How hyperbolic knots with homeomorphic cyclic branched coverings are related

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Abstract

We determine the exact geometric relation between two hyperbolic knots $K$ and $K'$ such that the $n$-fold cyclic branched covering of $K$ coincides with the $m$-fold cyclic branched covering of $K'$. If $m$ and $n$ are not powers of two the solution of the problem is known (complete solution for the case $m = n$, partial results for the case $n$ different from $m$). In the present work, we give a complete solution to the problem for branching orders which are powers of two, and thus in particular also for the most basic case of 2-fold branched coverings. © 2002 Elsevier Science B.V. All rights reserved.

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Introduction

An interesting and much considered class of 3-manifolds is constituted by the $n$-fold cyclic branched coverings of knots in the 3-sphere. Such a representation of a 3-manifold is not unique, in general. The reciprocal determination of knots and their branched coverings is a problem largely studied in low dimensional topology.

The general problem we are interested in is the following: what is the relation between two inequivalent knots $K$ and $K'$ with the same $n$-fold cyclic branched covering? Or, more generally, how are two inequivalent knots $K$ and $K'$ related such that the $n$-fold cyclic branched covering of $K$ and the $m$-fold cyclic branched covering of $K'$ are the same 3-manifold?

If $m$ and $n$ are not powers of two, the problem has been considered in [16] (the case $m = n$) and [11]. Under this hypothesis, it has been shown in [16] that two hyperbolic
knots with the same \( n \)-fold cyclic branched covering are related in a very simple way, and that there are at most two such knots. The case \( n \) different from \( m \) has been considered in [11]. It turns out that the case of 2-fold cyclic branched coverings, and of branching orders which are powers of two, is more complicated, see [10] for a general approach to this case. In the present work, we give a complete solution of the problem in the hyperbolic case for branching orders which are powers of two and so in particular for the most basic case of 2-fold branched coverings.

A knot \( K \) in \( S^3 \) is \textit{hyperbolic} if its complement \( S^3 - K \) is a hyperbolic manifold of finite volume. We denote by \( O_n(K) \) the orbifold with underlying topological space \( S^3 \) and with singular set, of singularity index \( n \), the knot \( K \) (see [13] for the general theory of orbifolds). A knot \( K \) is 2\( \pi/n \)-\textit{hyperbolic} if \( O_n(K) \) is a hyperbolic orbifold; equivalently, \( K \) is 2\( \pi/n \)-hyperbolic if the \( n \)-fold cyclic branched covering of \( K \) is a hyperbolic manifold and the covering transformations are isometries. For every \( n \), 2\( \pi/n \)-hyperbolicity implies hyperbolicity. On the other hand, by Thurston’s orbifold geometrization theorem [14,1,3], any hyperbolic knot is 2\( \pi/n \)-hyperbolic for every \( n \geq 3 \) with the exception of the figure-8 knot which is hyperbolic and 2\( \pi/3 \)-Euclidean. In fact if \( K \) is a hyperbolic knot the orbifold \( O_n(K) \) is geometric for \( n \geq 3 \) because it does not contain any incompressible spherical or Euclidean 2-suborbifold and we can apply the orbifold geometrization theorem. By Dunbar’s list of geometric non-hyperbolic orbifolds with underlying topological space the 3-sphere, the only hyperbolic knot such that \( O_n(K) \) is non-hyperbolic is the the figure-8 knot with \( n = 3 \). For \( n = 2 \) the situation is different; for example most 2-bridge knots are hyperbolic but the 2-fold branched covering of a 2-bridge knot is a lens space and thus no 2-bridge knot is \( \pi \)-hyperbolic. In the present paper we always work in the hyperbolic setting, that is we will assume that the cyclic branched coverings are hyperbolic 3-manifolds, and that the covering transformations are isometries. Thus, by Thurston’s orbifold geometrization theorem for branching order larger than two it is sufficient to assume that the knots are hyperbolic, for branching order two we will assume that they are \( \pi \)-hyperbolic.

Now we give a basic example of how two knots \( K \) and \( K' \) may be related such that the \( n \)-fold cyclic branched covering of \( K \) coincides with the \( m \)-fold cyclic branched covering of \( K' \).

\textit{The standard Abelian construction}

Let \( M \) be the \( n \)-fold and \( m \)-fold cyclic branched covering of two knots \( K \) and \( K' \), respectively. We denote by \( C \) and \( C' \) the cyclic covering groups of \( K \) and \( K' \), respectively; the preimage \( \tilde{K} \) (respectively, \( \tilde{K}' \)) of \( K \) (respectively, \( K' \)) in \( M \) is the fixed point set of \( C \) (respectively, \( C' \)). The groups \( C \) and \( C' \) commute and they generate a group \( G \) of diffeomorphisms of \( M \) isomorphic to \( \mathbb{Z}_n \times \mathbb{Z}_m \). Each element of the covering group \( C \) (respectively, \( C' \)) induces a rotation on \( \tilde{K}' \) (respectively, \( \tilde{K} \)), and the quotient orbifold \( M/G \) is the 3-sphere whose singular set is a link with two components of singularity indices \( n \) and \( m \).
This construction occurs in many situations. For example, if \( m = n \) is not a power of two, \( K \) and \( K' \) arise from the standard Abelian construction [16]. Also, in the more general case in which \( m \) and \( n \) are divided by a prime number \( p \) different from two, the two knots arise from the standard abelian construction; moreover there exist at most two such knots [11].

The case \( n = 2^a \) and \( m = 2^b \) of powers of two is more difficult because in general the Sylow 2-subgroup of the orientation-preserving isometry group of a hyperbolic 3-manifold which occurs as a cyclic branched covering is more complicated than the other Sylow subgroups. In [10], an analysis of the Sylow 2-subgroup for such a manifold is given together with the proof that we have at most nine inequivalent \( \pi \)-hyperbolic knots with the same 2-fold branched covering (see also [9]); sets of four such knots are known, and there is some evidence that nine may be the exact upper bound.

To solve the problem for powers of two we consider three different cases:

1. \( m = n = 2 \);
2. \( m > 2 \) and \( n > 2 \);
3. \( m = 2, n > 2 \).

The lucky case is the second one; in fact we prove the following theorem.

**Theorem 2.** For \( a \geq 2 \) and \( b \geq 2 \), let \( M \) be the \( 2^a \)-fold and \( 2^b \)-fold cyclic branched covering of the inequivalent knots \( K \) and \( K' \), respectively. Suppose that \( K \) is hyperbolic. Then \( K \) and \( K' \) arise from the standard Abelian construction. Moreover there are at most two such knots.

This formulation of Theorem 2 uses the orbifold geometrization theorem. By the orbifold geometrization theorem the hyperbolicity of \( K \) implies that \( K \) is \( 2\pi/2^a \)-hyperbolic that is \( M \) is hyperbolic and the covering transformations of \( K \) are isometries. Since \( M \) is hyperbolic by the orbifold geometrization theorem and by Mostow’s rigidity theorem we can suppose that also the covering group of \( K' \) is a group of isometries. Instead the hyperbolicity of one of the knots we can assume the hyperbolicity of the manifold \( M \) and by Thurston’s orbifold geometrization theorem we obtain that the covering transformations of \( K \) and \( K' \) are isometries. If we want to avoid the orbifold geometrization theorem we have to suppose that \( K \) is \( 2\pi/2^a \)-hyperbolic and that \( K' \) is \( 2\pi/2^b \)-hyperbolic.

We have to make precise that we shall work in the category of oriented manifolds and of orientation-preserving diffeomorphisms; so two knots \( K \) and \( K' \) are inequivalent if there is no orientation-preserving diffeomorphism of \( S^3 \) which maps \( K \) to \( K' \). In any case, if \( a \) is different from \( b \), the two knots are inequivalent. In fact, the volumes of the hyperbolic orbifolds \( O_n(K) \) increase monotonously with \( n \) (see [7]), so different cyclic branched coverings of the same knot have different volumes.

The other two cases are more complicated: the two knots can arise from different constructions. To avoid technical details in this introduction, we present these constructions and the theorems concerning the cases (1) and (3) in Section 1. In Section 2 we present some preliminary results, and in Sections 3, 4 and 5 we present the proofs of the theorems related to the cases (1), (2) and (3), respectively.
1. Basic constructions and theorems

In this section we describe the further constructions which appear in the analyses of the cases (1) and (3) (that is when at least one among $m$ and $n$ is equal to two).

First we present two types of graphs which appear in these constructions.

– *Theta-curve*. A theta-curve is a graph with two vertices $v_1, v_2$ and three edges $l_1, l_2, l_3$. All edges join $v_1$ with $v_2$. The loops $l_1 \cup l_2, l_2 \cup l_3$ and $l_1 \cup l_3$ are the three constituent knots of the theta-curve.

– *Pince-nez graph*. A pince-nez graph has two vertices $v_1, v_2$ and three edges $l_1, l_2, l_3$. The edge $l_1$ joins $v_1$ with itself, the edge $l_2$ joins $v_2$ with itself, so these two edges form two loops. The last edge $l_3$ joins $v_1$ with $v_2$.

Let $M$ be the $n$-fold and $m$-fold branched covering of two knots $K$ and $K'$, respectively. We denote by $C$ and $C'$ the cyclic covering groups of $K$ and $K'$, respectively; the preimage $\tilde{K}$ (respectively, $\tilde{K}'$) of $K$ (respectively, $K'$) in $M$ is the fixed point set of $C$ (respectively, $C'$).

**Standard dihedral construction I.** We suppose that $n = 2$ and that the covering involution of $K$ acts as a strong inversion (reflection) on $\tilde{K}'$. The covering groups $C$ and $C'$ generate a group $D$ of diffeomorphisms of $M$ isomorphic to the dihedral group of order $2m$; the covering involution of $K$ operates on the normal subgroup $C$ by sending each element to its inverse. The quotient orbifold $M/D$ is the 3-sphere whose singular set is a theta-curve; two edges have singularity index two, the remaining one has singularity index $m$.

**$\mathbb{Z}_2 \ltimes (\mathbb{Z}_m \times \mathbb{Z}_m)$ construction.** We suppose that $n = 2$. We denote by $h$ the covering involution of $K$ and by $h'$ a generator of $C'$. Suppose that $C$ and $C'$ generate a group $G$ which has the following presentation:

$$\langle h, h'| h^2 = (h')^m = 1, \ (hh'h^{-1})h' = h'(hh'h^{-1}) \rangle.$$

The group $G$ is a semidirect product of $C$ and the normal subgroup isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_m$ generated by $C'$ and $hC'h^{-1}$. In the group generated by $C'$ and $hC'h^{-1}$ there exist exactly two maximal cyclic subgroups with non-empty fixed point set which are $C'$ and $hC'h^{-1}$. The quotient orbifold $M/G$ is the 3-sphere, the singular set is a link with two components of singularity indices $m$ and two.

In the following three cases we suppose that $n = 2$ and $m = 2$. Then the covering groups $C$ and $C'$ generate a group $D$ of diffeomorphisms of $M$ isomorphic to a dihedral group. We assume that the order of $D$ is $2^{e+1}$ for some $e \geq 1$. We denote by $F$ the cyclic subgroup of order $2^e$ contained in $D$. The three constructions are different from a geometric point of view because the elements in the cyclic group $F$ may act freely or not.

**Standard dihedral construction II.** Each element of $F$ acts freely on $M$. The quotient orbifold $M/D$ is the 3-sphere whose singular set is a link with two components of singularity index two. For $e = 1$ the standard dihedral construction II coincides with the standard abelian construction.
Standard dihedral construction III. The subgroup $F$ has non-empty connected fixed point set $L$ (i.e., each element of $F$ fixes pointwise $L$). The reflections in $D$ act as reflections on $L$. The quotient orbifold $M/D$ is the 3-sphere whose singular set is a theta-curve; two edges have singularity index two, the remaining one has singularity index $2c$.

We note that in the standard dihedral constructions I and III the elements of maximal order in $D$ have non-empty fixed point set but there is a difference between these two construction. In the standard construction I the elements of maximal order of $D$ are covering transformation of $K$ and the quotient of $M$ for these elements is the 3-sphere; on the contrary in the standard construction III the covering transformations of $K$ and $K'$ are involutions and in general the quotient of $M$ for the group $F$ is not the 3-sphere.

Standard dihedral construction IV. The group $F$ has no global fixed points but $F$ contains a proper subgroup with non-empty fixed point set.

We denote by $2d$ the order of the maximal subgroup of $F$ with non-empty fixed point set and we denote by $L$ its connected fixed point set. The reflections in $D$ act as reflections on $L$. The quotient orbifold $M/D$ is the 3-sphere whose singular set is a pince-nez graph; the two loops have singularity index two, the remaining edge has singularity index $2d$.

Other details about these constructions appear in the proofs of the theorems.

Now we are able to state the theorems for the remaining cases.

Theorem 1. Let $M$ be the 2-fold cyclic branched covering of the inequivalent knots $K$ and $K'$. Suppose that $K$ is $\pi$-hyperbolic then $K$ and $K'$ arise from the standard dihedral constructions II, III or IV.

Theorem 3. For $b \geq 2$, let $M$ be the 2-fold and $2^b$-fold cyclic branched covering of the knots $K$ and $K'$, respectively. Suppose that $K$ is $\pi$-hyperbolic. Then $K$ and $K'$ arise from the standard abelian construction, from the standard dihedral construction I or from the $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2^b)$ construction.

This formulation of the theorems uses Thurston’s geometrization orbifold theorem. Also in this cases the conditions we really need are that $M$ is hyperbolic and that the covering transformations of the knots are isometries. To get this situation we have some different possibilities to state the theorems assuming different hypothesis as in Theorem 2; we can also avoid the use of Thurston’s geometrization orbifold theorem assuming in Theorem 1 that both $K$ and $K'$ are $\pi$-hyperbolic and in Theorem 3 that $K$ is $\pi$-hyperbolic and $K'$ is $2\pi/2^b$-hyperbolic.

2. Preliminaries

In this section we present some preliminary results concerning finite group actions on 3-manifolds.
Proposition 1. Let $\mathcal{M}$ be the $2^a$-fold cyclic branched covering of a knot, for $a \geq 1$. Then $\mathcal{M}$ is a $\mathbb{Z}_2$-homology 3-sphere [4, p. 16].

Proposition 2. Let $f$ be a periodic orientation-preserving diffeomorphism of a closed orientable $\mathbb{Z}_2$-homology 3-sphere whose period is a power of two. Then the fixed point set of $f$ is connected, that is empty or a simple closed curve. (By classical Smith theory, see [2] for a review of this theory.)

Let $K$ be a simple closed geodesic in a closed hyperbolic 3-manifold and $I$ a group of isometries that fixes setwise the geodesic. The elements of $I$ induce on $K$ reflections (strong inversions) or rotations; if an element of $I$ induces on $K$ a reflection we call it a $K$-reflection otherwise we call it a $K$-rotation.

Proposition 3. Let $I$ be a finite group of orientation-preserving isometries of a closed orientable hyperbolic 3-manifold which map a given simple closed geodesic $K$ to itself. Then $I$ is isomorphic to a subgroup of a semidirect product $\mathbb{Z}_2 \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$, for some nonnegative integers $n$ and $m$, where $\mathbb{Z}_2$ operates on the normal subgroup $\mathbb{Z}_n \times \mathbb{Z}_m$ by sending each element to its inverse.

Proof. The subgroup of $K$-rotations is Abelian: it contains the cyclic subgroup of all elements fixing $K$ pointwise, with cyclic factor group acting faithfully by rotations on $K$. The subgroup of $K$-rotations has index 1 or 2. In the second case $I$ contains a $K$-reflection and any $K$-reflection acts on the normal subgroup of $K$-rotations by inverting each element.

This finishes the proof. \(\Box\)

Proposition 4. Let $I$ be a group of isometries isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ of a closed orientable hyperbolic $\mathbb{Z}_2$-homology 3-sphere. Then either $I$ has two global fixed points or $I$ contains exactly one involution acting freely.

Proof. The group $I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ cannot act freely on a $\mathbb{Z}_2$-homology sphere [2, Theorem 8.1, p. 148]; thus there exists in $I$ an involution $h$ with non-empty fixed point set $K$.

We consider the quotient $\overline{\mathcal{M}} = \mathcal{M}/h$ and we denote by $\overline{K}$ the projection of $K$ in $\mathcal{M}/h$. Since $I$ is Abelian, $I$ fixes setwise $K$ and the group $I$ projects to a group $\overline{I} \cong \mathbb{Z}_2$ that maps $\overline{K}$ to itself. It is easy to see that $\mathcal{M}/h$ is a $\mathbb{Z}_2$-homology sphere; therefore the fixed point set of $\overline{I}$ is connected.

Suppose first that in $I$ there is another involution $h'$ with non-empty fixed point set $K'$. If $h'$ is a $K$-reflection, $K'$ has two points in common with $K$ and $I$ has two global fixed points.

On the other hand if $h'$ is a $K$-rotation, $K'$ is disjoint from $K$ and $h$ acts as a $K'$-rotation. Since the fixed point set of $\overline{I}$ is connected its preimage is exactly $K'$. If $hh'$ has non-empty fixed point set, $hh'$ fixes pointwise $K'$ but this fact is impossible because there is only one involution fixing pointwise a simple closed geodesic.
Finally we suppose that \( h \) is the unique involution with non-empty fixed point set in \( I \) then \( \overline{I} \) acts freely on \( M/h \). Consider the quotient \( N := \overline{M}/\overline{I} \); we denote by \( L \) the projection of \( \overline{K} \) in \( N \). We note that \( M − K \) is the unbranched covering of \( N − L \) with covering group \( I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). On the other hand \( H_1(N − L) \) has Betti number equal to one and the torsion subgroup of \( H_1(N − L) \) has odd order (see [6, Theorem 2.1] and [5, p. 92]). This fact implies that \( N − L \) cannot have a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) unbranched covering and this case does not occur.

This finishes the proof. \( \Box \)

3. Proof of Theorem 1

The covering involutions \( h \) and \( h' \) of \( K \), respectively \( K' \), are elements of the orientation-preserving isometry group \( \text{Iso}^+(M) \) of \( M \) and they are not conjugate because, by hypothesis, \( K \) and \( K' \) are not equivalent. By the Sylow theorems we can assume that \( h' \) and \( h \) are two distinct involutions of the same Sylow 2-subgroup \( S \) of \( \text{Iso}^+(M) \). We denote by \( D \) the subgroup of \( S \) which is generated by \( h' \) and \( h \). Since \( D \) is generated by two involutions, \( D \) is dihedral, say of order \( 2(c+1) \) for some \( c \geq 1 \); so \( 2^c \) is the order of the product \( (hh') \). We note that in \( D \) the only involution which can act freely on \( M \) is \( (hh')^{2c-1} \) because the other involutions are conjugated either with \( h \) or with \( h' \) which have non-empty fixed point set.

To prove the theorem it is enough to construct the quotient orbifold \( M/D \). This will be done by first considering the underlying topological space of \( M/D \) and then the structure of the singular set.

Claim A. The underlying topological space of \( M/D \) is the 3-sphere \( S^3 \).

Proof. We get the quotient \( M/D \) as the output of a sequence of \((c+1)\) orbifolds which are successive quotients of \( M \) and all have \( S^3 \) as underlying topological space.

We note that all the reflections of \( D \) have non-empty fixed point set. The first orbifold in our sequence is \( O := M/h \), which by construction has underlying topological space \( S^3 \) and singular set the knot \( K \) with singularity index two. The normalizer \( N_Dh \) in \( D \) of the covering involution \( h \) of \( O \) is isomorphic to a dihedral group \( D' \subset D \) of order four; the group \( D' \) projects in \( O \) to the factor group \( D'/h \) which is isomorphic to \( \mathbb{Z}_2 \) and is generated by an involution, say \( t \), with connected fixed point set by Proposition 2; since at least one lift of \( t \) to \( D \) has non-empty fixed point set also \( t \) has non-empty fixed point set.

The second orbifold we consider is the quotient \( O/t \). By the positive solution of the Smith Conjecture for involutions (see [15]), \( O/t \) has also underlying topological space \( S^3 \). Again the normalizer \( N_DD' \) of the covering group \( D' \) is isomorphic to a dihedral group of order eight; this dihedral group projects in \( O/t \) to a group isomorphic to \( \mathbb{Z}_2 \) and it contains an involution with non-empty connected fixed point set.

This construction can be iterated \((c+1)\) times and, after applying \((c+1)\) times the positive solution of the Smith Conjecture for involutions, we finally end up with the
quotient $M/D$ which has underlying topological space $S^3$. This finishes the proof of Claim A. \[\square\]

To complete the proof of the theorem it is enough to find the singular set of the quotient orbifold $M/D$ which is the projection of the fixed point sets of the elements of $D$. We denote by $F$ the cyclic subgroup generated by $hh'$; the structure of the singular set depends on the fixed point sets of the elements of $F$. There are three cases, which correspond to the three dihedral situations II, III and IV.

**Claim B1.** If each element of $F$ acts freely the singular set of $M/D$ is a link with two components with singularity index two at each point; the two knots $K$ and $K'$ arise from the standard dihedral construction II.

In this case the only elements of $D$ with non-empty fixed point sets are the involutions which lie in the conjugacy classes of $h$ and $h'$. By passing to the quotient we find a link with two components as singular set.

**Claim B2.** If each element of $F$ has non-empty fixed point set the singular set of $M/D$ is a theta-curve with singular index $2^c$ on one edge and two on the left two edges; the two knots $K$ and $K'$ arise from the standard dihedral construction III.

The quotient of $M$ by $F$ is an orbifold whose underlying topological space is a $\mathbb{Z}_2$-homology sphere and whose singular set is a knot with singularity index $2^c$. The group $D$ descends to an involution of $M/F$ which acts as a strong inversion on the knot. So the quotient $M/D$ is an orbifold with underlying topological space $S^3$ and singular set a theta-curve with singularity index $2^c$ on one edge and two on the left two edges.

**Claim B3.** Suppose that the subgroup of elements of $F$ with non-empty fixed point set is a proper subgroup of $F$ of order $2^d$. Then the singular set of $M/D$ is a pince-nez graph with singular index two on the two loops and $2^d$ on the connecting edge; the two knots $K$ and $K'$ arise from the standard dihedral construction IV.

We denote by $E$ the subgroup of the elements of $F$ with non-empty fixed point set. The quotient of $M$ by $E$ is a $\mathbb{Z}_2$-homology sphere whose singular set is a knot with singularity index $2^d$. The group $D$ normalizes the covering group $E$ and it descends to a dihedral group of order $2^{c-d+1}$; in particular we have the projection of $h$ to $M/E$ which acts as a strong inversion on the knot. We denote by $D'$ the subgroup of $D$ generated by $E$ and by $h$. The quotient $M/D'$ is an orbifold with underlying topological space a $\mathbb{Z}_2$-homology sphere and singular set a theta-curve with singularity index $2^d$ on one edge, say $e_1$, and two on the left two edges, say $e_2$ and $e_3$. We note that in $D'$ each element has non-empty fixed point set; moreover an element of $D'$ induces on the fixed point set of $E$ either a strong reflection (the elements not contained in $E$) or a trivial action (the elements in $E$). In the successive steps the geometric situation will be different.
The normalizer \( N_D D' \) in \( D \) of \( D' \) is a dihedral group \( D'' \) of order \( 2^{d+2} \). The group \( N_D D' \) descends in \( M/D' \) to a group isomorphic to \( \mathbb{Z}_2 \); this group contains an involution \( t \) of \( M/D' \) with connected fixed point set. We note that the set of the lifts of \( t \) in \( D \) contains \((hh')^{d-1}\) that is an element which acts freely and which induces a rotation of period two on the fixed point set of \( E \); thus \( t \) acts as a reflection on \( e_1 \) and exchanges \( e_2 \) and \( e_3 \). So the quotient \( M/D'' \) is an orbifold with underlying topological space a \( \mathbb{Z}_2 \)-homology sphere and singular set a pince-nez graph with singularity index two on the two loops and \( 2^d \) on the connecting edge.

In the successive step we find an involution of \( M/D'' \) with connected fixed point set which exchanges the two loops of the pince-nez graph and acts as a reflection on the connecting edge. By factoring by this involution we get an orbifold with underlying topological space a \( \mathbb{Z}_2 \)-homology sphere and singular set again a pince-nez graph with the same singularity indices as the previous one. The left steps are analogous until we finally end up with the quotient \( M/D \) with singularity graph a pince-nez graph with singularity index two on the two loops and \( 2^d \) on the connecting edge.

We have considered all the possible cases and the proof is complete.

4. Proof of Theorem 2

We need the following.

**Proposition 5.** Let \( H \) be a subgroup of a finite \( p \)-group \( G \). Then either \( H \) is normal, or a conjugate \( xhx^{-1} \) different from \( H \) is contained in the normalizer \( N_G(H) \) of \( H \) in \( G \) [12, (1.5), p. 88].

**Proof of Theorem 2.** The covering groups \( C \), respectively \( C' \), of \( K \), respectively \( K' \), are cyclic subgroups of the orientation-preserving isometry group \( \text{Iso}^+(M) \) of \( M \) and they are not conjugate because, by hypothesis, \( K \) and \( K' \) are not equivalent. By the Sylow theorems we can assume that \( C \) and \( C' \) are contained in the same Sylow 2-subgroup, say \( S \), of \( \text{Iso}^+(M) \). Each element of \( C \) (respectively, \( C' \)), acts as a rotation around its fixed point set \( \tilde{K} \) (respectively, \( \tilde{K}' \)) which is the preimage in \( M \) of \( K \) (respectively, \( K' \)); since \( K \) and \( K' \) are distinct sets, \( C \) and \( C' \) have trivial intersection.

Suppose first that \( C' \) normalizes \( C \); we prove that in this case \( K \) and \( K' \) arise from the standard abelian construction. The cyclic group \( C' \) fixes setwise the fixed point set \( \tilde{K} \) of \( C \) and, having order \( 2^{2b} \geq 4 \), it acts as a group of rotations on \( \tilde{K} \). By Proposition 3 and its proof this implies that \( C' \) commutes with \( C \), so \( C \) and \( C' \) generate a subgroup \( \mathbb{Z}_{2b} \times \mathbb{Z}_{2b} \) of \( \text{Iso}^+(M) \). In this case \( K \) and \( K' \) arise from the standard Abelian construction.

Now we want to prove that the previous situation is the only one that can occur; we suppose that the normalizer \( N_S C \) of \( C \) in \( S \) does not contain \( C' \) and we get a contradiction. In Claims A and B we apply Proposition 5 to obtain new groups with non-empty fixed point set and finally using these groups we get a contradiction.
Claim A. There exists a subgroup $I \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ of $\text{Iso}^+(M)$ which contains $C$; moreover $I$ contains exactly two cyclic subgroups of order $2^a$ with non-empty fixed point set.

Proof. Since $C$ is not normal in $S$, by Proposition 5, the normalizer $N_S C$ contains a conjugate subgroup $gCg^{-1}$ different from $C$. The group $gCg^{-1}$ is a cyclic group $\mathbb{Z}_{2^n}$ of local rotations around the fixed point set $g(\tilde{K})$; $\tilde{K}$ and $g(\tilde{K})$ are different, so $gCg^{-1}$ has trivial intersection with $C$. By construction $gCg^{-1}$ normalizes $C$ and, having order $2^a$, each element of $gCg^{-1}$ acts as a rotation on $\tilde{K}$. By Proposition 3 and its proof this implies that $C'$ commutes with $C$, so $C$ and $C'$ generate a subgroup $I \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ of $\text{Iso}^+(M)$.

We note that $C$ and $gCg^{-1}$ are the only two cyclic subgroups of $I$ of order $2^a$ with non-empty fixed point sets. Indeed if $t$ is an element of $I$ with non-empty fixed point set, its power $t^{2^{a-1}}$ is an involution with non-empty fixed point set. By Proposition 4 the fixed point set of this involution is either $\tilde{K}$ or $g(\tilde{K})$. This implies that the fixed point set of the group generated by $t$ is also $\tilde{K}$ or $g(\tilde{K})$, so the group generated by $t$ coincides with $C$ or $gCg^{-1}$.

This concludes the proof of Claim A. \(\square\)

Claim B. If $C'$ does not normalize $C$, the normalizer of $I$ in $S$ contains a cyclic subgroup $C''$ of order strictly greater than 2 with non-empty fixed point set and trivial intersection with $I$.

Proof. We apply Proposition 5 to the group $I$.

By Proposition 5 either $C'$ normalizes $I$ or the normalizer $N_S I$ of $I$ in $S$ contains a conjugate subgroup $g_1g_1^{-1}$ different from $I$.

In the first case we simply set $C'' := C'$. Indeed $C'$ has order $2^b$ and non-empty fixed point set. Moreover $C'$ has trivial intersection with $I$; in fact if $C'$ contains a non-trivial element of $I$, it also contains an involution $u$ of $I$. But $u$, like any element of $C'$, has fixed point set $\tilde{K}$; by Proposition 4 either $\tilde{K}$ coincides with $\tilde{K}$ or with $g(\tilde{K})$ and in any case we get a contradiction because this implies that the fixed point set of $C'$ is also $\tilde{K}$ or $g(\tilde{K})$, so $C'$ commutes with $C$.

In the second case, the group $g_1g_1^{-1}$ contains exactly two cyclic subgroups of order $2^a$ with non-empty fixed point sets. If both such cyclic subgroups have non-trivial intersection with $I$, then their fixed point sets are $\tilde{K}$ and $g(\tilde{K})$ because the fixed point sets of the involutions of $I$ are exactly $\tilde{K}$ and $g(\tilde{K})$ by Proposition 4. But this implies that $g_1g_1^{-1}$ coincides with $I$, a contradiction. Therefore there exists at least one cyclic subgroup $C''$ of $g_1g_1^{-1}$ of order $2^a$ which has non-empty fixed point set and has trivial intersection with $I$.

This concludes the proof of Claim B. \(\square\)

We have thus proved that there always exists in $N_S I$ a cyclic subgroup $C''$ of order $2^c$ for $c \geq 2$ which has non-empty fixed point set and trivial intersection with $I$ and we use the existence of $C''$ to get a contradiction.

Since $C''$ normalizes $I$, it normalizes also the union $C \cup gCg^{-1}$ of the unique two cyclic subgroups of $I$ of order $2^a$ with non-empty fixed point sets. By construction the order
of \( C'' \) is \( 2^c \geq 4 \), hence \( C'' \) contains a non-trivial subgroup which normalizes both \( C \) and \( gCg^{-1} \). In particular \( C'' \) contains an involution \( r \) which normalizes both \( C \) and \( gCg^{-1} \) and consequently \( r \) fixes setwise both \( \bar{K} \) and \( g(\bar{K}) \).

We note that, by Proposition 3, all the \( \bar{K} \)-rotation of order two and all the \( g(\bar{K}) \)-rotation of order two are contained in \( I \). This fact implies that \( r \) acts as a reflection on the fixed point sets \( \bar{K} \) and \( g(\bar{K}) \) of \( C \), respectively \( gCg^{-1} \), so the fixed point set of \( r \) intersects non-trivially both \( \bar{K} \) and \( g(\bar{K}) \). Since all the elements of \( C'' \) have the same simple closed curve as fixed point set, also the fixed point set of \( C'' \) intersects non-trivially \( \bar{K} \) and \( g(\bar{K}) \).

By construction \( C'' \) normalizes the union \( C \cup gCg^{-1} \), so it fixes setwise the set \( \bar{K} \cup g(\bar{K}) \); since the fixed point set of \( C'' \) contains some points of \( \bar{K} \cup g(\bar{K}) \) the only possibility is that \( C'' \) has order two and its only non-trivial element is \( r \). This fact contradicts our hypothesis.

The first part of the theorem is proved.

Now we suppose the existence of a third inequivalent knot \( K'' \) such that its \( 2^c \)-fold branched covering is \( M \), with \( c > 1 \). We denote by \( C'' \) the cyclic covering group of \( K'' \). Since the fixed point sets of \( C, C' \) and \( C'' \) are different subsets of \( M \), \( C'' \) has no non-trivial element in common with \( C' \) and with \( C \). By the previous part of the proof \( C' \) and \( C'' \) generate a subgroup of \( \text{Iso}^+(M) \) isomorphic to \( \mathbb{Z}_{2^c} \oplus \mathbb{Z}_{2b} \oplus \mathbb{Z}_{2f} \). Any element of this subgroup fixes setwise \( \bar{K} \) but, by Proposition 3, this is impossible. This concludes the proof of Theorem 2.

5. Proof of Theorem 3

Let \( h \) be the covering involution of \( K \) and \( C' \) the covering group of \( K' \). We denote by \( \bar{K} \) (respectively, \( \bar{K'} \)) the fixed point set of \( h \) (respectively, \( C' \)). By the Sylow theorems we suppose that \( h \) and \( C' \) are contained in the same Sylow 2-subgroup, say \( S \), of \( \text{Iso}^+(M) \). We note that by hypothesis the quotient \( M/C' \) is the 3-sphere and the singular set, of singularity index \( 2^b \), is the knot \( K' \). We consider different possibilities.

(a) If \( h \) and \( C' \) commute elementwise, \( K \) and \( K' \) arise from the standard abelian construction.

The elements \( h \) and \( C' \) generate a group \( G \) isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_{2^b} \). Since \( h \) commutes elementwise with \( C' \), \( h \) is a \( \bar{K}' \)-rotation by Proposition 3 and it projects to an isometry \( \bar{h} \) of \( M/C' \) which acts as a rotation on \( K' \); \( \bar{h} \) has connected fixed point set by Proposition 2. The quotient \( M/G = (M/C')/\bar{h} \) is the 3-sphere by the positive solution of the Smith Conjecture for involutions and the singular set is a link with two components of singularity indices two and \( 2^b \).

(b) If \( h \) normalizes \( C' \) but \( h \) does not commute elementwise with \( C' \), \( K \) and \( K' \) arise from the standard dihedral construction I.

Since \( h \) normalizes \( C' \), \( h \) fixes setwise \( \bar{K}' \). By Proposition 3, \( h \) and \( C' \) generate a dihedral group \( D \) of order \( 2^b+1 \) and \( h \) is a \( \bar{K}' \)-reflection. We consider the projection \( \bar{h} \) of \( h \) to \( M/C' \); \( \bar{h} \) induces a reflection on \( K' \). The quotient \( M/D = (M/C')/\bar{h} \) is an orbifold whose underlying topological space is \( S^3 \) by the positive solution of the Smith Conjecture for
involutions and whose singular set is a theta-curve with three edges of singularity indices two, two and $2^b$.

(c) If $h$ does not normalize $C'$, $K$ and $K'$ arise from a $\mathbb{Z}_2 \times (\mathbb{Z}_{2^b} \times \mathbb{Z}_{2^b})$ construction.

We consider the group $C'' := h^{-1}C'h$ which is different from $C'$ by hypothesis. The fixed point set of $C''$ is $h(\tilde{K}')$; since $\tilde{K}'$ and $h(\tilde{K}')$ are distinct, $C''$ and $C'$ have trivial intersection. Since we supposed that $h$ and $C'$ are contained in the same Sylow subgroup $S$, we have that $C''$ is contained in $S$ too. We can suppose that $C''$ normalize $C'$ otherwise applying iteratively Proposition 5 as in in the proof of Theorem 2 we get a contradiction. By Proposition 3, $C$ and $C''$ commute and they generate an Abelian group $C'' \oplus C'$ isomorphic to $\mathbb{Z}_{2^b} \times \mathbb{Z}_{2^b}$. Analogously to the case (a) we obtain that $M/(C'' \oplus C')$ is an orbifold whose underlying topological space is $S^3$ and whose singular set, of singularity index $2^b$, is a link with two components. The involution $h$ normalizes $C'' \oplus C'$ and it projects to an involution $\tilde{h}$ of $M/(C'' \oplus C')$. Since $h$ conjugates $C''$ and $C'$, $\tilde{h}$ exchanges the two components of the singular set of $M/(C'' \oplus C')$. The quotient is an orbifold whose underlying topological space is $S^3$, by the positive solution of the Smith conjecture for involutions, and the singular set is a link with two components of singularity indices two and $2^b$.

We have considered all the possible cases and the proof is complete.

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References