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Oscillation of Even Order Delay Differential Equations*

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Some oscillation criteria for even order functional differential equations $x^{(n)}(t) + q(t)f(x(t), x(g_1(t)),..., x(g_m(t))) = 0$ and $x^{(n)}(t) + p(t)x^{(n-1)}(t) + q(t)f(x(g_1(t)),..., x(g_m(t))) = 0$ are given. These criteria are an extension of some recent results established by S. R. Grace and B. S. Lalli. © 1987 Academic Press, Inc.

Recently, Grace and Lalli [1] considered

$$x^{(n)}(t) + q(t)f(x(t), x(g(t))) = 0, \tag{1}$$

where n is an even positive integer. They established an oscillation criterion for (1). In this note we extend Grace and Lalli's work to the nth-order equations

$$x^{(n)}(t) + q(t)f(x(t), x(g_1(t)),..., x(g_m(t))) = 0 \tag{2a}$$

and obtain results that are an improvement over their work. By applying the same method to

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) + q(t)f(x(g_1(t)),..., x(g_n(t))) = 0, \tag{2b}$$

we obtain a result that is better than results in presented recently in [6].

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In what follows we restrict our discussion to nontrivial solutions of (1) which are indefinitely continuable to the right. A solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros, and nonoscillatory if it is eventually of constant sign. Equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

The proof of our main result, Theorem 1, is based on the proof of Theorem 1 in [1].

THEOREM 1. *Assume that*

(i) $q, g_i \in C[t_0, +\infty)$, $i = 1, 2, \ldots, m$, $f \in C(R^{m+1})$, and $y_i f(y_1, \ldots, y_{m+1}) > 0$ *when* $y_i \neq 0$ *has the same sign for* $i = 1, 2, \ldots, m+1$;

(ii) *there exists a continuously differentiable function* $\sigma \in C[t_0, +\infty)$ *such that* $0 < \sigma(t) \leqslant g_i(t)$, $1 \geqslant \dot{\sigma}(t) > 0$ *for* $t \geqslant T$ *and* $\sigma(t) \to \infty$ *as* $t \to \infty$;

(iii) *there exists a positive number* $c > 0$ *such that for every increasing function* $|y(t)|$

$$\liminf_{|y| \to \infty} \left| \frac{f(y_1, \ldots, y_{m+1})}{y} \right| \geqslant c > 0,$$

where $|y_i| \geqslant |y|$, $i = 1, 2, \ldots, m+1$;

(iv) $q(t) \geqslant 0$ *and* $q(t)$ *is not eventually identically equal to zero on any subinterval* (t_1, ∞);

(v) *there exists a positive continuously differentiable function* $\rho(t)$ *on* $[t_0, +\infty)$ *such that*

$$\lim_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{(t-s)^{m-3}(\dot{\rho}(s)(t-s) - (m-1)\,\rho(s))^2}{\dot{\sigma}(s)\,\sigma^{n-2}(s)\,\rho(s)}\,ds < \infty \tag{3}$$

$$\lim_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds = +\infty, \tag{4}$$

where $m > 2$ *is some integer number. Then every solution of* (2a) *oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (2a). Assume that $x(t) > 0$ for $t \geqslant t_0$ and choose a $t_1 \geqslant t_0$ so that $g_i(t) \geqslant t_0$ for $t \geqslant t_1$, $i = 1, 2, \ldots, m$. By Lemma 1 of [1], there exists a $t_2 \geqslant t_1$ such that $x^{(n-1)}(t) > 0$ and $\dot{x}(t) > 0$ for $t \geqslant t_2$. Choose a $t_3 \geqslant t_2$ so that $\sigma(t) \geqslant 2t_2$ for $t > t_3$. We apply Lemma 2 of [1] for $u = \dot{x}$, $\lambda = \frac{1}{2}$ and conclude that there exist $M_1 > 0$ and $t_4 \geqslant t_3$ such that

$$\dot{x}[\tfrac{1}{2}\sigma(t)] \geqslant M_1 \sigma^{n-2}(t)\, x^{(n-1)}[\sigma(t)] \geqslant M_1 \sigma^{n-2}(t)\, x^{(n-1)}(t), \qquad t \geqslant t_4.$$

Let $w(t) = x^{(n-1)}(t)/x[\frac{1}{2}\sigma(t)]$. Thus $w(t)$ satisfies

$$\dot{w}(t) = -q(t)\frac{f(x(t), x(g_1(t)),..., x(g_m(t)))}{x[\frac{1}{2}\sigma(t)]} - \frac{1}{2}\dot{\sigma}(t)\, w(t)\frac{\dot{x}(\frac{1}{2}\sigma(t))}{x(\frac{1}{2}\sigma(t))}.$$

Since $\dot{x}(t) > 0$ for $t \geqslant t_4$, $\lim_{t\to\infty} x(t)$ exists either as a finite or an infinite limit.

If $\lim_{t\to\infty} x(t) = b$ is finite, then

$$\lim_{t\to\infty}\frac{f(x(t), x(g_1(t)),..., x(g_m(t)))}{x(\frac{1}{2}\sigma(t))} = \frac{f(b, b,..., b)}{b} > 0$$

If $\lim_{t\to\infty} x(t) = +\infty$, then

$$\frac{f(x(t), x(g_1(t)),..., x(g_m(t)))}{x(\frac{1}{2}\sigma(t))} \geqslant \frac{c}{2} > 0$$

for sufficiently large t. On the other hand,

$$\frac{1}{2}\dot{\sigma}(t)\, w(t)\frac{\dot{x}(\frac{1}{2}\sigma(t))}{x(\frac{1}{2}\sigma(t))} \geqslant \frac{1}{2}\dot{\sigma}(t)\, w(t)\left(M_1\sigma^{n-2}(t)\frac{x^{(n-1)}(t)}{x(\frac{1}{2}\sigma(t))}\right) = \frac{M_1}{2}\dot{\sigma}(t)\,\sigma^{n-2}(t)\, w^2(t).$$

So, we have

$$\dot{w}(t) \leqslant -\frac{c_0}{2}q(t) - \frac{M_1}{2}\dot{\sigma}(t)\,\sigma^{n-2}(t)\, w^2(t),$$

where $c_0 = \min(c, f(b,..., b)/b)$.

Therefore,

$$\frac{c_0}{2}q(t)\,\rho(t) \leqslant -\rho(t)\,\dot{w}(t) - \frac{M_1}{2}\dot{\sigma}(t)\,\sigma^{n-2}(t)\,\rho(t)\, w^2(t), \qquad t \geqslant t_4.$$

Or

$$\frac{c_0}{2}\int_{t_4}^{t}(t-s)^{m-1}\rho(s)\, q(s)\, ds \leqslant -\int_{t_4}^{t}(t-s)^{m-1}\rho(s)\,\dot{w}(s)\, ds - \frac{M_1}{2}\int_{t_4}^{t}(t-s)^{m-1}\dot{\sigma}(s)\,\sigma^{n-2}(s)\,\rho(s)\, w^2(s)\, ds$$

$$= (t-t_4)^{m-1}\rho(t_4)\, w(t_4)$$

$$+ \int_{t_4}^{t} (t-s)^{m-2} w(s)(\dot{\rho}(s)(t-s) - (m-1)\,\rho(s))\, ds$$

$$- \frac{M_1}{2} \int_{t_4}^{t} (t-s)^{m-1} \dot{\sigma}(s)\, \sigma^{n-2}(s)\, \rho(s)\, w^2(s)\, ds$$

$$= (t-t_4)^{m-1}\rho(t_4)\, w(t_4)$$

$$- \int_{t_4}^{t} \left(\frac{\sqrt{M_1}}{2} (t-s)^{(m-1)/2} \sqrt{\dot{\sigma}(s)\, \sigma^{n-2}(s) \rho(s)\, w(s)} \right.$$

$$\left. - \frac{1}{2} \frac{\sqrt{2}}{M_1} \frac{(t-s)^{(m-3)/2}}{\sqrt{\dot{\sigma}(s)\, \sigma^{n-2}(s)\, \rho(s)}} (\dot{\rho}(s)(t-s) - (m-1)\,\rho(s)) \right)^2 ds$$

$$+ \frac{1}{2M_1} \int_{t_4}^{t} \frac{(t-s)^{m-3}}{\dot{\sigma}(s)\, \sigma^{n-2}(s)\, \rho(s)} (\dot{\rho}(s)(t-s) - (m-1)\,\rho(s))^2\, ds$$

$$\leqslant (t-t_4)^{m-1}\rho(t_4)\, w(t_4)$$

$$+ \frac{1}{2M_1} \int_{t_4}^{t} \frac{(t-s)^{m-3}}{\dot{\sigma}(s)\, \sigma^{n-2}(s)\, \rho(s)} (\dot{\rho}(s)(t-s) - (m-1)\,\rho(s))^2\, ds.$$

But, for every $t \geqslant t_4$,

$$\int_{t_0}^{t} (t-s)^{m-1}\rho(s)\, q(s)\, ds - \int_{t_4}^{t} (t-s)^{m-1}\rho(s)\, q(s)\, ds$$

$$= \int_{t_0}^{t_4} (t-s)^{m-1}\rho(s)\, q(s)\, ds \leqslant (t-t_0)^{m-1} \int_{0}^{t_4} \rho(s)\, q(s)\, ds$$

so

$$\frac{c_0}{2t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1}\rho(s)\, q(s)\, ds$$

$$\leqslant \frac{c_0}{2}\left(1-\frac{t_0}{t}\right)^{m-1} \int_{t_0}^{t_4} \rho(s)\, q(s)\, ds + \left(1-\frac{t_4}{t}\right)^{m-1} \cdot \rho(t_4)\, w(t_4)$$

$$+ \frac{1}{2M_1 t^{m-1}} \int_{t_0}^{t} \frac{(t-s)^{m-3}(\dot{\rho}(s)(t-s) - (m-1)\,\rho(s))^2}{\dot{\sigma}(s)\, \sigma^{n-2}(s)\, \rho(s)}\, ds$$

for all $t \geqslant t_4$. This gives

$$\lim_{t\to\infty} \frac{c_0}{2t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds$$

$$\leqslant \rho(t_4)\, w(t_4) + \frac{c_0}{2} \int_{t_0}^{t_4} \rho(s)\, q(s)\, ds$$

$$+ \lim_{t\to\infty} \frac{1}{2M_1 t^{m-1}} \int_{t_0}^{t} \frac{(t-s)^{m-3}(\dot\rho(s)(t-s)-(m-1)\,\rho(s))^2}{\dot\sigma(s)\,\sigma^{n-2}(s)\,\rho(s)}\, ds, \qquad (5)$$

which contradicts condition (4).

A similar proof holds if $x(t) < 0$, for $t \geqslant t_0$.

COROLLARY 1. *Under the conditions of Theorem* 1 *and the assumptions that* $1 \geqslant \dot\sigma(t) \geqslant k > 0$ *and that there exists an* $\alpha \in [0, n-1)$ *such that*

$$\lim_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} s^{\alpha} q(s)\, ds = \infty, \qquad (6)$$

every solution of (2a) *oscillates.*

Proof. We choose $\rho(s) = s^{\alpha}$. By direct computations one can show that for $\alpha \in [0, n-1)$, $\rho(t)$ satisfies condition (3). By application of Theorem 1 we obtain the conclusion of Corollary 1.

Remark 1. Let $\rho(t) \equiv 1$, $1 \geqslant \dot\sigma(t) \geqslant k > 0$. Then Corollary 1 leads to Grace and Lalli's Theorem 1 in [1]. This shows that conditions (3) and (4) are more widely applicable than the existing ones.

EXAMPLE 1. Consider the equation

$$x^{(n)}(t) + f(x(\ln t)) = 0, \qquad n \text{ even}; \quad t > 1, \qquad (7)$$

where

$$f(x) = \begin{cases} x \exp[(x(1+\sin x)], & \text{for } x \geqslant 0 \\ x, & \text{for } x \leqslant 0 \end{cases} \qquad (8)$$

$$q(t) = 1, \qquad \sigma(t) = \ln t, \qquad \dot\sigma(t) = 1/t.$$

The theorem of Grace and Lalli is not valid for Eq. (7). But Theorem 1 is valid for (7). In fact, $\rho(t) = t^{-\alpha}$, $0 < \alpha < 1$, satisfies conditions (3) and (4), so all solutions of (7) are oscillatory according to Theorem 1.

EXAMPLE 2. Consider the equation

$$x^{(n)}(t) + t^{-\alpha} f\left(x\left[\frac{t}{2}\right]\right) = 0, \qquad n \text{ even}, t > 0, 1 < \alpha < n-1. \tag{9}$$

$f(x)$ is defined in (8).

It is easy to see that the oscillation theorem of paper [1] cannot be applied to Eq. (9), but if we take $\rho(t) = t^{\alpha}$, (9) satisfies the condition of Corollary 1. Therefore, all solutions of (9) are oscillatory.

Under the modification of the hypotheses of Theorem 1, we can obtain the following result:

THEOREM 2. *In Theorem* 1, *condition* (v) *is replaced by the following condition*:

(v)′ *There exists a positive continuously differentiable function* $\rho(t)$ *on* $[t_0, +\infty)$ *such that*

$$\int^{\infty} \rho(t)\, q(t)\, dt = \infty \tag{10}$$

and

$$\int^{\infty} \frac{\dot{\rho}^2(s)}{\rho(s)\, \dot{\sigma}(s)\, \sigma^{n-2}(s)}\, ds < \infty. \tag{11}$$

Then the conclusion of Theorem 1 *is true.*

COROLLARY 2. *Under the conditions of Theorem* 1 *and the assumption that* $1 \geqslant \dot{\sigma}(t) \geqslant k > 0$, *if*

$$\int^{\infty} t^{\alpha} q(t)\, dt = \infty, \qquad \alpha \in [0, n-1), \tag{12}$$

then every solution of (1) *oscillates.*

Remark 2. Theorem 2 includes some results of paper [5].

EXAMPLE 3. We consider

$$y^{(4)}(t) + t^{-4} \ln t\, y^{1/3}\left(\frac{t}{2}\right) y^{2/3}\left(\frac{t}{3}\right) = 0. \tag{13}$$

It does not satisfy condition (12), but we can choose $\rho(t) = t^3/(\ln t)^2$, which satisfies conditions (10) and (11). Therefore all solutions of (13) oscillate.

The results of paper [5] are not applicable for (13).

Remark 3. Let $n=2, f(x, y)=F(x)$, $xF(x)>0$, $F'(x)\geqslant k>0$. Then $q(t)$ need not be a positive function to ensure the oscillation of (2). In that case Theorem 1 includes Philos' theorem [2] as a special case. Theorem 1 also includes Theorem 1 in [3] as a particular case.

Now we apply the above method to a more general equation,

$$x^{(n)}(t)+p(t)\,x^{(n-1)}(t)+q(t)\,f(x(g_1(t)),..., x(g_m(t)))=0, \tag{14}$$

where n is even, $f\in C(R^m)$, g_i, p, $q\in C(R_+, R)$, and $i=1, 2,..., m$ and such that

(i) $p(t)\geqslant 0$, $q(t)\geqslant 0$ on R_+ and $q(t)\not\equiv 0$ on any ray$[T, \infty)$;

(ii) if $y_i\leqslant z_i$, $i=1, 2,..., m$, then $f(y_1, y_2,..., y_m)\leqslant f(z_1, z_2,..., z_m)$ and

$$f(y_1, y_2,..., y_m)>0 \qquad \text{if } y_i>0 \text{ for all } i,$$

$$f(y_1, y_2,..., y_m)<0 \qquad \text{if } y_i<0 \text{ for all } i;$$

(iii) $g_i(t)\to\infty$, $i=1, 2,..., m$, and there exists $\sigma_i\in C^1[R_+, R_+]$, $i=1, 2,..., m$, such that

$$0<\sigma_i(t)=\inf_{s\geqslant t}\min\{s, g_i(s)\},$$

$$\dot{\sigma}_i(t)>0, \qquad \sigma_i(t)\to\infty \qquad \text{as} \quad t\to\infty;$$

(iv) $\partial f/\partial y_i$ exist and $\partial f/\partial y_i\geqslant\alpha_i>0$ for $y_i\neq 0$, $i=1, 2,..., m$;

(v) $\lim_{t\to\infty}\int_a^t \exp(-\int_a^s p(\tau)\,d\tau)\,ds=\infty$ for any $a\geqslant t_0$;

(vi) there exists a positive continuously differentiable function $\rho(t)$ on R_+ such that

$$\lim_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^t (t-s)^{m-1}\rho(s)\,q(s)\,ds=\infty$$

and

$$\lim_{t\to\infty}\frac{1}{t^{m-1}}\int_{t_0}^t (t-s)^{m-3}\rho(s)^{-1}\left(\sum_{i=1}^m \alpha_i\sigma_i^{n-2}(s)\,\dot{\sigma}_i(s)\right)^{-1}$$
$$\times\{(\dot{\rho}(s)-\rho(s)\,p(s))(t-s)-(m-1)\,\rho(s)\}^2\,ds<\infty$$

for some integer $m\geqslant 3$.

THEOREM 3. *Under conditions* (i)–(vi) *every solution of* (14) *oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (14), and without loss of generality, assume that $x(t)>0$ for $t \geqslant t_0$. Then $t_1 \geqslant t_0$ exists so that $x[\sigma_i(t)]>0$ for $t \geqslant t_i$, $i=1, 2,..., m$.

Define

$$w(t)=\frac{x^{(n-1)}(t)}{f(x(\frac{1}{2}\sigma_i(t)),..., x(\frac{1}{2}\sigma_m(t)))}$$

according to the procedure of paper [6]; we have

$$\dot{w}(t) \leqslant -q(t)-p(t)\,w(t)-\frac{2^{1-2n}}{(n-1)!}\,w^2(t)\sum_{i=1}^{m}\alpha_i\sigma_i^{n-2}(t)\,\dot{\sigma}_i(t)$$

so

$$\rho(t)\,q(t) \leqslant -\rho(t)\,\dot{w}(t)-\rho(t)\,p(t)\,w(t) -\frac{2^{1-2n}}{(n-1)!}\,\rho(t)\,w^2(t)\sum_{i=1}^{m}\alpha_i\sigma_i^{n-2}(t)\,\dot{\sigma}_i(t). \tag{15}$$

Integrating (15), we get

$$\begin{aligned}
&\int_{t_5}^{t}(t-s)^{m-1}\rho(s)\,q(s)\,ds\\
&\quad\leqslant -\int_{t_5}^{t}(t-s)^{m-1}\rho(s)\,\dot{w}(s)\,ds-\int_{t_5}^{t}(t-s)^{m-1}\rho(s)\,p(s)\,w(s)\,ds\\
&\qquad-\frac{2^{1-2n}}{(n-1)!}\int_{t_5}^{t}(t-s)^{m-1}\rho(s)\,w^2(s)\left(\sum_{i=1}^{m}\alpha_i\sigma_i^{n-2}(s)\,\dot{\sigma}_i(s)\right)ds\\
&\quad=(t-t_5)^{m-1}\rho(t_5)\,w(t_5)\\
&\qquad+\int_{t_5}^{t}(t-s)^{m-2}w(s)(\dot{\rho}(s)(t-s)-(m-1)\,\rho(s))\,ds\\
&\qquad-\int_{t_5}^{t}(t-s)^{m-1}\,\rho(s)\,p(s)\,w(s)\,ds-\frac{2^{1-2n}}{(n-1)!}\int_{t_5}^{t}(t-s)^{m-1}\,\rho(s)\,w^2(s)\\
&\qquad\times\left(\sum_{i=1}^{m}\alpha_i\sigma_i^{n-2}(s)\,\dot{\sigma}_i(s)\right)ds=(t-t_5)^{m-1}\rho(t_5)\,w(t_5)\\
&\qquad-\int_{t_5}^{t}\left\{w(s)\sqrt{\frac{2^{1-2n}}{(n-1)!}}\,(t-s)^{(m-1)/2}\rho(s)^{1/2}\left(\sum_{i=1}^{m}\alpha_i\,\sigma_i^{n-2}(s)\,\dot{\sigma}_i(s)\right)^{1/2}\right.\\
&\qquad-\sqrt{\frac{(n-1)!}{2^{3-2n}}}\,(t-s)^{m-s)/2}\rho(s)^{-1/2}\left(\sum_{i=1}^{m}\alpha_i\sigma_i^{n-2}(s)\,\dot{\sigma}_i(s)\right)^{-1/2}
\end{aligned}$$

$$\times ((\dot{\rho}(s) - p(s)\,\rho(s))(t-s) - (m-1)\,\rho(s))\Big\}^2\, ds$$

$$+ \frac{(n-1)!}{2^{3-2n}} \int_{t_5}^{t} (t-s)^{m-3} \rho(s)^{-1}$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s) \right)^{-1} ((\dot{\rho}(s) - \rho(s)\, p(s))(t-s) - (m-1)\,\rho(s))^2\, ds$$

$$\leqslant (t-t_5)^{m-1} \rho(t_5)\, w(t_5) + \frac{(n-1)!}{2^{3-2n}} \int_{t_5}^{t} (t-s)^{m-3} \rho(s)^{-1}$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s) \right)^{-1} ((\dot{\rho}(s) - \rho(s)\, p(s))(t-s) - (m-1)\,\rho(s))^2\, ds,$$

where $t_5 \geqslant t_0$ is some constant. We have

$$\int_{t_0}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds - \int_{t_5}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds$$

$$= \int_{t_0}^{t_5} (t-s)^{m-1} \rho(s)\, q(s)\, ds \leqslant (t-t_0)^{m-1} \int_{t_0}^{t_5} \rho(s)\, q(s)\, ds$$

so

$$\frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds$$

$$\leqslant \left(1 - \frac{t_0}{t}\right)^{m-1} \int_{t_0}^{t_5} \rho(s)\, q(s)\, ds + \left(1 - \frac{t_5}{t}\right)^{m-1} \rho(t_5)\, w(t_5)$$

$$+ \frac{(n-1)!}{2^{3-2n}} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)^{-1} \times \left(\sum_{i=1}^{m} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s) \right)^{-1}$$

$$\times \{(\dot{\rho}(s) - \rho(s)\, p(s))(t-s) - (m-1)\,\rho(s)\}^2\, ds.$$

Therefore

$$\lim_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)\, q(s)\, ds$$

$$\leqslant C + \lim_{t \to \infty} \frac{(n-1)!}{2^{3-2n}} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \rho(s)^{-1}$$

$$\times \left(\sum_{i=1}^{m} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s) \right)^{-1}$$

$$\times \{(\dot{\rho}(s) - \rho(s)\, p(s))(t-s) - (m-1)\,\rho(s)\}^2\, ds.$$

This is a contradiction. The proof is complete.

If we integrate (15) directly, we have the following conclusion.

THEOREM 4. *In Theorem* 3 *condition* (vi) *is replaced by conditions*

$$\int^{\infty} \rho(s)\, q(s)\, ds = \infty \tag{16}$$

$$\int^{\infty} \frac{(\dot{\rho}(s) - \rho(s)\, p(s))^2}{\rho(s)(\sum_{i=1}^{m} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s))}\, ds < \infty. \tag{17}$$

Then every solution of (14) *oscillates.*

Remark 4. If $\rho(t) \equiv t$, then Theorem 3 becomes Theorem 1 of paper [6]. The other results of paper [6] can be improved with the above method also.

The following example shows that Theorem 3 is an improvement of Theorem 1 in [6].

EXAMPLE 4. We consider

$$y^{(4)}(t) + t^{-1} y^{(3)}(t) + t^{-4} \ln t \left(\sinh y\left(\frac{t}{3}\right) + y\left(\frac{t}{2}\right) \exp\left(y^2\left(\frac{t}{2}\right)\right)\right) = 0, \qquad t > 1, \tag{18}$$

where $p(t) = t^{-1}$, $q(t) = t^{-4} \ln t$.

It is easy to see that (18) does not satisfy condition (8) of Theorem 1 in [6]. We choose $\rho(t) = t^3/(\ln t)^2$; then

$$\int_{t_0}^{t} \rho(s)\, q(s)\, ds = \int_{t_0}^{t} \frac{1}{s(\ln s)}\, ds \to \infty \qquad \text{as} \quad t \to \infty.$$

On the other hand,

$$\int_{t_0}^{t} \frac{(\dot{\rho}(s) - \rho(s)\, p(s))^2}{\rho(s)(\sum_{i=1}^{2} \alpha_i \sigma_i^{n-2}(s)\, \dot{\sigma}_i(s))}\, ds \leqslant C_1 \int_{t_0}^{t} \frac{d(\ln s)}{(\ln s)^2} < \infty,$$

where C_1 is some positive number. That is, (18) satisfies the conditions of Theorem 4. Therefore, every solution of (18) oscillates.

REFERENCES

1. S. R. GRACE AND B. S. LALLI, An oscillation criterion for *n*th order non-linear differential equations with functional arguments, *Bull. Canad. Math.* **26**, No. 1, (1983), 35–40.

2. Ch. G. Philos, "Oscillation of second order linear ordinary differential equations in alternating coefficients, *Bull. Austral. Math. Soc.* **27**, No. 2 (1983), 307–314.
3. C.-C. Yeh, An oscillation criterion for second order nonlinear differential equations with functional arguments, *J. Math. Anal. Appl.* **76** (1980), 72–76.
4. M. K. Grammatikopoulos, Y. G. Sficas, and V. A. Staikos, Oscillatory properties of strongly superlinear differential equations with deviating arguments, *J. Math. Anal. Appl.* **67** (1979), 171–187.
5. S. R. Grace and B. S. Lalli, Oscillation theorems for *n*th-order delay differential equations, *J. Math. Anal. Appl.* **91** (1983), 352–366.
6. S. R. Grace and B. S. Lalli, Oscillation theorems for damped differential equations of even order with deviating arguments, *SIAM Math. Anal.* **2** (1984), 308–316.