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SPACES WITH σ-POINT FINITE BASES

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Theorem. Let X be a T_1 space. The following are equivalent:
(1) X has a σ-disjoint base.
(2) X is quasi-developable and has a base that is the union of a sequence of rank 1 collections.
(3) X has a quasi-development $(\mathscr{G}_n)$ with the property that for each x, $\{st^2(x, \mathscr{G}_n) : x \in st^2(x, \mathscr{G}_n)$, n a positive integer$\}$ is a base for $\mathscr{N}(x)$.

Theorem. Let X be a T_1 space. The following are equivalent:
(1) X has a σ-point finite base.
(2) X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ well ranked.
(3) X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ Noetherian of sub-infinite rank.
(4) X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ Noetherian of point finite rank.

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σ-disjoint base	rank 1 collection
σ-point finite base	point finite rank
quasi-development	sub-infinite rank
Noetherian	screenable

1. Introduction

Aull [2] has shown that a space with a σ-point finite base is quasi-developable. Lutzer [13] proved the converse of this result for any space whose topology is generated by a linear order; however, Example 2.6 of [5] is a quasi-developable space that does not have a σ-point finite base. Bennett and Lutzer [6] characterized quasi-developability in terms of a "point finiteness" condition by showing that a space is quasi-developable if, and only if, it has a θ-base. This note investigates what kinds of quasi-developments exist for spaces having σ-point finite and σ-disjoint bases. Arhangelskii [1] has shown that every perfectly normal collectionwise normal T_1 space with a σ-point finite base is metrizable. For further information and references the reader is referred to [2] and [3]. In what follows all spaces are assumed to be T_1, and N denotes the set of all positive integers.

2. σ-disjoint bases

A topological space X is *screenable* [7] if every open cover has a σ-disjoint open refinement that covers X. A collection $\mathcal{K}$ of subsets of X is a *point-star-refinement* [11] of a collection $\mathcal{H}$ if for each $p \in X$, $\mathrm{st}(p, \mathcal{K})$ is a subset of some element of $\mathcal{H}$. A collection $\mathcal{H}$ of subsets of X is of *rank* 1 [14] provided that for any sets A, B in $\mathcal{H}$, if $A \cap B \neq \emptyset$ then either $A \subset B$ or $B \subset A$ obtains. A space X is *quasi-developable* [5] if there exists a sequence $(\mathcal{G}_n)$ of collections of open subsets of X such that for each x, $\{\mathrm{st}(x, \mathcal{G}_n) : x \in \mathrm{st}(x, \mathcal{G}_n),\ n \in \mathbb{N}\}$ is a base for $\mathcal{N}(x)$. The sequence $(\mathcal{G}_n)$ is called a *quasi-development*. The authors owe a substantial debt to Heath whose paper [11] provides the central idea of the proof of (iii) implies (i) in the following theorem.

Theorem 2.1. *Let X be a T_1-space. The following are equivalent*:

(i) *X has a σ-disjoint base.*

(ii) *X is quasi-developable and has a base that is the union of a sequence of rank 1 collections.*

(iii) *X has a quasi-development $(\mathcal{G}_n)$ with the property that for each x, $\{\mathrm{st}^2(x, \mathcal{G}_n) : x \in \mathrm{st}^2(x, \mathcal{G}_n),\ n \in \mathbb{N}\}$ is a base for $\mathcal{N}(x)$.*

Proof. (i) implies (ii). If X has a base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a disjoint collection, then Theorem 3 of [3] shows that X is quasi-developable. It is obvious that each $\mathcal{B}_n$ is a rank 1 collection.

(ii) implies (iii). Let $(\mathcal{G}_n)$ be a quasi-development for X and let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a base for X where each $\mathcal{B}_n$ is a rank 1 collection. For each $m, n \in \mathbb{N}$, set

$$\mathcal{H}_{m,n} = \{B \in \mathcal{B}_m : \text{for some } G \in \mathcal{G}_n,\ B \subset G\}.$$

Let $p \in X$ and $R \in \mathcal{N}(p)$. Then there exist $m, n \in \mathbb{N}$, $B \in \mathcal{B}_m$ and $G \in \mathcal{G}_n$ such that $p \in B \subset G \subset \mathrm{st}(p, \mathcal{G}_n) \subset R$. Let $y \in \mathrm{st}^2(p, \mathcal{H}_{m,n})$. Then there exist $B', B'' \in \mathcal{B}_m$ such that $y \in B'$, $p \in B''$ and $B' \cap B'' \neq \emptyset$. Since $\mathcal{B}_m$ is a rank 1 collection either $B' \subset B''$ or $B'' \subset B'$. If $B' \subset B''$, then $y \in B'' \subset \mathrm{st}(p, \mathcal{H}_{m,n}) \subset \mathrm{st}(p, \mathcal{G}_n) \subset R$. If $B'' \subset B'$ then $y \in B' \subset \mathrm{st}(p, \mathcal{H}_{m,n}) \subset \mathrm{st}(p, \mathcal{G}_n) \subset R$.

(iii) implies (i). We show first that if X satisfies condition (iii), then it satisfies:

$(*)$ Any collection $\mathcal{H}$ of open subsets of X has an open σ-point star refinement that covers $\bigcup \mathcal{H}$.

Let $\mathcal{H}$ be an open collection of subsets of X. Set $\mathcal{V}_n = \{\mathrm{st}(p, \mathcal{G}_n) : \text{for some } H \in \mathcal{H},\ p \in \mathrm{st}^2(p, \mathcal{G}_n) \subset H\}$. Then $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is an open σ-point star refinement covering $\bigcup \mathcal{H}$.

We complete the proof by showing that a quasi-developable space satisfying $(*)$ is hereditarily screenable. Since $(*)$ is an open hereditary condition, it suffices to show that X is screenable.

Let X be a space satisfying $(*)$ and let $\mathcal{H}$ be an open cover of X. Let $(\mathcal{G}_i)$ be a quasi-development for X such that each $G \in \mathcal{G}_i$ is contained in some element of $\mathcal{H}$. Note that there exists a sequence $\mathcal{A}_i = \{A(H, i) : H \in \mathcal{H}\}$ of collections of subsets of X such that

(A) For each i, no element of $\mathscr{G}_i$ meets two elements of $\mathscr{A}_i$

(B) For each i and each $H \in \mathscr{H}$, $\operatorname{st}(A(H,i), \mathscr{G}_i) \subset H$ and

(C) For each $x \in X$, there exists $H \in \mathscr{H}$ and an integer i such that $x \in A(H,i) \subset H$, and $x \in \operatorname{st}(x, \mathscr{G}_i)$.

In order to construct the sequence $(\mathscr{A}_i)$, well order $\mathscr{H}$. For each $H \in \mathscr{H}$, let $A(H,i) = \{x : \text{if } x \in H' \in \mathscr{H} \text{ then } H' \text{ follows } H \text{ and } x \in \operatorname{st}(x, \mathscr{G}_i) \subset H\}$. Let $\mathscr{A}_i$ be as above, then $\mathscr{A}_i$ satisfies (A), (B) and (C). For each i, let $\mathscr{D}_i = \bigcup_{n=1}^{\infty} \mathscr{D}_{in}$ be an open σ-point star refinement of $\mathscr{G}_i$. If $A(M,i) \neq A(N,i)$ then for each n, $\operatorname{st}(A(M,i), \mathscr{D}_{in}) \cap \operatorname{st}(A(N,i), D_{in}) = \emptyset$. Therefore for each i, n $\mathscr{C}_{in} = \{\operatorname{st}(A(H,i), \mathscr{D}_{in}) \colon A(H,i) \in \mathscr{A}_i\}$ is a collection of pairwise disjoint open sets. It follows from (B) and (C) above that $\bigcup_i \bigcup_n \mathscr{C}_{in}$ covers X and refines $\mathscr{H}$. This completes the proof.

The referee has noted that there is a close connection between the sequence $(\mathscr{A}_i)$ constructed above and the sequence $(\mathscr{P}_n)$ constructed in [6, Proposition 7].

It is natural to inquire whether certain weakenings of condition (ii) imply that X has a σ-disjoint base. For instance suppose X has a base $\mathscr{B} = \bigcup_n \mathscr{B}_n$ where each $\mathscr{B}_n$ has rank 1. [12, Example 5.3] shows that such a space need not be first countable, and Gruenhage [9] gives an example of a first countable space with a rank 1 base that does not have a σ-point finite base since it is not quasi-metrizable. Aull [3] gives an example of a quasi-metrizable space with a rank 1 base that does not have a σ-point finite base. Heath [11, Example 1] gives an example of a non screenable space with quasi-development $(\mathscr{G}_n)$ where each $(\mathscr{G}_n)$ is of rank 2.

3. σ-point finite bases

A collection $\mathscr{A}$ of subsets of a set X has *rank n at $x \in X$*, denoted by $r_x(\mathscr{A}) = n$ if every collection of $n+1$ elements of $\mathscr{A}$ each containing x has a pair related under inclusion, and $\mathscr{A}$ contains an incomparable subcollection of n members, each of which contains x. $\mathscr{A}$ is of *rank n* if $\max\{r_x(\mathscr{A}) \colon x \in X\} = n$. $\mathscr{A}$ is of *point finite rank* if for each $x \in X$, $r_x(\mathscr{A})$ is finite. $\mathscr{A}$ is of *sub-infinite rank* at x if every collection of incomparable members of $\mathscr{A}$ containing x is finite. $\mathscr{A}$ is *Noetherian* if every nonempty subcollection has a maximal element (relative to set inclusion). $\mathscr{A}$ is *well-ranked* if $\mathscr{A} = \bigcup \mathscr{A}_n$ where each $\mathscr{A}_n$ is a Noetherian collection of sub-infinite rank. For a detailed discussion of these concepts, see [10] and [12].

Theorem 3.1. *Let X be a T_1 space. The following are equivalent:*

(i) *X has a σ-point finite base.*

(ii) *X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ well-ranked.*

(iii) *X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ Noetherian of sub-infinite rank.*

(iv) *X has a quasi-development $(\mathscr{G}_n)$ with each $\mathscr{G}_n$ Noetherian of point finite rank.*

Proof. (iv) implies (iii) and (iii) implies (ii) are obvious.

(ii) implies (i). It follows from the proof of [12, Theorem 3.3] that X is hereditarily σ-metacompact. (Every open cover has an open σ-point finite refinement.) Clearly a hereditarily σ-metacompact quasi-developable space has a σ-point finite base.

(i) implies (iv). Since a σ-point finite base for X is a θ-base, X has a quasi-development $(\mathcal{G}_n)$ such that each $(\mathcal{G}_n)$ is point finite. This completes the proof.

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