AN INTERSECTION-UNION THEOREM FOR INTEGER SEQUENCES

Konrad ENGEL and Hans-Dietrich O.F. GRONAU


Received 3 February 1984
Revised 23 August 1984

Let $E_n^k := \{x = (x_1, \ldots, x_n) : x_i \in \{0, \ldots, k-1\}, i = 1, \ldots, n\}$ ($k \geq 2$). A subset $\mathcal{F} \subseteq E_n^k$ is called an intersection-union-family if for all $x, y \in \mathcal{F}$ there are coordinates $i$ and $j$ such that $x_i, y_j \neq 0$ and $x_i, y_j \neq k-1$. All maximum intersection-union-families and their size (exactly and asymptotically) are determined.

1. Introduction

Let $E_n^k$ be the set of all $n$-tuples $x = (x_1, \ldots, x_n)$, where $x_i \in \{0, \ldots, k-1\}$ for all $i$ ($k \geq 2$). The elements of $E_n^k$ can be considered as characteristic vectors of subsets of $N := \{1, \ldots, n\}$. In the following we identify $x \in E_n^k$ with $\{i : x_i = 1\}$.

For $k \geq 2$ we call a subset $\mathcal{F} \subseteq E_n^k$ an intersection-family, briefly int-family (a union-family, briefly un-family) if for all $x, y \in \mathcal{F}$ there is an index $i$ such that $x_i, y_i \neq 0$ (an index $j$ such that $x_i, y_j \neq k-1$). The subset $\mathcal{F} \subseteq E_n^k$ is called an intun-family if it is both an int-family and an un-family. Let

$$e_{\text{intun}}(k, n) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is a intun-family in } E_n^k\},$$

where (xxx) stands for int, un, and intun, respectively. $\mathcal{F} \subseteq E_n^k$ is called a maximum (xxx)-family if $|\mathcal{F}| = e_{\text{xxx}}(k, n)$.

It is easy to prove that $e_{\text{int}}(2, n) = e_{\text{un}}(2, n) = 2^{n-1}$ (see, for instance, [4, p. 319]). The following theorem was proved by Schönheim [8], Daykin and Lovasz [2], Seymour [9], Anderson [1], and Kleitman [6].

**Theorem 1.** $e_{\text{intun}}(2, n) = 2^{n-2}$.

A result of Erdős and Schönheim [5] reads in the special case of $E_n^k$:

**Theorem 2.**

$$e_{\text{int}}(k, n) = e_{\text{un}}(k, n) = \frac{1}{2} \sum_{i=0}^{k-1} \binom{n}{i} (k-1)^{\max(i, n-i)}.$$

It is worthwhile to note that because of this theorem the obvious lower bound of $(k-1)k^{n-1} = \{|x \in E_n^k : x_1 > 0|\}$ is not correct if $k > 2$. 

Anderson [1] inferred from the Erdős–Schönheim result [5] via an FKG-type-inequality a bound for intun-families which reads in the special case of $E^n_k$ as follows:

**Theorem 3.**

$$e_{\text{intun}}(k,n) \leq \frac{1}{4k^n} \left( \sum_{i=0}^{n} \binom{n}{i}(k-1)^{\max(i,n-i)} \right)^2.$$

In contrast with the $k=2$ case there is no equality in general. We will determine the exact value of $e_{\text{intun}}(k,n)$, list all maximum intun-families and give an asymptotic value for $e_{\text{intun}}(k,n)$ as $n \to \infty$.

2. Notations and a preliminary lemma

We define the functions $\text{rest}: E^n_k \to E^n_2$ and $\text{supp}: E^n_k \to E^n_2$ by $\text{rest}(x):=\{i: x_i = k-1\}$ and $\text{supp}(x):=\{i: x_i \neq 0\}$. For $\mathcal{F} \subseteq E^n_k$ let $\text{supp}(\mathcal{F}):=\{\text{supp}(x): x \in \mathcal{F}\}$ and let $\text{rest}(\mathcal{F})$ be defined correspondingly.

**Lemma 1.** $\mathcal{F} \subseteq E^n_k$ is an int-family (an un-family) iff $\text{supp}(\mathcal{F})$ is an int-family in $E^n_2$ (rest(\mathcal{F}) is an un-family in $E^n_2$).

The proof is trivial and can be omitted.

For $A, B \subseteq N$ we define

$$V^B_A := \{x \in E^n_k: \text{rest}(x) = A \text{ and } \text{supp}(x) = B\},$$

and if $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of $N$ let

$$V^\mathcal{B}_\mathcal{A} := \bigcup_{A \in \mathcal{A}} \bigcap_{B \in \mathcal{B}} V^B_A.$$

Obviously,

$$|V^B_A| = \begin{cases} (k-2)^{|B|-|A|} & \text{if } B \supseteq A, \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

and

$$|V^\mathcal{B}_\mathcal{A}| = \sum_{A \in \mathcal{A}} |V^B_A|. \quad (2)$$

We set

$$K_n := \begin{cases} \frac{1}{2} \binom{n}{n/2} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$
A family $\mathcal{M}$ of $n/2$-element subsets of $N$ for which $A \in \mathcal{M}$ iff $N \setminus A \notin \mathcal{M}$ is called a middle-level-without-complement-family (briefly milewic-family). If $n$ is odd then only $\emptyset$ is a milewic-family. Obviously, a milewic-family is an intun-family in $E_2^n$, and it consists of $K_n$ subsets of $N$. Each int-family (un-family) of $E_2^n$ which consists only of $n/2$-element subsets of $N$ has at most $K_n$ elements. There are $2^K$-milewic-families. Finally let

$$\mathcal{R} := \{A \subseteq N : |A| < n/2\} \quad \text{and} \quad \mathcal{G} := \{B \subseteq N : |B| > n/2\}.$$ 

3. The main results

**Theorem 4.** Let $k \geq 3$. Exactly the families $V_{\mathcal{R} \cup \mathcal{M}}$, where $\mathcal{M}$ is a milewic-family, are the maximum intun-families in $E_2^n$. In particular,

$$e_{\text{intun}}(k, n) = \sum_{i > n/2} \sum_{i < n/2} \binom{n}{i} \binom{k-2}{i} 2^{K_n} \sum_{i < n/2} \binom{n/2}{i} (k-2)^{n/2-i} + K_n.$$ 

**Proof.** Let $\mathcal{M}$ be a milewic-family. Then $\mathcal{R} \cup \mathcal{M}$ and $\mathcal{G} \cup \mathcal{M}$ are an un-family and an int-family in $E_2^n$, respectively. By Lemma 1 $V_{\mathcal{R} \cup \mathcal{M}}$ is an intun-family. Further, using (1) and (2) we obtain that the size of $V_{\mathcal{R} \cup \mathcal{M}}$ is equal to the right-hand side of the formula above.

Now assume that $\mathcal{R}$ is a maximum intun-family, but $\mathcal{R} \neq V_{\mathcal{R} \cup \mathcal{M}}$ for all milewic-families $\mathcal{M}$. We will show that there is an intun-family $\mathcal{R}'$ such that $|\mathcal{R}'| > |\mathcal{R}|$. By this contradiction the proof of our theorem will be complete. Let

$$I := \{i \in \{0, \ldots, n\} : \text{there is an } A \subseteq N \text{ such that } |A| = i \text{ and } A \notin \text{rest}(\mathcal{R})\}.$$ 

$$J := \{j \in \{0, \ldots, n\} : \text{there is a } B \subseteq N \text{ such that } |B| = j \text{ and } B \notin \text{supp}(\mathcal{R})\}.$$ 

We set $i^* := \min\{i : i \in I\}$ and $j^* := \max\{j : j \in J\}$. Obviously, we have $0 \leq i^* \leq \frac{1}{2}(n + 1)$ and $n \geq j^* \geq \frac{1}{2}(n - 1)$. (If, for instance, $i^* \geq \frac{1}{2}(n + 1)$, then rest($\mathcal{R}$) contains all $[\frac{1}{2}(n + 1)]$-element subsets of $N$, i.e. rest($\mathcal{R}$) cannot be an un-family which contradicts Lemma 1.)

**Case 1.** $i^* = 0$ or $j^* = n$. Then $\mathcal{R}' := \mathcal{R} \cup V_{\mathcal{R}}^N$ is a larger intun-family.

**Case 2.** $i^* = j^* = \frac{1}{2}n$. Then $\mathcal{R} = V_{\mathcal{R} \cup \mathcal{M}}$, where $\mathcal{A}$ and $\mathcal{B}$ are an un-family and an int-family of $n/2$-element subsets of $N$, respectively. But we have

$$|V_{\mathcal{R} \cup \mathcal{M}}| = |V_{\mathcal{R}}| + |V_{\mathcal{A}}| + |V_{\mathcal{B}}| + |V_{\mathcal{M}}| < |V_{\mathcal{R}}| + |V_{\mathcal{A}}| + |V_{\mathcal{B}}| + |V_{\mathcal{M}}| = |V_{\mathcal{R} \cup \mathcal{M}}|,$$

where $\mathcal{M}$ is an arbitrary milewic-family (note that $|\mathcal{A}|, |\mathcal{B}| \leq K_n$ and $|V_{\mathcal{R}}| = |\mathcal{A} \cap \mathcal{B}| < K_n = |V_{\mathcal{R}}|$).

**Case 3.** $i^* = \frac{1}{2}(n + 1), j^* = \frac{1}{2}(n - 1)$. But then $\mathcal{R} = V_{\mathcal{R} \cup \mathcal{M}}$, which contradicts our assumption.
Case 4. Not Cases 1, 2, or 3. Then there are three subsidiary cases:
Case 4.1. \( i^* < n - j^* \) (hence, \( 1 < i^* < \frac{1}{2}n \)).
Case 4.2. \( i^* > n - j^* \) (hence, \( n > j^* > \frac{1}{2}n \)).
Case 4.3. \( 1 < i^* = n - j^* < \frac{1}{2}n \).

Let
\[
\mathcal{A} := \{ A \subseteq N : |A| = i^* \text{ and } A \notin \text{rest}(\mathcal{F}) \},
\]
\[
\mathcal{B} := \{ B \subseteq N : |B| = j^* \text{ and } B \notin \text{supp}(\mathcal{F}) \}.
\]
By definition of \( i^* \) and \( j^* \) we have \( \mathcal{A}, \mathcal{B} \neq \emptyset \). For \( A \subseteq N \) let \( \tilde{A} := N \setminus A \). Further let
\[
\tilde{\mathcal{A}} := \{ \tilde{A} : A \in \mathcal{A} \} \text{ and } \tilde{\mathcal{B}} := \{ \tilde{B} : B \in \mathcal{B} \}.
\]
We put
\[
\tilde{\mathcal{F}}' = \begin{cases}
(\mathcal{F} \setminus \text{supp}(\mathcal{F})) \cup \text{supp}(\mathcal{F}) & \text{in Case 4.1,} \\
(\mathcal{F} \setminus \text{rest}(\mathcal{F})) \cup \text{rest}(\mathcal{F}) & \text{in Case 4.2,} \\
(\mathcal{F} \setminus (\text{supp}(\mathcal{F}) \cup \text{rest}(\mathcal{F}))) \cup (\text{supp}(\mathcal{F}) \setminus \mathcal{F}) \cup \text{rest}(\mathcal{F}) \setminus \mathcal{F} & \text{in Case 4.3.}
\end{cases}
\]

(For an illustration of the definition of \( \tilde{\mathcal{F}}' \) in Case 4.1 see Fig. 1.)

Then
\[
\text{rest}(\tilde{\mathcal{F}}') = \begin{cases}
(\text{rest}(\mathcal{F}) \setminus \tilde{\mathcal{A}}) \cup \tilde{\mathcal{A}} & \text{in Cases 4.1, 4.3,} \\
\text{rest}(\mathcal{F}) & \text{in Case 4.2}
\end{cases}
\]
and
\[
\text{supp}(\tilde{\mathcal{F}}') = \begin{cases}
\text{supp}(\mathcal{F}) & \text{in Case 4.1,} \\
(\text{supp}(\mathcal{F}) \setminus \tilde{\mathcal{B}}) \cup \tilde{\mathcal{B}} & \text{in Cases 4.2, 4.3.}
\end{cases}
\]

Now \( \text{rest}(\tilde{\mathcal{F}}') \) is an un-family in \( E_2^{n} \). To show this assume that \( A_1, A_2 \in \text{rest}(\tilde{\mathcal{F}}') \) but
An intersection-union theorem for integer sequences

157

A₁ ∪ A₂ = N. If A₁, A₂ ∈ rest(𝒜) or A₁, A₂ ∈ 𝒮 we obtain a contradiction by Lemma 1 or because of |A₁| = |A₂| = i* < ½n, respectively. If A₁ ∈ rest(𝒜) \ 𝒮 but A₂ ∈ 𝒮, then we have because of our assumption A₂ ∉ A₁, i.e. |A₁| < |A₂| = i*. By definition of i* it is A₁ ∈ rest(𝒜). But A₁ is also contained in rest(𝒜), which contradicts Lemma 1.

In the same way we can prove that supp(𝒜') is an int-family. In the following we consider only Case 4.3. The other cases can be treated analogously. It holds

|V_{supp(𝒜’)}| = |𝒜| \sum_{l=0}^{i*} \binom{n-i*}{l} (k-2)^l,

|V_{rest(𝒜’)}| = |𝒜| \sum_{l=0}^{n-j*} \binom{n-j*}{l} (k-2)^{n-j*-l}.

By definition of i*, j*, 𝒮, and 𝒮 we have

B ∈ (supp(𝒜) \ 𝒮) ∪ 𝒮 if |B| ≡ j*,

A ∈ rest(𝒜) \ 𝒮 if |A| < i*.

Thus

|V_{(supp(𝒜) \ 𝒮) ∪ 𝒮}| ≥ |𝒜| \sum_{l=0}^{i*} \binom{n-i*}{l} (k-2)^{n-2i*+l},

|V_{rest(𝒜) \ 𝒮}| ≥ |𝒜| \sum_{l=0}^{n-j*} \binom{j*}{l} (k-2)^{j*-l}.

Because of 1 ≤ i* = n - j* < ½n and (j*) ≤ (j* + 2), (j*) ≤ (j) if a > b, a, b, c ∈ N,

\binom{i*}{l}(k-2)^l ≤ \binom{n-i*}{n-2i*+l}(k-2)^{n-2i*+l} if l ∈ \{1, ..., i*\},

\binom{i*}{0}(k-2)^0 < \binom{n-i*}{n-2i*}(k-2)^{n-2i*}.

Moreover we have

\binom{n-j*}{l}(k-2)^{n-j*-l} ≤ \binom{i*}{l}(k-2)^{i*-l} if l ∈ \{0, ..., n - j* - 1\},

and here equality holds for all l only if k = 3 and j* = n - 1.

Consequently, if k > 3 or i* < n - 1, then

|𝒜'| ≥ |𝒜| + (|V_{(supp(𝒜) \ 𝒮) ∪ 𝒮}| - |V_{supp(𝒜')}|)

+ (|V_{rest(𝒜) \ 𝒮}| - |V_{rest(𝒜')}|) > |𝒜|.

If k = 3 and j* = n - 1 (i.e. i* = 1) we conclude as follows: If |𝒜| > 1, say \{1\}, \{2\} ∈ 𝒮, then 𝒮 ∪ V₃ \(1\) or 𝒮 ∪ V₃ \(2\) is a larger int-family since rest(𝒜) \{1\} or rest(𝒜) \{2\} is an un-family. If |𝒮| > 1 we argue analogously. Thus we can suppose |𝒜| = |𝒮| = 1. But then

|V_{supp(𝒜')}| ≤ 2, |V_{rest(𝒜')}| ≤ 2,

|V_{(supp(𝒜) \ 𝒮) ∪ 𝒮}| ≥ n and |V_{rest(𝒜) \ 𝒮}| ≥ n - 1,
i.e.
\[ |\mathfrak{F}'| \geq |\mathfrak{F}| - 4 + (2n - 1) > |\mathfrak{F}| \]
because \( n \geq 3 \) (in Case 4.3 we have \( 1 < \frac{3}{4}n \)).

**Theorem 5.**

\[ e_{\text{intun}}(k, n) = \begin{cases} 
   k^{n-2} & \text{if } k = 2, \\
   k^n \left(1 - O\left(\frac{1}{n}\right)\right) & \text{if } k \geq 3 \text{ and } n \to \infty.
\end{cases} \]

**Proof.** Because of Theorem 1 we can suppose \( k \geq 3 \). Obviously,

\[ e_{\text{intun}}(k, n) \leq k^n. \quad (3) \]

By Theorem 4 we have

\[ e_{\text{intun}}(k, n) \geq \sum_{|i-j(k-1)| \leq n/2k^2} \binom{n}{j} (k-2)^i \left( \sum_{|i-j(k-1)| \leq n/2k^2} \binom{j}{i} (k-2)^{-i} \right). \quad (4) \]

Now consider the identically distributed and independent random variables \( \xi_1, \ldots, \xi_j \) with

\[ P(\xi_l = 0) = \frac{k-2}{k-1}, \quad P(\xi_l = 1) = \frac{1}{k-1}, \quad l = 1, \ldots, j. \]

Further let \( \zeta_l = \xi_1 + \cdots + \xi_j. \) Obviously, \( j/(k-1) \) and \( j(k-2)/(k-1)^2 \) are the expected value and variance of \( \zeta_l \), respectively. Using Čhebyšev's inequality we obtain

\[ \left( \frac{k-2}{k-1} \right)^i \sum_{|i-j(k-1)| \leq n/2k^2} \binom{j}{i} (k-2)^{-i} = P \left( \left| \frac{\zeta_l}{k-1} \right| \geq \frac{n}{2k^2} \right) \leq \frac{j(k-2)4k^4}{(k-1)^2 n^2} = O\left(\frac{1}{n}\right), \]

uniformly relative to \( j \leq n \).

Thus

\[ \sum_{|i-j(k-1)| \leq n/2k^2} \binom{j}{i} (k-2)^{-i} = \sum_{i=0}^{j} \binom{j}{i} (k-2)^{-i} - \sum_{|i-j(k-1)| \approx n/2k^2} \binom{j}{i} (k-2)^{-i} \approx \left( \frac{k-1}{k-2} \right)^i \left(1 - O\left(\frac{1}{n}\right)\right), \quad (5) \]

uniformly relative to \( j \leq n \).

Considering random variables \( \eta_1, \ldots, \eta_n \) with

\[ P(\eta_l = 0) = \frac{1}{k}, \quad P(\eta_l = 1) = \frac{k-1}{k}, \quad l = 1, \ldots, n, \]
we obtain in the same way

\[ \sum_{|j-(k-1)n/k|<n/2k} \binom{n}{j}(k-1)^j \geq k^n \left(1 - \frac{1}{n}\right). \]  

(6)

Now the theorem can be derived from (3)–(6). \(\square\)

References