Any Diophantine quintuple contains a regular Diophantine quadruple

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1. Introduction

Diophantus raised the problem of finding a set of four (rational) numbers which has the property that the product of any two numbers in the set increased by one is a square, and found such a set \{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\} of four positive rational numbers. Fermat first found a set of four positive integers with the above property, which was \{1, 3, 8, 120\}. A set \{a_1, \ldots, a_m\} of \(m\) distinct positive integers is called a Diophantine \(m\)-tuple if \(a_ia_j + 1\) is a perfect square for all \(i, j\) with \(1 \leq i < j \leq m\). It is conjectured that if \{a, b, c, d\} is a Diophantine quadruple with \(a < b < c < d\), then \(d = d_+\), where \(d_+ = a + b + c + 2abc + 2rst\) and \(r = \sqrt{ab + 1}, s = \sqrt{ac + 1}, t = \sqrt{bc + 1}\). In this paper, we show that if \(\{a, b, c, d, e\}\) is a Diophantine quintuple with \(a < b < c < d < e\), then \(d = d_+\).
and thirdly, Dujella [6] showed that if \( \{F_{2k}, F_{2k+2}, F_{2k+4}\} \), where \( k \geq 1 \) and \( F_v \) denotes the \( v \)th Fibonacci number, is a Diophantine quadruple, then \( d = 4F_{2k+1}F_{2k+2}F_{2k+3} \). Furthermore, the first two results have been generalized as follows: if \( \{k-1, k+1, c, d\} \) is a Diophantine quadruple, then \( d = c_{v-1} \) or \( c_{v+1} \), where

\[
c = c_v = \frac{1}{2(k^2 - 1)} \left( (k + \sqrt{k^2 - 1})^{2v+1} + (k - \sqrt{k^2 - 1})^{2v+1} - 2k \right) \quad (v = 1, 2, \ldots)
\]

(cf. [10] and [4]). In general, Dujella [8] showed that there does not exist a Diophantine sextuple and that there exist only finitely many Diophantine quintuples. At this stage, no example of a Diophantine quintuple is known.

The following is stronger than the Diophantine quintuple conjecture.

**Conjecture 1.** (Cf. [1].) If \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d \), then \( d = d_+ \), where

\[
d_+ = a + b + c + 2abc + 2rst \quad \text{and} \quad r = \sqrt{ab + 1}, \quad s = \sqrt{ac + 1}, \quad t = \sqrt{bc + 1}.
\]

We say that a Diophantine quadruple \( \{a, b, c, d\} \) with \( a < b < c < d \) is regular if \( d = d_+ \). The above-mentioned results mean that the Diophantine quadruples containing the pairs \( \{k-1, k+1\} \) or the triples \( \{F_{2k}, F_{2k+2}, F_{2k+4}\} \) are regular. Besides them, it has been known that the Diophantine quadruples containing the following triples are regular:

\[
\{1, 8, 15\}, \quad \{1, 8, 120\}, \quad \{1, 15, 24\}, \quad \{1, 24, 35\}, \quad \{2, 12, 24\} \quad \text{(by Kedlaya [13]),}
\]
\[
\{4, 12, 30\} \quad \text{(by Dujella [8, p. 213]).}
\]

Thus, one may easily check that if \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d_+ < d \), then

\[
b > 8, \quad c > 33, \quad ac > 48 \quad (= 1 \cdot 48), \quad bc > 528 \quad (= 16 \cdot 33).
\]

These lower bounds take us in the situation where we can prove our main theorem.

Suppose that \( \{a, b, c, d, e\} \) is a Diophantine quintuple with \( a < b < c < d < e \). Since \( d_+ \) is the smallest among the \( d \)'s such that \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d \) (cf. [8, Proof of Lemma 6]), all the quadruples contained in the quintuple, other than \( \{a, b, c, d\} \), are irregular. We assert that the quadruple \( \{a, b, c, d\} \) is always regular.

**Theorem 2.** If \( \{a, b, c, d, e\} \) is a Diophantine quintuple with \( a < b < c < d < e \), then \( d = d_+ \).

Theorem 2 immediately implies the following, the latter of which is the above-mentioned theorem of Dujella.

**Corollary 3.** An irregular Diophantine quadruple \( \{a, b, c, d\} \) cannot be extended to a Diophantine quintuple \( \{a, b, c, d, e\} \) with \( e > \max(a, b, c, d) \).

**Corollary 4.** (See [8, Theorem 2].) There does not exist a Diophantine sextuple.

The proof of Theorem 2 is reduced to examining irregular Diophantine quadruples in detail. Suppose that \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d_+ < d \). One may transform this assumption into the condition that a system of Diophantine equations has "non-trivial" solutions. Some congruence relations then give lower bounds for the solutions. Using Baker’s method or Padé approximation method, one may get upper bounds for the solutions, which together with the lower bounds yield upper bounds for \( d \). While Baker’s method is applicable to any triple \( \{a, b, c\} \) (cf. [4,6,9]),
Padé approximation method (a theorem of Bennett [3, Theorem 3.2] or of Rickert [16, Theorem]) is applicable only to certain triples \( \{a, b, c\} \) (with \( ac > b^8 \) or \( b = a + 2, \ c = 4(a + 1) \)); however, if applicable, it gives much better bounds for \( d \) than Baker's method (cf. [5,7], [8, Section 10], [10]).

Our strategies are the following. First, we examine the possibilities for small solutions more precisely, and obtain better gap principles for irregular Diophantine quadruples \( \{a, b, c, d\} \) (Proposition 16) than Proposition 1 in [8]. Secondly, we divide the cases \( b \geq 2a \) and \( b < 2a \) so that we obtain a better gap principle (Proposition 16) in the case of \( b < 2a \) (unless \( c = a + b + 2r \)), and better lower bounds for solutions (Lemma 20) in the case of \( b \geq 2a \). Thirdly, we slightly modify the theorem of Bennett (ibid.) so that the assumption is satisfied if \( “c > b^7” \) (Theorem 21), which together with the gap principles allows us to apply this theorem to the quadruple \( \{b, c, d, e\} \) in the case of \( b < 2a \) and \( c = a + b + 2r \) and to the quadruple \( \{a, b, d, e\} \) in the other cases, given a Diophantine quintuple \( \{a, b, c, d, e\} \) with \( a < b < c < d_+ < d < e \) (cf. Proof of Theorem 2). Thus, we finally arrive at a contradiction in any case.

2. Gap principles

In this section, we give gap principles between \( b \) and \( d \) for irregular Diophantine quadruples \( \{a, b, c, d\} \) with \( a < b < c < d \). We first rephrase the problem in terms of a system of Diophantine equations, which induces recurrent sequences. Then, by examining lower terms of the sequences in detail, we obtain the desired gap principles.

Let \( \{a, b, c\} \) be a Diophantine triple with \( a < b < c \) such that \( ab + 1 = r^2, \ ac + 1 = s^2, \ bc + 1 = t^2 \), where \( r, s, t \) are positive integers. Assume that \( \{a, b, c, d\} \) is a Diophantine quadruple. Then there exist integers \( x, y, z \) such that \( ad + 1 = x^2, \ bd + 1 = y^2, \ cd + 1 = z^2 \), from which eliminating \( d \), we obtain the system of Diophantine equations

\[
\begin{align*}
ax^2 - cx^2 &= a - c, \\
by^2 - cy^2 &= b - c.
\end{align*}
\]

**Lemma 5.** (Cf. [8, Lemma 1]) Let \((z, x), (z, y)\) be positive solutions of (1), (2), respectively. Then there exist solutions \((z_0, x_0)\) of (1) and \((z_1, y_1)\) of (2) in the ranges

\[
1 \leq x_0 < \sqrt{\frac{s + 1}{2}} < 0.76\sqrt[4]{ac}, \quad (3)
\]

\[
1 \leq |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < 0.269c, \quad (4)
\]

\[
1 \leq y_1 < \sqrt{\frac{t + 1}{2}} < 0.723\sqrt[4]{bc}, \quad (5)
\]

\[
1 \leq |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < 0.148c \quad (6)
\]

such that

\[
\begin{align*}
z\sqrt{a} + x\sqrt{c} &= (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m, \quad (7)
z\sqrt{b} + y\sqrt{c} &= (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n \quad (8)
\end{align*}
\]

for some integers \( m, n \geq 0 \).
Proof. This is exactly Lemma 1 in [8], except the right-hand sides of inequalities, which are obtained from \( ac \geq 48 \) and \( bc \geq 528. \) \( \square \)

By (7) we may write \( z = v_m, \) where
\[
v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m,
\]
and by (8) we may write \( z = w_n, \) where
\[
w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n.
\]

Lemma 6. (Cf. [8, Lemma 8], [7, Lemma 3].)

1. If \( v_{2m} = w_{2n} \) has a solution, then \( z_0 = z_1. \) Moreover, \( |z_0| = 1, \) \( |z_0| = cr - st \) or \( |z_0| < \min(0.869a^{-5/14}b^{9/14}, 0.972b^{-0.3}c^{0.7}). \)
2. If \( v_{2m+1} = w_{2n+1} \) has a solution, then \( |z_0| = t, \) \( |z_1| = cr - st \) and \( z_0z_1 < 0. \) Moreover, \( b < 2a \) and \( c \geq 4ab + a + b. \)
3. If \( v_{2m} = w_{2n+1} \) has a solution, then \( |z_0| = cr - st, \) \( |z_1| = s \) and \( z_0z_1 < 0. \)
4. If \( v_{2m+1} = w_{2n+1} \) has a solution, then \( |z_0| = t, \) \( |z_1| = s \) and \( z_0z_1 > 0. \)

Proof. This is exactly Lemma 8 in [8], except the second statement of (2). We first show that \( c \geq 4ab + a + b. \) By Lemma 3 in [7], we have
\[
cx_0 - s|z_0| = |z_1|.
\]
Suppose that \( c = a + b + 2r. \) Then
\[
|z_1| = cr - st = 1.
\]
On the other hand, we see from (4), \( ac \geq 48 \) and \( c > 4a \) that
\[
c^2 - ac - z_0^2 > \left( 1 - \frac{1}{4} - \frac{1}{2\sqrt{48}} \right) c^2 > 0.67c^2,
\]
and from (3) that \( cx_0 + s|z_0| < 2cx_0 < 1.52c\sqrt{ac}. \) Hence we have
\[
cx_0 - s|z_0| = \frac{c^2 - ac - z_0^2}{cx_0 + s|z_0|} > \frac{0.67c^2}{1.52c\sqrt{ac}} > 0.44\sqrt{c} > 1,
\]
which contradicts (9) and (10). Hence by Lemma 4 in [12] we have \( c \geq 4ab + a + b. \) Note that we may assume \( ac \geq 360 \) in this case (this lower bound comes from the triple \( \{a, b, c\} = (3, 8, 120)\). Suppose now that \( b \geq 2a. \) Since \( c/a \geq 4b + 1 + b/a \geq 35, \) we have
\[
c^2 - ac - z_0^2 > \left( 1 - \frac{1}{35} - \frac{1}{2\sqrt{360}} \right) > 0.94c^2,
\]
which yields
\[
cx_0 - s|z_0| > \frac{0.94c^2}{1.52c\sqrt{ac}} > 0.6\frac{c\sqrt{c}}{\sqrt{a}}.
\]
On the other hand, by (6) we have

$$|z_1| < \frac{c\sqrt{c}}{2\sqrt{b}} \leq \frac{1}{\sqrt{2\sqrt{2}}} \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} < 0.6 \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}.$$

which contradicts (9) and (11). Therefore, we obtain $b < 2a$. □

**Lemma 7.** If $v_m = w_n$ has a solution, then $n - 1 \leq m \leq 2n + 1$.

**Proof.** This is exactly Lemma 3 in [8]. □

**Lemma 8.** Suppose that $(a, b, c, d)$ is a Diophantine quadruple with $a < b < c < d + < d$. If $v_m = w_n$ has a solution, then $m, n \geq 3$ and $(m, n) \neq (3,3)$.

**Proof.** This is a direct corollary of Lemmas 5 and 7 in [8]. □

Our first goal in this section is to show that if $(a, b, c, d)$ is a Diophantine quadruple with $a < b < c < d$ and if either $m$ or $n$ is odd, then $n \geq 5$. This is done by examining carefully the equation $v_m = w_n$ for $n = 3$ or 4 in each case of (2) to (4) in Lemma 6. In view of Lemma 8, we may assume that either $m$ or $n$ is not less than 4.

**Lemma 9.** $v_{2m+1} \neq w_4$ for all $m$.

**Proof.** Suppose that $v_{2m+1} = w_4$ for some $m$. By Lemma 7, we may assume that $m \geq 1$. By Lemma 6, we have $|z_0| = t, |z_1| = cr - st, z_0z_1 < 0$ and $x_0 = r, y_1 = rt - bs$. It suffices to show the following:

(i) $v_3 \neq w_4$,

(ii) $v_5 \neq w_4$,

(iii) $v_7 > w_4$.

(i) We have

$$v_3 = (4ac + 1)sz_0 + (4ac + 3)cx_0 \leq (4ac + 1)st + (4ac + 3)cr$$

$$= (4ac + 1)(cr + st) + 2cr,$$

$$w_4 = (8b^2c^2 + 8bc + 1)z_1 + 4(2bc + 1)tcy_1$$

$$\geq -(8b^2c^2 + 8bc + 1)(cr - st) + 4c(2bc + 1)(b(cr - st) + r)$$

$$= (4bc + 1)(cr + st) + 2cr.$$

It follows that $v_3 < w_4$.

(ii) If $z_0 > 0$ and $z_1 < 0$, then

$$v_5 = (16a^2c^2 + 12ac + 1)sz_0 + (16a^2c^2 + 20ac + 5)cx_0$$

$$= (16a^2c^2 + 12ac + 1)(cr + st) + 4(2ac + 1)cr,$$

$$w_4 = (4bc + 1)(cr + st) + 2cr.$$

It follows that $v_5 > w_4$. 
If \( z_0 < 0 \) and \( z_1 > 0 \), then
\[
\begin{align*}
  v_5 &= (16a^2c^2 + 12ac + 1)(cr - st) + 4(2ac + 1)cr, \\
  w_4 &= (16b^2c^2 + 12bc + 1)(cr - st) + 4(2bc + 1)cr.
\end{align*}
\]
It follows that \( v_5 < w_4 \). Therefore, we obtain \( v_5 \neq w_4 \).

(iii) We have
\[
\begin{align*}
  v_7 &= (64a^3c^3 + 80a^2c^2 + 24ac + 1)sz_0 + (64a^3c^3 + 112a^2c^2 + 56ac + 7)cx_0 \\
  &\geq (64a^3c^3 + 80a^2c^2 + 24ac + 1)(cr - st) + 2(16a^2c^2 + 16ac + 3)cr, \\
  w_4 &\leq (16b^2c^2 + 12bc + 1)(cr - st) + 4(2bc + 1)cr.
\end{align*}
\]
Since \( b < 2a \) by Lemma 6, we easily see that \( v_7 > w_4 \). 

**Lemma 10.** \( v_{2m} \neq w_3 \) for all \( m \) with \( m \geq 2 \).

**Proof.** Suppose that \( v_{2m} = w_3 \) for some \( m \geq 2 \). By Lemma 6, we have \( |z_0| = cr - st \), \( |z_1| = s \), \( z_0z_1 < 0 \) and \( x_0 = rs - at \), \( y_1 = r \). It suffices to show the following:

(i) \( v_4 \neq w_3 \), (ii) \( v_6 \neq w_3 \).

(i) If \( z_0 > 0 \) and \( z_1 < 0 \), then
\[
\begin{align*}
  v_4 &= (8a^2c^2 + 8ac + 1)z_0 + 4(2ac + 1)csx_0 \\
  &= (16a^2c^2 + 12ac + 1)(cr - st) + 4(2ac + 1)cr, \\
  w_3 &= (4bc + 1)tz_1 + (4bc + 3)cy_1 \\
  &= (4bc + 1)(cr - st) + 2cr.
\end{align*}
\]
It follows that \( v_4 > w_3 \).

If \( z_0 < 0 \) and \( z_1 > 0 \), then
\[
\begin{align*}
  v_4 &= (4ac + 1)(cr + st) + 2cr, \\
  w_3 &= (4bc + 1)(cr + st) + 2cr.
\end{align*}
\]
It follows that \( v_4 < w_3 \). Therefore, we obtain \( v_4 \neq w_3 \).

(ii) We have
\[
\begin{align*}
  v_6 &= (32a^3c^3 + 48a^2c^2 + 18ac + 1)z_0 + 2(16a^2c^2 + 16ac + 3)csx_0 \\
  &\geq (16a^2c^2 + 12ac + 1)(cr + st) + 4(2ac + 1)cr, \\
  w_3 &\leq (4bc + 1)(cr + st) + 2cr.
\end{align*}
\]
It follows that \( v_6 > w_3 \). 

**Lemma 11.** \( v_{2m+1} \neq w_3 \) for all \( m \) with \( m \geq 2 \).
Proof. Suppose that \( v_{2m+1} = w_3 \) for some \( m \geq 2 \). By Lemma 6, we have \( |z_0| = t \), \( |z_1| = s \), \( z_0 z_1 > 0 \) and \( x_0 = y_1 = r \). It suffices to show that \( v_5 > w_3 \). However, since

\[
v_5 = (16a^2c^2 + 12ac + 1)(cr + s) + 4(2ac + 1)cr,
\]

\[
w_3 = (4bc + 1)(cr + s) + 2cr,
\]

it is clear that \( v_5 > w_3 \). \( \Box \)

Suppose that \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d_+ < d \). Putting Lemmas 8–11 together, we see that if \( v_m = w_n \) has a solution with either of \( m \) and \( n \) odd, then \( n \geq 5 \); hence, in general we have \( n \geq 4 \).

We next consider the case \( v_{2m} = w_4 \). We could not prove \( v_{2m} \neq w_4 \) in general, whence we examine this case more precisely.

Lemma 12. Assume that \( c < 3.918a^{2.5}b^{3.5} \). If \( v_{2m} = w_{2n} \) has a solution, then \( |z_0| = 1 \) or \( cr - st \).

Proof. Suppose that \( |z_0| \neq 1, cr - st \). Putting \( d_0 = (z_0^2 - 1)/c \), we find that \( \{a, b, d_0, c\} \) is an irregular Diophantine quadruple with \( d_0 < c \) as in the proof of Lemma 8 in [8]. Thus, letting \( v'_m \) and \( w'_n \) be the recurrent sequences attached to the quadruple \( \{a, b, d_0, c\} \), we have \( n \leq 4 \), which together with the proof of Proposition 1 in [8] implies that \( c > 3.918a^{2.5}b^{3.5} \), a contradiction. \( \Box \)

Lemma 13. If \( c < 3.918a^{2.5}b^{3.5} \), then \( v_4 \neq w_4 \).

Proof. By Lemma 12, it suffices to show that if \( v_4 = w_4 \), then \( |z_0| \neq 1, cr - st \). Suppose that \( v_4 = w_4 \). Then \( z_0 = z_1 \) yields

\[-2(b - a)(b + a)c + 1)z_0 = (2bc + 1)t y_1 - (2ac + 1)s x_0 > 0,\]

whence \( z_0 < 0 \). If \( z_0 = -1 \), then

\[2(b - a)((b + a)c + 1) = 2(bt - as)c + t - s.\]

However, by \( c \geq a + b + 2r \) we have \( (b + a)c + 1 < tc \) and \( bt - as > (b - a)t \), which lead to a contradiction.

If \( z_0 = -(cr - st) \), then

\[2(b - a)((b + a)c + 1)(cr - st) = (b - a)[(2(b + a)c + 1)(cr - st) + 2cr],\]

that is, \( cr + st = 0 \), a contradiction. Therefore, we obtain \( |z_0| \neq 1, cr - st \). \( \Box \)

Lemma 14. Assume that \( c < 3.918a^{2.5}b^{3.5} \).

(1) If \( c \geq 0.25a^{-2.5}b^2 \), then \( v_{2m} \neq w_4 \) for all \( m \). In particular, if \( b < 2a \), then \( v_{2m} \neq w_4 \) for all \( m \).

(2) If \( v_{2m} = w_4 \) has a solution, then \( z_0 = z_1 \geq -1 \). Moreover, if \( c = a + b + 2r \), then \( z_0 = z_1 = 1 \).

Proof. (1) Suppose that \( v_{2m} = w_4 \) for some \( m \). We may assume by Lemma 12 that \( |z_0| = 1 \) or \( cr - st \) and by Lemmas 7 and 13 that \( m \geq 3 \). It suffices to show that \( v_6 > w_4 \) as long as \( c \geq 0.25a^{-2.5}b^2 \).
If \( z_0 = 1 \), then
\[
\nu_6 = (32a^2c^3 + 32ac^2 + 6c)(s + a) + 16a^2c^2 + 12ac + 1 > 32a^2c^3(s + a),
\]
\[
w_4 = 8bc^2(t + b) + 4c(t + 2b) + 1 < 16bc^2t + 12ct.
\]

It is easy to see from \( c \geq 0.25a^{-2.5}b^2 \) and \( b \geq 8 \) that \( 2a^2cs > bt \), that is, \( \nu_6 > w_4 \).

If \( z_0 = -1 \), then
\[
\nu_6 = 32a^2c^3(s - a) + 32ac^2(s - 2a) + 16a^2c^2 - 18ac - 1
\]
\[
> 32a^2c^3(s - a) + 32ac^2,
\]
\[
w_4 = 8bc^2(t - b) + 4c(t - 2b) - 1.
\]

Since \( s - a > \sqrt{ac}/2 \) and \( t - b < \sqrt{bc} \), it follows from \( c \geq 0.25a^{-2.5}b^2 \) and \( b \geq 8 \) that \( \nu_6 > w_4 \).

If \( z_0 = cr - st \), then
\[
\nu_6 = (64a^3c^3 + 80a^2c^2 + 24ac + 1)(cr - st) + 2(16a^2c^2 + 16ac + 3)cr,
\]
\[
w_4 = (16b^2c^2 + 12bc + 1)(cr - st) + 4(2bc + 1)cr.
\]

It follows from \( c \geq 0.25a^{-2.5}b^2 \) that \( \nu_6 > w_4 \).

If \( z_0 = -(cr - st) \), then
\[
\nu_6 = (16a^2c^2 + 12ac + 1)(cr - st) + 4(2ac + 1)cr,
\]
\[
w_4 = (4bc + 1)(cr + st) + 2cr.
\]

Thus, \( \nu_6 > w_4 \) holds without the assumption \( c \geq 0.25a^{-2.5}b^2 \).

If \( b < 2a \), then \( 0.25a^{-2.5}b^2 < a^{-0.5} < c \). Hence, the second assertion immediately follows from the first assertion.

(2) Suppose that \( \nu_{2m} = w_4 \) for some \( m \). By Lemma 12 we have \( |z_0| = 1 \) or \( cr - st \), and we have seen in the proof of (1) that if \( z_0 = -(cr - st) \), then \( \nu_6 > w_4 \) holds without the assumption \( c \geq 0.25a^{-2.5}b^2 \).

It follows that \( z_0 \geq -1 \).

If \( c = a + b + 2r \), then \( |z_0| = 1 \) by Theorem 8 in [12]. If \( z_0 = -1 \), then
\[
\nu_6 > 32a^2c^3r, \quad w_4 < 8bc^2r.
\]

Hence \( \nu_6 > w_4 \), and we must have \( \nu_4 = w_4 \), which contradicts Lemma 13. Therefore, we obtain \( z_0 = 1 \). This completes the proof of Lemma 14.  \( \square \)

**Lemma 15.**

(1) If \( z \geq w_5 \), then \( d > 62b^{3.5}c^{4.5} \).

(2) If \( z \geq w_4 \), then \( d > 16b^{2.5}c^{3.5} \).

Moreover, if either \( z_1 \geq -1 \) and \( c > 5b \) or \( z_1 > 0 \) and \( c = a + b + 2r \), then \( d > 120b^7 \).

**Proof.** (1) If \( c \geq 4ab + a + b > 5b \), then (5) and (6) imply that
\[
w_1 \geq cy_1 - t|z_1| = \frac{c^2 - bc - z_1^2}{cy_1 + t|z_1|} > \frac{1 - 1/5 - 0.148^2}{2 \cdot 0.723 \sqrt{bc}} c > 0.538b^{-1/4}c^{3/4}.
\]
If \( c = a + b + 2r < 4b \), then \( |z_1| = 1 \) by Theorem 8 in [12] and
\[
w_1 \geq c - t = s > \sqrt{ac} > \frac{1}{\sqrt{2}} a^{1/2} b^{-1/4} c^{3/4} > 0.538 b^{-1/4} c^{3/4}.
\]
Hence we have
\[
w_5 > 0.538 b^{-1/4} c^{3/4} (2t - 1)^4 > 7.88 b^{7/4} c^{11/4},
\]
which yields
\[
d > \frac{w_5^2 - 1}{c} > \frac{7.88^2 b^{7/2} c^{11/2} - 1}{c} > 62 b^{3.5} c^{4.5}.
\]
(2) In exactly the same way as (1), we have
\[
d > \frac{w_4^2 - 1}{c} > \frac{4.02^2 b^{5/2} c^{9/2} - 1}{c} > 16 b^{2.5} c^{3.5}.
\]
If \( z_1 \geq -1 \) and \( c > 5b \), then \( w_1 \geq c - t > 0.551 c \). Hence we have
\[
w_4 > 0.551 c (2t - 1)^3 > 4.12 b^{3/2} c^{5/2},
\]
which yields
\[
d > \frac{4.12^2 b^3 c^5 - 1}{c} > 16 b^2 c^4 > 10000 b^7.
\]
If \( z_1 > 0 \) and \( c = a + b + 2r \), then \( w_1 \geq c + t = c + b + r > 3c/2 \). Hence we have
\[
w_4 > \frac{3}{2} c (2t - 1)^3 > 11.2 b^{3/2} c^{5/2},
\]
which yields
\[
d > \frac{11.2^2 b^3 c^5 - 1}{c} > 120 b^3 c^4 > 120 b^7.
\]
This completes the proof of Lemma 15. \( \square \)

We are now ready to state the gap principles.

**Proposition 16.** Let \( \{a, b, c, d\} \) be a Diophantine quadruple with \( a < b < c < d \). Then either \( d > 60b^8 \) or \( d \geq \max\{400a^2 b^6, 100b^7\} \). In case \( b < 2a \), we have \( d > 1400b^{12.5} \) unless \( c = a + b + 2r \).

**Proof.** Assume first that \( b < 2a \). Since Lemmas 8–11 and 14 together imply \( z \geq w_5 \), we have \( d > 62 b^{3.5} c^{4.5} > 60b^8 \) by Lemma 15. If further \( c \neq a + b + 2r \), then \( c > 4ab > 2b^2 \) and \( d > 1400b^{12.5} \).

Assume next that \( b \geq 2a \). The above-mentioned lemmas together imply \( z \geq w_5 \) or \( z = v_{2m} = w_4 \) and either

(i) \( c > 3.918 a^{2.5} b^{3.5} \) or

(ii) \( c < 0.25 a^{-2.5} b^2 \).
If \( z \geq w_5 \), then Lemma 15 again implies \( d > 60b^8 \). It suffices to show that if \( z = v_{2m} = w_4 \), then \( d > \max(400a^5b^6, 100b^7) =: M \).

In the case of (i), by Lemma 15 we have
\[
d > 16b^{2.5}c^{3.5} > 16 \cdot 3.918^{3.5} a^{8.75} b^{14.75} > M.
\]

In the case of (ii), we have \( z \geq v_6 \) by Lemmas 7, 8 and 13. If \( c > 5b \), then \( z_0 = z_1 \geq -1 \) by Lemma 14 and \( v_1 = c - s > 0.682c \). Hence we have
\[
v_6 > 0.682c(2s - 1)^5 > 15a^{5/2}c^{7/2},
\]
and
\[
d > \frac{15^2 a^5 c^7 - 1}{c} > 400a^5 b^6,
\]
which together with Lemma 15 implies \( d > M \). If \( c = a + b + 2r \), then \( z_0 = z_1 = 1 \) by Lemma 14 and \( v_1 = s + c > c \). Hence we have
\[
v_6 > c(2s - 1)^5 > 22a^{5/2}c^{7/2},
\]
and
\[
d > \frac{22^2 a^5 c^7 - 1}{c} > 400a^5 b^6,
\]
which together with Lemma 15 implies \( d > M \). This completes the proof of Proposition 16.

3. Lower bounds for solutions

In this section, we give lower bounds for solutions, assuming \( c > b^7 \), by considering the equation \( v_m = w_n \) as a congruence relation. It is crucial to consider the cases \( b \geq 2a \) and \( b < 2a \) separately.

Lemma 17. Assume that \( c > b^7 \). If \( v_m = w_n \) has a solution with \( m, n \geq 4 \), then \( m \leq 8n/7 + 6/7 \).

Proof. The proof proceeds along the same lines as Lemma 4 in [8]. Since \( m, n \geq 4 \) and \( c > b^7 \geq 8^7 \), the inequality (20) in [8] holds:
\[
1.999^{m-1}a^{(2m-3)/4}c^{m/2} < 2.004^n b^{(2n-1)/4}c^{(n+1)/2},
\]
which yields
\[
1.999^{m-1}c^{m/2} < 2.004^n c^{(16n+13)/28}.
\]
(Note that the exponent of \( c \) in the right-hand side of (20) in [8] should be \( n/2 \).) Hence, either \( m/2 < (16n + 13)/28 \) or \( m - 1 < 1.004n \) holds. If the former holds, then \( m \leq 8n/7 + 6/7 \). If the latter holds, then \( m \leq 251n/250 + 249/250 \). Therefore, we obtain \( m \leq 8n/7 + 6/7 \).

Lemma 18. Assume that \( c > b^7 \). Then \( v_{2m+1} \neq w_2n \) and \( v_{2m} \neq w_{2n+1} \). Moreover, if \( v_{2m} = w_{2n} \) has a solution with \( m, n \geq 2 \), then either \( |z_0| = 1 \) or \( |z_0| < 0.823b^{-5/14}c^{9/14} \).
Proof. One may prove the first assertions \( v_{2m+1} \neq w_{2n} \) and \( v_{2m} \neq w_{2n+1} \) in the same way as Lemma 10 in \([8, (2)\) and (3), p. 200].

Let \( d_0 = (z_0^2 - 1)/c \). If \( d_0 = 0 \), then \( |z_0| = 1 \). If \( d_0 \neq 0 \), then one may prove \(|z_0| \neq cr - st\) in the same way as Lemma 10 in \([8, (1.2), p. 198]\). Hence, as in the proof of Lemma 8 in \([8]\), we see that \((a, b, c, d_0)\) is an irregular Diophantine quadruple and that \(0.9999z_0^2/c < d_0 < c\). Since \(w_{2n} \geq w_4\), by the proof of Proposition 1 in \([8, l. 2, p. 194]\) we have

\[
c > 3.918d_0^{3.5}b^{-2.5} > 3.916|z_0|^7c^{-3.5}b^{-2.5},
\]

that is, \(|z_0| < 0.823b^{-5/14}c^{9/14}\). \(\square\)

Lemma 19.

(1) \( v_{2m} \equiv z_0 + 2c(az_0m^2 + sx_0m) \pmod{8c^2} \).

(2) \( v_{2m+1} \equiv z_0 + c(2asz_0m(m + 1) + x_0(2m + 1)) \pmod{4c^2} \).

(3) \( w_{2n} \equiv z_1 + 2c(bz_1n^2 + ty_1n) \pmod{8c^2} \).

(4) \( w_{2n+1} \equiv t_1 + c(2bt_1n(n + 1) + y_1(2n + 1)) \pmod{4c^2} \).

Proof. This is exactly Lemma 4 in \([7]\). \(\square\)

Lemma 20. Assume that \(c > b^7\).

(i) If \( v_{2m} = w_{2n} \) has a solution with \( m, n \geq 2 \), then the following hold:

(I) In the case of \( |z_0| = 1 \), if \( b \geq 2a \), then \( n > 0.0418a^{1/2}b^{-1}c^{1/2} \); otherwise, \( n > a^{-1/2}c^{1/8} \).

(II) In the case of \( |z_0| < 0.823b^{-5/14}c^{9/14} \), if \( b \geq 2a \), then \( n > 1.588b^-9/28c^{5/28} \); otherwise,

\[
n > \min \{ 0.999a^{-1/2}b^{-1/8}c^{1/8}, 0.625b^{-11/28}c^{3/28} \}.
\]

(ii) If \( v_{2m+1} = w_{2n+1} \) has a solution with \( m, n \geq 2 \), then

\[
n > \min \{ 0.586a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4} \}.
\]

Proof. (i) By Lemma 19 with \( z_0 = z_1 \), we have

\[
a z_0m^2 + sx_0m \equiv bz_1n^2 + ty_1n \pmod{4c}.
\]

(1) In this case, we have

\[
\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{4c}.
\]

Assume first that \( b \geq 2a \). Suppose that \( n \leq 0.0418a^{1/2}b^{-1}c^{1/2} \). Since \( m \leq 19n/14 \) by Lemma 17 and \( c > b^7 \geq 8^7 \), we easily see that

\[
am^2 < c, \quad sm < c, \quad bn^2 < c, \quad tn < c.
\]

It follows from (12) that

\[
\pm am^2 + sm = \pm bn^2 + tn.
\]

(13)
If \( z_0 = 1 \), then \( am^2 + sm = bn^2 + tn \). Since \( b \geq 2a \), we have \( bn^2 + tn > 2am^2 + 1.414sn \). On the other hand, \( m \leq 19n/14 \) implies that \( am^2 + sm < 1.9am^2 + 1.4sn \), which is a contradiction. If \( z_0 = -1 \), then \( bn/m - am/n = t/m - s/n \). We now have \( bn/m - am/n < b \) and

\[
\frac{t}{m} - \frac{s}{n} \geq \left( \frac{14}{19} \cdot 1.414 - 1 \right) s \cdot \frac{b}{0.0418\sqrt{ac}} > b,
\]

which are contradictions. Therefore, if \( b \geq 2a \), then \( n > 0.0418a^{1/2}b^{-1}c^{1/2} \).

Secondly, assume that \( b < 2a \). Suppose that \( n \leq a^{-1/2}c^{1/8} \). Squaring (13) twice, we have

\[
\{(am^2 - bn^2)^2 - (m^2 + n^2)\}^2 \equiv 4m^2n^2 \pmod{c}.
\]

Since

\[
\{(am^2 - bn^2)^2 - (m^2 + n^2)\}^2 < (an^2)^4 < c,
\]

\[
4m^2n^2 \leq 4\left( \frac{19}{14} \right)^2 a^{-2}c^{1/2} < 8 \cdot 4b^{-2}c^{1/2} < c,
\]

(14) is in fact an equation, that is,

\[
am^2 - bn^2 = \mp(m + n).
\]

By (13) and (15), we have \( m(s \mp 1) = n(t \pm 1) \), which together with (13) implies that

\[
\pm\left\{ a\left( \frac{t \pm 1}{s \mp 1} \right)^2 - b \right\} n = t - \frac{s(t \pm 1)}{s \mp 1},
\]

where the signs are taken simultaneously (note that the plus sign in (15) implies the minus signs in (13)). If (16) holds with the upper signs, then \( n \) has to be negative. Hence we obtain

\[
n = \frac{(s + 1)(t + s)}{2(at + bs - a + b)} > \frac{ac}{b(2s + 1)} > a^{-1/2}c^{1/8},
\]

which is a contradiction. Therefore, if \( b < 2a \), then \( n > a^{-1/2}c^{1/8} \).

(II) Assume first that \( b \geq 2a \). Suppose that \( n \leq 1.558b^{-9/28}c^{5/28} \). Since \( |az_0m^2| < 2c \), \( |bz_0n^2| < 2c \), and

\[
sx_0m < \left( a|z_0| + \frac{c}{|z_0|} \right) m < 6.6ab^{-9/28}c^{23/28} < 2c,
\]

\[
ty_1n < \left( b|z_0| + \frac{c}{|z_0|} \right) n < 3.2b^{9/28}c^{23/28} < 2c,
\]

it follows from (12) that

\[
az_0m^2 + sx_0m = bz_0n^2 + ty_1n.
\]
Noting \( z_0^2 > 3c/a \), we have

\[
0 \leq \frac{s x_0}{a |z_0|} - 1 = \frac{x_0^2 + ac - a^2}{a |z_0| (s x_0 + a |z_0|)} < \frac{1.001ac}{2a^2 z_0^2} < 0.1669,
\]

and obtain

\[
0 \leq \frac{t y_1}{b |z_0|} - 1 = \frac{y_1^2 + bc - b^2}{b |z_0| (t y_1 + b |z_0|)} < \frac{1.001bc}{2b^2 z_0^2} < 0.0835.
\]

If \( z_0 > 0 \), then (17) implies that \( az_0(m + 1.6669) > b z_0(n + 1) \), that is, \( m(m + 1.6669) > 2n(n + 1) \). By Lemma 17 we have

\[
\left( \frac{8}{7} n^3 + \frac{3}{7} \right) \left( \frac{8}{7} n^3 + \frac{3}{7} + 1.6669 \right) > 2n(n + 1).
\]

Hence we obtain \( n < 1.3 \), which is a contradiction. If \( z_0 < 0 \), then in the same way as above we have

\[
m(m - 1) > 2n(n - 1.0835),
\]

and obtain \( n < 2.8 \), that is, \( n = 2 \). We now claim that if \( c > b^7 \), then \( v_6 > w_4 \). Indeed, this immediately follows from \( t y_1 < (bc)^{3/4} < c^{6/7} \). Hence we must have \( m = 2 \), which together with (18) implies that \( 2 > 4(2 - 1.0835) > 3 \), a contradiction. Therefore, if \( b \geq 2a \), then \( n > 1.558b^{-9/28}c^{5/28} \).

Secondly, assume that \( b < 2a \). Suppose that \( n \leq \min(0.999a^{-1/2}b^{-1/8}c^{1/8}, 0.625b^{-11/28}c^{3/28}) \). Then we also have Eq. (17), which implies that

\[
(\frac{a m^2 - b n^2}{c})^2 \equiv x_0^2 m^2 + y_1^2 n^2 - 2s t x_0 y_1 m n \pmod{c}.
\]

Multiplying the both sides by \( s \) and by \( t \) respectively, we have

\[
C s \equiv -2t x_0 y_1 m n \pmod{c}, \quad C t \equiv -2s x_0 y_1 m n \pmod{c},
\]

where \( C = (a m^2 - b n^2)^2 - (x_0^2 m^2 + y_1^2 n^2) \). Since

\[
y_1^2 < \frac{b}{c} z_0^2 + 1 < 0.686 b^{2/7} c^{2/7},
\]

we have \( x_0^2 m^2 + y_1^2 n^2 < 1.95 b^{2/7} c^{2/7} n^2 \). Hence we have

\[
C t < \min\{a^2 n^4 \sqrt{bc + 1}, 1.95 b^{2/7} c^{2/7} n^2 \sqrt{bc + 1}\} < \min\{1.001 a^2 b^{1/2} c^{1/2} n^4, 1.96 b^{11/14} c^{11/14} n^2\} < c.
\]

Since

\[
2t x_0 y_1 m n < 2t y_1^2 n^2 < \frac{19}{14} b^{11/14} c^{11/14} n^2 < \frac{c}{2},
\]

it follows from (19) and (20) that either

\[
C s = -2t x_0 y_1 m n, \quad C t = -2t x_0 y_1 m n
\]
Theorem 21.

Suppose that (22) and (23) are in fact equations. Hence we have

\[ -2t^2x_0y_1mn + 2s^2x_0y_1mn = 0. \]

that is, \( t^2 = s^2 \), a contradiction. If the latter holds, then

\[ (t - s)c - 2(t^2 - s^2)x_0y_1mn = 0, \]

that is, \( c = 2(t + s)x_0y_1mn \), which contradicts (21). Therefore, if \( b < 2a \), then \( n > \min(0.999a^{-1/2}b^{-1/8}c^{1/8}, 0.625b^{-11/28}c^{5/28}) \).

(ii) By Lemma 19 with \( |z_0| = t, |z_1| = s, z_0z_1 > 0 \) and \( x_0 = y_1 = r \), we have

\[ \pm atm(m + 1) + rm \equiv \pm bsn(n + 1) + rn \pmod{2c}. \]

Multiplying the both sides by \( s \) and by \( t \) respectively, we have

\[ \pm atm(m + 1) + rsm \equiv \pm bsn(n + 1) + rsn \pmod{2c}, \quad (22) \]

\[ \pm asm(m + 1) + rtm \equiv \pm bsn(n + 1) + rtn \pmod{2c}. \quad (23) \]

Suppose that \( n \leq \max(0.586a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}) \). Since \( m \leq 39n/28 \) by Lemma 17 and \( c > 8^7 \), we see that

\[ atm(m + 1) < c, \quad bsn(n + 1) < c, \quad rtm < c, \]

and that (22) and (23) are in fact equations. Hence we have

\[ rm(s^2 - t^2) = rsn(s^2 - t^2), \]

\[ atm(m + 1)(t^2 - s^2) = bsn(n + 1)(t^2 - s^2), \]

that is, \( m = n, atm(m + 1) = bsn(n + 1) \), which together yield \( m = 0 \), a contradiction. Therefore, we obtain \( n > \min(0.586a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}) \). This completes the proof of Lemma 20. \( \square \)

4. Application of a theorem of Bennett

In this section, we reduce the assumption in a theorem of Bennett (see [3, Theorem 3.2]) to the one that is satisfied if \( "c > b^7." \) Then applying this theorem to the quadruple \( (a, b, d, e) \) if either \( b \geq 2a \) or \( c \neq a + b + 2r \) and to the quadruple \( (b, c, d, e) \) if \( b < 2a \) and \( c = a + b + 2r \), we complete the proof of Theorem 2.

Theorem 21. (Cf. [3,16].) Let \( a, b \) and \( N \) be integers with \( 0 < a < b, b \geq 8 \) and \( N > 2.4a' a^2b^4(b - a)^2 \), where \( a' = \max(b - a, a) \). Then the numbers \( \theta_1 = \sqrt{1 + b/N} \) and \( \theta_2 = \sqrt{1 + a/N} \) satisfy

\[ \max \left\{ \left| \frac{\theta_1 - p_1}{q} \right|, \left| \frac{\theta_2 - p_2}{q} \right| \right\} > \left( 16.01a' b^2 N \right)^{-1} q^{-\lambda}. \]
for all integers \( p_1, p_2, q \) with \( q > 0 \), where

\[
\lambda = 1 + \frac{\log(8.01a'b^2N)}{\log(3.37a^2b^{-2}(b-a)^{-2}N^2)} < 2.
\]

**Proof.** The assumptions immediately imply \( \lambda < 2 \). All we have to do is find those real numbers satisfying the assumptions in the following lemma.

**Lemma 22.** (Cf. [3, Lemma 3.1], [16, Lemma 2.1].) Let \( \theta_1, \ldots, \theta_m \) be arbitrary real numbers and \( \theta_0 = 1 \). Assume that there exist positive real numbers \( l, p, L, P \) and positive integers \( D, f \) with \( f \) dividing \( D \) and with \( L > D \), having the following property. For each positive integer \( k \), we can find rational numbers \( p_{ijk} \) \((0 \leq i, j \leq m)\) with nonzero determinant such that \( f^{-1}D^kp_{ijk} \) \((0 \leq i, j \leq m)\) are integers and

\[
|p_{ijk}| \leq p^k \quad (0 \leq i, j \leq m), \quad \sum_{j=0}^{m} p_{ijk}\theta_j \leq IL^{-k} \quad (0 \leq i \leq m).
\]

Then

\[
\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \ldots, \left| \theta_m - \frac{p_m}{q} \right| \right\} > cq^{-\lambda}
\]

holds for all integers \( p_1, \ldots, p_m, q \) with \( q > 0 \), where

\[
\lambda = 1 + \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad c^{-1} = 2mf^{-1}pDP (\max\{1, 2f^{-1}l\})^\lambda.
\]

Note that \( l, p, L, P, p_{ijk} \) in Lemma 3.1 in [3] denote \( f^{-1}l, f^{-1}p, L/D, DP, f^{-1}D^kp_{ijk} \) in the above lemma, respectively. In our situation, we take \( m = 2 \) and \( \theta_1, \theta_2 \) as in Theorem 21. By the arguments following Lemma 3.1 in [3], we have

\[
\sum_{j=0}^{2} p_{ijk}\theta_j = |I_i(1/N)| < \frac{27}{64} \left( 1 - \frac{b}{N} \right)^{-1} \left\{ \frac{27}{4} \left( 1 - \frac{b}{N} \right)^2 N^3 \right\}^{-k}
\]

and

\[
|p_{ijk}| \leq \max_{z \in \Gamma_j} \left\{ \frac{(1 + z/N)^{k+1/2}}{|A(z)|^k} \right\} \quad (0 \leq j \leq 2),
\]

where \( A(z) = \prod_{i=0}^{2} (z - a_i) \) and the contours \( \Gamma_j \) \((0 \leq j \leq 2)\) are defined by

\[
|z - a_j| = \min_{i \neq j} \left\{ \frac{|a_j - a_i|}{2} \right\}
\]

with \( a_0 = 1, a_1 = a, a_2 = b \). The inequality (24) enables us to take

\[
l = \frac{27}{64} \left( 1 - \frac{b}{N} \right)^{-1}, \quad L = \frac{27}{4} \left( 1 - \frac{b}{N} \right)^2 N^3.
\]
In the inequality (25), we have

\[ \left| 1 + \frac{z}{N} \right| \leq \begin{cases} 
1 + \frac{1}{2N} & \text{on } \Gamma_0, \\
1 + \frac{a+b}{2N} & \text{on } \Gamma_1, \\
1 + \frac{3b-a}{2N} & \text{on } \Gamma_2.
\end{cases} \]

Moreover, again by the arguments following Lemma 3.1 in [3], we have \( \min |A(z)| \geq \frac{\zeta}{8} \), where

\[ \zeta = \begin{cases} 
a^2(2b-a) & \text{if } b - a \geq a, \\
(b-a)^2(a+b) & \text{if } b - a < a.
\end{cases} \]

Hence we obtain

\[ |p_{ijk}| \leq \frac{\max_{x \in \Gamma_j} |1 + z/N|^{|+1/2}}{\theta_j \cdot \min_{x \in \Gamma_j} |A(z)|^{|}} \]

\[ \leq \begin{cases} 
(1 + \frac{b-a}{2N})^{1/2} \left( \frac{8(1 + \frac{3b-a}{2N})}{\zeta} \right)^{|} & \text{if } b - a \geq a, \\
(1 + \frac{a}{2N})^{1/2} \left( \frac{8(1 + \frac{3b}{2N})}{\zeta} \right)^{|} & \text{if } b - a < a.
\end{cases} \]

Therefore, we may take

\[ p = \left( 1 + \frac{a^l}{2N} \right)^{1/2}, \quad P = \frac{8(1 + \frac{3b-a}{2N})}{\zeta}. \]

Finally, we may take \( f = 2 \) and \( D = 2a^2b^2(b-a)^2N \). Indeed, let \( p_{ij}(x) \) be those polynomials appearing in Lemma 3.2 in [16], which have rational coefficients of degree at most \( k \) (cf. [16, (3.7)]). Note that we take \( p_{ijk} = p_{ij}(1/N) \) for varying values of \( k \). Then we see from the expression (3.7) in [16] of \( p_{ij}(1/N) \) that

\[ 2^l \left( \frac{ab(b-a)}{2} \right)^{l_2} N^k p_{ij}(1/N) \in \mathbb{Z} \]

for some integers \( l_1, l_2 \). By a consideration similar to the proof of Lemma 4.3 in [16], we may take \( l_1 = 3k - 1, l_2 = 2k \). Hence we obtain

\[ 2^{-1} \left\{ 2a^2b^2(b-a)^2N \right\}^k p_{ij}(1/N) \in \mathbb{Z}, \]

and we may take \( f = 2, D = 2a^2b^2(b-a)^2N \).

From the assumptions in Theorem 21, we obtain

\[ DP < 8.01a'b^2N, \quad \frac{L}{D} > \frac{3.37N^2}{a^2b^2(b-a)^2}, \quad c^{-1} < 16.01a'b^2N. \]

Theorem 21 now follows immediately from Lemma 22. \( \square \)
Lemma 23. (See [7, Lemma 12].) Let \(N = abc\) and let \(\theta_1, \theta_2\) be as in Theorem 21. Then all positive integer solutions of the system of Diophantine equations (1) and (2) satisfy

\[
\max \left\{ \left| \frac{\theta_1 - \frac{sby}{abz}}{abz} \right|, \left| \frac{\theta_2 - \frac{tay}{abz}}{abz} \right| \right\} < \frac{c}{2a} z^{-1}.
\]

Theorem 21 and Lemma 23 together yield an upper bound for \(z\).

Lemma 24. Let \((a, b, c, d)\) be a Diophantine quadruple with \(a < b < c < d\). Assume that \(c > b^7\). Then

\[
\log z < \frac{4 \log(2.83ab^3c) \log(1.836(b - a)^{-1}c)}{\log(0.4207a^{-1}b^{-6}c)}.
\]

**Proof.** The assumption \(c > b^7\) implies \(N := abc > 2.4a^2b^4(b - a)^2\). Indeed, it suffices to show that 
\(2.4a^2b^4(b - a)^2 < b^4\). The inequality is clearly satisfied if \(b \geq 2.4a\). Thus, we may assume that \(b < 2.4a\).

If \(b > 2a\), then \(2.4a^2(b - a)^2/b^4 \leq 1.2(b - a)^3/b^3 < 1\); if \(b < 2a\), then \(2.4a^2(b - a)^2/b^4 < 0.6a^2/b^2 < 1\). Hence, we may apply Theorem 21 with \(N = abc\).

Putting \(q = abz\), \(p_1 = sbx\), \(p_2 = tay\), we see from Theorem 21 and Lemma 23 that

\[
z^{2-\lambda} < 8.005a^2b^5c^2 < (2.83ab^3c)^2.
\]

Since

\[
\frac{1}{2 - \lambda} = \frac{\log(3.37(b - a)^{-2}c^2)}{\log\left(\frac{3.37c}{8.01a^2b^4 - a^2c}b^3\right)} < \frac{2 \log(1.836(b - a)^{-1}c)}{\log(0.4207a^{-1}b^{-6}c)},
\]

we obtain the desired inequality. \(\square\)

The following lemma translates an upper bound for \(z\) into the one for \(n\).

Lemma 25. Assume that \(c > b^7\). If \(z = \nu_m = w_n\), then

\[
\log z > \frac{n}{2} \log(4bc).
\]

**Proof.** One may prove that if \(c > b^7\), then \(y_1 \sqrt[\lambda]{c} - |z_1| \sqrt[\lambda]{b} > 3 \sqrt[\lambda]{b}\) and \(w_n > (t + \sqrt{bc})^n > (4bc)^{n/2}\) along the same lines as in the proof of Theorem 3 in [7]. This immediately shows the lemma. \(\square\)

We are now ready to bound \(c\) for irregular \((a, b, c, d)\) with \(a < b < c < d\).

**Proposition 26.** Let \((a, b, c, d)\) be a Diophantine quadruple with \(a < b < c < d\). Assume that \(b \geq 2a\). (1) If \(c \geq \max\{100a^5b^6, 100b^7\}\), then \(d = d_+\).

(2) If \(c \geq 20b^8\), then \(d = d_+\).

**Proof.** Suppose that \(d > d_+\). Since \(c > b^7\) in each case of (1) and (2), Lemmas 24 and 25 together imply that

\[
\frac{n}{8} < \varphi(a, b, c),
\]

(26)
where
\[ \varphi(a, b, c) = \frac{\log(2.83ab^3c) \log(1.836(b-a)^{-1}c)}{\log(4bc) \log(0.4207a^{-1}b^{-6}c)}. \]

(1) The assumption on \( c \) implies \( a^{-1}b^{-6}c \geq 100b^{4/5} \). Noting \( a \leq b/2 \) and \( b - a \geq 5 \), we see from \( c \geq 100b^7 \) that
\[ \varphi(a, b, c) < \frac{\log(141.5b^{11}) \log(36.72b^7)}{\log(400b^5) \log(42.07b^{6/5})} = \frac{11 \cdot 7}{8 \cdot 0.8} f_1(b) < \frac{385}{32}. \]

where
\[ f_1(b) = \frac{\log(141.51/b) \log(36.72^{1/7}b)}{\log(400^{1/8}b) \log(42.07^{5/4}b)}. \]

(i) Assume that both \( m \) and \( n \) are even. If \( |z_0| = 1 \), then (26), (27) and Lemma 20 together imply that \( 0.0418a^{1/2}b^{-1}c^{-1/2} < 385/8 \). Since \( c \geq 100b^7 \), we obtain \( b < 116^{3/5} < 7 \), a contradiction. If \( |z_0| \neq 1 \), \( cr - st \), then in the same way as above we have \( 1.55b^{-9/28}c^{5/28} < 385/8 \) and \( b < 16.6 \), that is, \( b \leq 16 \). Since \( f_1(b) \) is increasing, we have \( f_1(b) \leq f_1(16) < 0.41 \), which yields \( 1.55b^{-9/28}c^{5/28} < 0.41 \cdot 385/8 < 19.8 \). Hence we obtain \( b < 7 \), a contradiction.

(ii) Assume that both \( m \) and \( n \) are odd. By Lemma 20 we have
\[ \frac{n - 1}{2} > \min\{0.586a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}\} \geq 0.816b^{-3/4}c^{1/4} > 2.58b, \]
which together with (26) and (27) implies that \( b < 18.5 \), that is, \( b \leq 18 \). In this case, \( f_1(b) \leq f_1(18) < 0.42 \), which yields \( 2.58b < 0.42 \cdot 385/8 - 1/2 < 19.8 \), that is, \( b < 8 \), a contradiction.

In any case, we obtain \( d = d_+ \).

(2) We see from \( c \geq 20b^8 \) that
\[ \varphi(a, b, c) < \frac{\log(28.3b^{12}) \log(7.344b^8)}{\log(80b^9) \log(16.828b)} = \frac{12 \cdot 8}{9} f_2(b) < \frac{32}{3}, \]
where
\[ f_2(b) = \frac{\log(28.3^{1/12}b) \log(7.344^{1/8}b)}{\log(80^{1/9}b) \log(16.828b)}. \]

Suppose that \( d > d_+ \). Since one may arrive at a contradiction along the same lines as (1), we only give an outline of the proof.

(i) Assume that both \( m \) and \( n \) are even. If \( |z_0| = 1 \), then we have \( n/2 > 0.186b^3 \) and \( b < 7 \), a contradiction. If \( |z_0| \neq 1 \), \( cr - st \), then we have \( n/2 > 2.66b^{31/28} \) and \( b \leq 12 \). Since \( f_2(b) \leq f_2(12) < 0.5 \), we obtain \( b < 7 \), a contradiction.

(ii) Assume that both \( m \) and \( n \) are odd. Then we have \( (n - 1)/2 > 1.72b^{5/4} \) and \( b \leq 12 \). Since \( f_2(b) \leq f_2(12) < 0.5 \), we obtain \( b < 8 \), a contradiction.

In any case, we obtain \( d = d_+ \), which completes the proof of Proposition 26. \( \square \)

**Proposition 27.** Let \( (a, b, c, d) \) be a Diophantine quadruple with \( a < b < c < d \). If \( c \geq 100b^{12.5} \), then \( d = d_+ \).
Proof. Suppose that \( d > d_+ \). By Proposition 26, we may assume that \( b < 2a \) and hence, may assume that \( b \geq 15 \) and \( b - a \geq 7 \) (these lower bounds come from the pair \( \{a, b\} = \{8, 15\} \). Since \( c \geq 100b^{12.5} \), we have (26) and

\[
\phi(a, b, c) < \frac{\log(283b^{16.5}) \log(26.23b^{12.5})}{\log(400b^{13.5}) \log(42.07b^{5.5})} < \frac{16.5 \cdot 12.5}{13.5 \cdot 5.5} = \frac{25}{9}.
\]

Suppose that \( d > d_+ \). One may also arrive at contradictions in the same lines as the proof of Proposition 26(1), except using lower bounds for \( n \) in the case of \( b < 2a \) in Lemma 20. Thus, we only give an outline of the proof.

(i) Assume that both \( m \) and \( n \) are even. If \( |z_0| = 1 \), then we have \( n/2 > 1.778b^{8.5/8} \) and \( b < 6 \), a contradiction. If \( |z_0| \neq 1 \), then we have \( n/2 > \min\{1.776b^{7.5/8}, 1.023b^{26.5/28}\} \) and \( b < 13 \), a contradiction.

(ii) Assume that both \( m \) and \( n \) are odd. Then we have \( (n - 1)/2 > 1.853b^{8.5/4} \) and \( b < 3 \), a contradiction.

In any case, we obtain \( d = d_+ \), which completes the proof of Proposition 27. \( \Box \)

Now, Theorem 2 immediately follows from Propositions 26, 27 and 16, except for the case \( b < 2a \) and \( c = a + b + 2r \).

Proof of Theorem 2. Suppose that \( d > d_+ \). In case \( b \geq 2a \), Propositions 16 and 26 together imply that \( \{a, b, d, e\} \) is a regular Diophantine quadruple, which is a contradiction. Assume that \( b < 2a \). If \( c > a + b + 2r \), then Propositions 16 and 26 together lead us to a contradiction, too. Hence, we may assume that \( c = a + b + 2r \). Then, \( c > (1/2 + 1 + 2/\sqrt{2})b > 2.9b \). Assume that \( z = v_m = w_n \).

Since \( |z_1| = 1 \) by Lemma 8 in [12], we see that \( w_1 \geq c - t > 0.411c \), and from Lemma 6 that \( n \) is even. Hence by Lemmas 9–11 and 14 ((1) with \( b < 2a \)) we have either \( n = 4 \) and \( c > 3.918a^{2.5}b^{3.5} \) or \( n \geq 6 \). If the former holds, then \( d > 16.2b^{2.5}c^{3.5} > 1900b^{34.75} \) and Proposition 27 together lead us to a contradiction. Hence \( n \geq 6 \), and

\[
d \geq \frac{w^2_1}{c} > \frac{0.411c(2t - 1)^3}{c} > 130b^5c^6,
\]

which together with \( c < 4b \) implies that \( d > 0.12c^{11} > 20c^8 \). It follows from Proposition 26(2) that \( \{b, c, d, e\} \) is a regular Diophantine quadruple, which is a contradiction. Therefore, we obtain \( d = d_+ \), which completes the proof of Theorem 2.

We conclude this paper with some remarks. Corollary 3 implies that in order to settle the Diophantine quintuple conjecture, it suffices to examine the extensibility of regular Diophantine quadruples. Let \( \{a, b, c\} \) be a Diophantine triple with \( a < b < c \). As already seen (cf. Propositions 26 and 27), if there is a big gap between \( b \) and \( c \), then it is not difficult to see that the fourth element \( d \) has to be \( d_+ \). It seems to be very hard to handle the case \( c = a + b + 2r \). However, if \( \{a, b, c, d, e\} \) is a Diophantine quintuple with \( c = a + b + 2r < d = d_+ < e \), then \( \{a, c, d\} \) is a “standard triple” (cf. Definition 1 in [8] or Definition 3.1 in [11]). Hence, combining the congruence method (cf. Section 3) with Baker’s theory (a theorem [14] of Matveev) yields \( d > b^5 \) whenever \( b > 10^{50} \), which contradicts \( c = a + b + 2r \) and \( d = d_+ \) (cf. Corollary 4.5 in [11]). The problem is that there are so many pairs \( \{a, b\} \) with \( b \leq 10^{50} \) that we cannot apply the reduction method (cf. [2] and [9]) to each of the pairs, using a computer of current performance. The key to solve the problem seems to be to develop the congruence method and Baker’s theory. For this purpose, the strategies of Bugeaud, Dujella and Mignotte [4] can be relevant, where they applied a more precise congruence method and a theorem (see [15]) of Mignotte on linear forms in three logarithms, combined with the theorem of Matveev, to the triples \( (k - 1, k + 1, 16k^3 - 4k) \). The other solution would be to find an alternative to the congruence method or Padé approximation method; otherwise there would be nothing to be done but wait for the development of computer technology.
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