptotic behavior of \(y(t)\).

**Theorem 4.2.** Assume that \(u(t) \geq 0\) for all \(t \in [0, \tau_2]\), then the solution \(y(t)\) of (4.1) is positive when \(\lambda > 0\). Furthermore, if \(u(t) \leq 1\) and if

\[
|\lambda| (\tau_2 - \tau_1) < 1
\]

then \(y(t)\) is positive and bounded, for all \(t \geq \tau_2\) and for any \(\lambda\).
Proof. We apply the method of steps and we examine the cases \( \lambda \) positive and negative separately. First of all let us suppose \( \lambda > 0 \) and consider the interval \([\tau_2, \tau_2 + \tau_1]\), the integration variable \( s \) in (4.1) runs in \([0, \tau_2]\) and thus \( y(s) = u(s) \). Since \( u \) is positive and bounded by 1, it is easy to show that \( 0 < y(t) < 1 + \lambda(\tau_2 - \tau_1), \forall t \in [\tau_2, \tau_2 + \tau_1] \).

In the next interval \([\tau_2 + \tau_1, \tau_2 + 2\tau_1]\), \( s \) runs in \([\tau_1, \tau_2 + \tau_1]\) and thus \( 0 < y(s) < 1 + \lambda(\tau_2 - \tau_1) \). Hence \( 0 < y(t) < 1 + \lambda(\tau_2 - \tau_1) + \lambda^2(\tau_2 - \tau_1)^2 \). Going on with the same procedure through the next adjacent intervals we come out with

\[
0 < y(t) < \sum_{j=0}^{+\infty} \lambda^j (\tau_2 - \tau_1)^j.
\]

(4.4)

Since the series at the right-hand side of (4.4) converges if \( \lambda(\tau_2 - \tau_1) < 1 \) the theorem is proved for \( \lambda > 0 \).

If \( \lambda < 0 \), then in \([\tau_2, \tau_2 + \tau_1]\) we have

\[
1 + \lambda(\tau_2 - \tau_1) < y(t) < 1.
\]

(4.5)

Thus, \( y(t) \) is obviously bounded and, if \( |\lambda|(\tau_2 - \tau_1) < 1 \), it is positive. Stepping in the next interval and going further by the method of steps, we have that (4.5) is satisfied in each interval and this yields the result stated in the theorem. \( \square \)

Remark 1. Observe that hypothesis \( u(t) \leq 1 \) is not restrictive for us, since it depends on the choice of the forcing function of the test equation, that in our case is 1.

Thus, the sign and the boundedness of the initial known function \( u(t) \in [0, \tau_2] \) influences the sign and the boundedness of \( y(t) \) on the entire interval, whenever (4.3) is satisfied. The theorem below shows that the same condition (4.3) is also responsible for the convergence of \( y(t) \) at infinity.

Theorem 4.3. Assume that the hypotheses of Theorem 4.2 hold, then

\[
\lim_{t \to \infty} y(t) = y^* = \frac{1}{1 - \lambda(\tau_2 - \tau_1)}.
\]

(4.6)

Proof. Let \( l' = \lim \inf_{t \to \infty} y(t) \leq \lim \sup_{t \to \infty} y(t) = l'' \), since \( y(t) \) is a continuous function, there exist two sequences \( \{t_n'\} \) and \( \{t_n''\} \) such that \( \lim_{n} y(t_n') = l' \) and \( \lim_{n} y(t_n'') = l'' \). On the other hand,

\[
y(t_n') = 1 + \lambda \int_{t_1}^{\tau_2} y(t_n' - x) \, dx.
\]

By applying the integral mean-value theorem and passing to the limit as \( n \) goes to infinity, we have that \( \lim_{n} y(t_n' - \xi_n') = \frac{l' - 1}{\lambda(\tau_2 - \tau_1)} \), with \( \xi_n' \in (t_1, \tau_2) \). Thus,

\[
l' \leq \frac{l' - 1}{\lambda(\tau_2 - \tau_1)} \leq l''.
\]

An analogous procedure on \( y(t_n'') \) leads to

\[
l' \leq \frac{l'' - 1}{\lambda(\tau_2 - \tau_1)} \leq l''.
\]

hence, by adding the left-hand side of the first inequality to the right-hand of the second for \( \lambda > 0 \) and vice versa in the case \( \lambda < 0 \), we have that

\[
\frac{l'' - l'}{|\lambda|(\tau_2 - \tau_1)} \leq l'' - l'.
\]

Since, for our assumption, \( |\lambda|(\tau_2 - \tau_1) \leq 1 \), then it is \( l' = l'' \). Let us denote by \( y^* \) this limit. The application of the integral mean-value theorem to (4.1) leads to \( y(t) = 1 + \lambda y(t - \xi)(\tau_2 - \tau_1) \), and thus, passing to the limit as \( t \to \infty \) we get \( y^* = 1 + \lambda(\tau_2 - \tau_1)y^* \), and (4.9) is proved. \( \square \)
Remark 2. If \( \lambda > 0 \), the right-hand side of (4.6) becomes negative when \( \lambda (\tau_2 - \tau_1) > 1 \). However, since \( y(t) > 0 \) (because of Theorem 4.2), \( \lim_{t \to \infty} y(t) \) cannot be negative. Hence, the limit of the solution \( y(t) \) of (4.1) cannot exist and then \( y(t) \) is either unbounded or oscillatory. Of course, also in the case \( \lambda (\tau_2 - \tau_1) = 1 \) the function \( y(t) \) cannot have a finite limit, therefore, in Theorem 4.2, when \( \lambda > 0 \), condition (4.3) is also necessary.

4.2. Qualitative behavior of the numerical solution

In this section we investigate on the numerical stability of the TDQ method for solving (4.1) according to the following definition.

Definition 1. A numerical method is stable when its linearized error (with respect to a constant perturbation on the forcing term) behaves like the corresponding linearized error of the continuous problem.

Of course, the difference equation representing the linearized error corresponds to the equation we obtain when the TDQ method is applied to (4.1), i.e.

\[
y_n = 1 + \lambda h \left[ y_{n-r_2} + 2 \sum_{j=n-r_2+1}^{n-r_1-1} y_j + y_{n-r_1} \right], \quad n = r_2 + 1, \ldots,
\]

with \( y_0 = 1 \), and \( y_1 = u_1, \ldots, y_{r_2} = u_{r_2} \) given and where, once again, \( h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2} \).

Hence, we look for the conditions on the parameters of (4.7) that lead to a numerical solution \( y_n \) which replicates the global properties obtained in the previous section for the analytical solution \( y(t) \).

As in the continuous case, it is easy to show that, when \( \lambda \) is negative, the numerical solution \( y_n \) oscillates. What is more, by a simple procedure based on the method of steps we can prove the following theorem which is the discrete analogue of Theorem 4.2 and gives conditions for the positiveness and boundedness of \( y_n \) for \( n > r_2 \).

Theorem 4.4. Assume that \( u_l \geq 0 \) for \( l = 0, \ldots, r_2 \) and \( h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2} \), then the solution \( y_n \) of (4.7) is positive when \( \lambda > 0 \). Furthermore, if \( u_l \leq 1 \) for \( l = 0, \ldots, r_2 \) and if

\[
h|\lambda|(r_2 - r_1) < 1
\]

then \( y_n \) is positive and bounded for all \( n \geq r_2 \) and for any \( \lambda \).

Since conditions (4.8) and (4.3) coincide, Theorem 4.4 states that the numerical solution of (4.1), obtained by the TDQ method, is positive and bounded whenever the analytical solution is. The same correspondence holds on the asymptotic behavior, as shown in the following theorem.

Theorem 4.5. Assume that (4.8) holds, then

\[
\lim_{n \to \infty} y_n = y^* = \frac{1}{1 - h\lambda(r_2 - r_1)}.
\]

Proof. Let us suppose that \( y_n \) is not regular, hence there exist \( \{k_n'\} \) and \( \{k_n''\} \) such that \( l' = \lim inf_n y_n = \lim y_{k_n'} < \lim y_{k_n''} = \lim sup_n y_n = l'' \). Set \( z_n = y_{n-r_2} + 2 \sum_{j=n-r_2+1}^{n-r_1-1} y_j + y_{n-r_1} \), then \( y_n = 1 + \lambda \frac{h}{2} z_n \) and thus,

\[
\lim_{k_n'} z_{k_n'} = \frac{l' - 1}{h\lambda}.
\]

\[
\lim_{k_n''} z_{k_n''} = \frac{l'' - 1}{h\lambda}.
\]
What is more, after some manipulation on the expression of $z_n$ we get

$$(r_2 - r_1)l' \leq \frac{l' - 1}{h\lambda} \leq (r_2 - r_1)l'',$$

$$(r_2 - r_1)l' \leq \frac{l'' - 1}{h\lambda} \leq (r_2 - r_1)l''.$$

Suitably combining these last two inequalities we have that

$$(1 - h|\lambda|(r_2 - r_1))(l'' - l') \leq 0, \quad (4.10)$$

and, since for our assumption $h|\lambda|(r_2 - r_1) < 1$, (4.10) is satisfied only for $l' = l'' = y^*$. Hence, the limit for $n \to \infty$ in Eq. (4.7) gives $y^* = 1 + h\lambda(r_2 - r_1)y^*$, this yields the result stated in the theorem. \(\square\)

**Remark 3.** Theorems 4.4 and 4.5 assure that, in the hypothesis (4.3), the analytical solution of (4.1) and its numerical solution furnished by the TDQ method have the same behavior. In this sense we can claim that if (4.3) is satisfied, the TDQ method is stable with respect to the test equation (4.1).

5. Numerical experiments

In Section 4.1 we say that the classical theory on the stability of (4.1)–(4.2) go through the analysis of the roots of the characteristic equation. In order to allow comparisons between our results in Section 4.1 and the stability results which could be obtained by investigating the roots of the characteristic equation of (4.1), we use the boundary locus technique in the particular case $\tau_1 = 0.5$, $\tau_2 = 1$. It is possible to prove that Eq. (4.1) is stable when $\lambda(\tau_2 - \tau_1) \in (A, B)$, where $A$ and $B$ are the points shown in Fig. 1. We can note that such a stability interval is in accordance with the results described in Section 4 which give sufficient conditions for the stability of (4.1).

Now, we report some numerical experiments that show the performances of the TDQ method (4.7) (with $h$ satisfying (2.1)) when applied to some problems of the form (1.7), with $\tau_1 = 0.5$, $\tau_2 = 1$ and, in order to relate the performances of the method to the stability condition (4.3), we assume (by first approximation) $\lambda \approx k(0)g'(y(0))$. First we consider equations with kernels of the type $(\sigma + \mu(t - s))y(s)$ and solutions $y(t) = \exp(-t)$, $y(t) = t$ and $y(t) = t \sin(t)$, respectively. Figures 2–4 show the behavior of the numerical solution (−.∗) with respect to the analytical one (solid line) for different values of the parameter $\sigma$ (we set $\mu = -1$). From these pictures it is clear that when condition (4.3) is satisfied with $\lambda \approx k(0)$, the numerical solution is highly reliable, also when the solution is highly oscillating (see Fig. 3, where the solution is $t \sin(t)$), while it may skip away from the expected behavior when (4.3) is not satisfied. Since we could observe the same behavior when integrating more complicated kernels (such as $(\sigma + \mu \exp(t - s))y(s)$, $(\sigma + \mu \sin(t - s))y(s)$, ...) we can assert that the TDQ method for (1.7) turns

![Partition in complex $\lambda(\tau_2 - \tau_1)$-plane.](image-url)
out to be reliable when $|λ| (τ_2 − τ_1) < 1$, however, whenever this condition is not satisfied we cannot assure a correct behavior. For the sake of completeness we report in Fig. 5 an analogous experiment on the nonlinear kernel $(σ + μ(t − s))(y(s) + 1)^2$. As a counterexample we report, in Fig. 6, the results of our simulations on the model problem (1.7) with $k(t − s) = (σ + μ(t − s))$, $g(y) = exp(−y)y$ and solution $y(t) = 1$. In this case the numerical and analytical solution show exactly the same behavior both when (4.3) is satisfied and when it is not. Which, according to Remark 3, means that (4.3) is a sufficient but not necessary condition for the stability of the TDQ method with respect to (4.1).
Fig. 4. \( k(t - s)g(y(s)) = (\sigma + \mu(t - s))y(s) \) with solution \( y(t) = t \sin(t) \).

Fig. 5. \( k(t - s)g(y(s)) = (\sigma + \mu(t - s)) (1 + y(s))^2 \) with solution \( y(t) = \sin(t) \).

6. Concluding remarks

We have approached the problem of studying the stability of the TDQ method for solving double delay VIEs by considering a parametric test equation. For the complete case (1.4) we prove a condition on the stepsize which assures that the numerical solution inherits the asymptotic properties of the continuous one. For the particular case (4.1) no restrictions on the stepsize are required in the sense that the analytical and numerical solution have the same global behavior provided that the condition (4.3), which characterizes the problem, is fulfilled. Whenever condition
(4.3) is not satisfied, we cannot assure this property anymore. The numerical experiments show the same behavior described above for the test equation (4.1) when integrating more complicated problems. In this sense, we can claim that condition (4.3) (with \( \lambda \approx k(0)g'(y(0)) \)) plays just the role of a stability condition.

References