Isospin chemical potential and the QCD phase diagram at nonzero temperature and baryon chemical potential

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Abstract

We use the Nambu–Jona-Lasinio model to study the effects of the isospin chemical potential on the QCD phase diagram at nonzero temperature and baryon chemical potential. We find that the phase diagram is qualitatively altered by a small isospin chemical potential. There are two first order phase transitions that end in two critical endpoints, and there are two crossovers at low baryon chemical potential. These results have important consequences for systems where both baryon and isospin chemical potentials are nonzero, such as heavy ion collision experiments. Our results are in complete agreement with those recently obtained in a random matrix model.

1. Introduction

QCD at nonzero temperature and baryon chemical potential plays a fundamental role in many different physical systems. Two important ones are neutron stars, which probe the low temperature and high baryon chemical potential domain, and heavy ion collision experiments, which probe the high temperature and low baryon chemical potential domain. In the last few years, various models have been used to predict the main characteristics of the QCD phase diagram at nonzero temperature and baryon chemical potential [1]. The existence of color superconducting phases at low temperature and high baryon chemical potential [2,3], as well as the presence of a tricritical point at intermediate temperature and baryon chemical potential [4,5] are among the most important results. These features have numerous phenomenological consequences. Some of the effects of a nonzero isospin chemical potential have also been studied, but only in the low temperature and high baryon chemical potential domain [6–11]. However systems such as heavy ion collision experiments that explore the high temperature and low baryon chemical potential domain also have a nonzero isospin chemical potential. Therefore, there is a clear need to study the effects of a nonzero isospin chemical potential on the whole QCD phase diagram at nonzero temperature and baryon chemical potential.

Most of our knowledge of the QCD phase diagram at nonzero temperature \( T \) is restricted to either zero baryon chemical potential \( \mu_B \), or to zero isospin chemical potential \( \mu_I \). At \( \mu_B = \mu_I = 0 \) numerical
lattice simulations and effective theories predict that
the ground state corresponds to a hadronic phase at
low temperature and to a quark–gluon-plasma phase
at high temperature. For nonzero quark masses, there
is no order parameter that distinguishes between these
two phases. For QCD with two flavors, a crossover
is expected at \( T \sim 170 \text{ MeV} \) [12]. This crossover
extends into the phase diagram at nonzero baryon and
isospin chemical potentials.

Numerical lattice simulations have explored the
high temperature and small baryon chemical potential
domain [13–18]. It was found that these high temper-
ature crossover lines at \( \mu_B \ll \Lambda_{\text{QCD}} \) and \( \mu_I = 0 \), and
at \( \mu_B = 0 \) and \( \mu_I \) much smaller than the pion mass are
identical [15,18].

At zero baryon chemical potential, both effective
theories and numerical lattice simulations predict the
existence of a superfluid pion condensation phase for
high enough \( \mu_I \) [18–24]. At zero temperature a second
order phase transition at a critical isospin chemical
potential, \( \mu_I^{\text{crit}} \), equal to half the pion mass separates
the hadronic phase from the pion condensation phase.
When the temperature is increased, this second order
phase transition line ends in a tricritical point and the
phase transition becomes first order [22,23].

At nonzero baryon chemical potential, standard
numerical lattice simulations do not work. Therefore
our knowledge of the QCD phase diagram at nonzero
\( T \) and \( \mu_B \) relies exclusively on effective theories, such
as Nambu–Jona-Lasinio and random matrix models.
At temperatures smaller than a few tens of MeV, an
increase in \( \mu_B \) leads to a crystalline LOFF phase
[7]. If \( \mu_B \) is further increased, the ground state
responds to a color superconductor. Both of these
phase transitions are of first order. If the temperature
is increased to a few tens of MeV the LOFF and
color superconducting phases disappear, and a first
order phase transition directly separates the hadronic
phase from the quark–gluon-plasma phase. At zero
quark mass, this first order line ends in a tricritical
endpoint at \( T \sim 100 \text{ MeV} \), where a second order phase
transition starts [4,5].

Finally, the only available theoretical study of the
whole phase diagram at nonzero \( T \), \( \mu_B \), and \( \mu_I \)
has been performed using a random matrix model
[24]. It was found that a small isospin chemical
potential induces two first order phase transitions
at low \( T \) that end in two critical endpoints, and
there are two crossovers at low \( \mu_B \). Because of the
phenomenological implications of these results, it is
essential to try to reproduce them within other models.

In this Letter, we study the QCD phase diagram at
nonzero \( T \), \( \mu_B \), and \( \mu_I \) for two quark flavors of equal mass
\( m \) in the Nambu–Jona-Lasinio model. As a first
step and for simplicity, we shall not study the domain
of low temperature and high baryon chemical potential
where the ground state corresponds to a LOFF crystal
or to a color superconductor. We shall also restrict
ourselves to small isospin chemical potential, \( \mu_I < \mu_I^{\text{crit}} \). A more complete analysis will be published
elsewhere.

### 2. Phase diagram in the Nambu–Jona-Lasinio

model

A crucial observation made in [24] is that when
both \( \mu_B \neq 0 \) and \( \mu_I \neq 0 \) in QCD, there is no reason to
expect that the quark–antiquark condensates are equal
for each flavor. Indeed if we define \( \mu_B = \frac{1}{2}(\mu_u + \mu_d) \)
and \( \mu_I = \frac{1}{2}(\mu_u - \mu_d) \), the QCD Lagrangian can be
written as

\[
\mathcal{L}_{\text{QCD}} = \sum_{f=u,d} \bar{\psi}_f (i\gamma \cdot \mathbf{D} - m + \mu_I \gamma_5) \psi_f, \tag{1}
\]

and there is no symmetry in the QCD Lagrangian that
constrains \( \langle \bar{u}u \rangle \) to be equal to \( \langle \bar{d}d \rangle \). Therefore, we
have to consider the quark–antiquark condensates for
each flavor separately. In this Letter, we shall concen-
trate on three observables: The quark–antiquark
condensates for each flavor \( \sigma_u = \langle \bar{u}u \rangle \) and \( \sigma_d = \langle \bar{d}d \rangle \),
as well as the pion condensate \( \rho = \frac{1}{2} \langle \bar{u} \gamma_5 d - \bar{d} \gamma_5 u \rangle \).

We use the same Nambu–Jona-Lasinio model with
an instanton-induced four-fermion interaction as Bergs
and Rajagopal who studied the QCD phase dia-
gram at nonzero \( T \) and \( \mu_B \) in [5]. After the
standard introduction of bosonic fields via a Hubbard–
Stratonovich transformation and integration over the
fermion fields [5,25], we find that the mean-field free
energy is given by

\[
\Omega = \frac{1}{8G_1} \left( \sigma_u^2 + \sigma_d^2 + 2\rho^2 \right) - \text{Tr} \log \begin{pmatrix}
\frac{h_u}{F^2(\hat{p})\rho \gamma_5} & -F^2(\hat{p})\rho \gamma_5 \\
-F^2(\hat{p})\rho \gamma_5 & \frac{h_d}{F^2(\hat{p})\rho \gamma_5}
\end{pmatrix}, \tag{2}
\]
where

\[ h_f = (i\omega_n + \mu_f)\gamma_0 + i\vec{p}\gamma_5 + m + F^2(\bar{p})\sigma_f, \]

(3)

with \( \omega_n = (2n + 1)\pi T \), and the form factor

\[ F(\bar{p}) = \frac{\Lambda^2}{\bar{p}^2 + \Lambda^2} \]

(4)

is introduced to mimic the effects of asymptotic freedom [5]. In the free energy (2), we have only kept the potentially nonvanishing condensates \( \sigma_u \) and \( \sigma_d \). We follow Berges and Rajagopal and take the scale \( \Lambda = 0.8 \) GeV and the coupling constant \( G_1 = 6.47/\Lambda^2 \) which are reasonable phenomenological choices [5]. They correspond to \( \sigma_u = \sigma_d = 0.4 \) GeV at \( m = T = \mu_B = \mu_I = 0 \).

In order to proceed we have to compute the excitation energies in (2), and in general solve a fourth order equation in \( \omega_n \). We have not been able to find a short expression for the general solution. We shall therefore analyze the most relevant particular cases separately. First at \( \mu_B = 0 \) (i.e., \( \mu_u = -\mu_d \)), relying on lattice simulations [18,20,21,24], we can assume that \( \sigma_u = \sigma_d = \sigma \).

We find that the excitation energies are given by

\[ E_{\pm} = \sqrt{(E \pm \mu_I)^2 + F^4(\bar{p})\rho^2}, \]

(5)

where \( E = \sqrt{\bar{p}^2 + (m + F^2(\bar{p})\sigma)^2} \). Thus the free energy (2) becomes

\[ \Omega = \frac{1}{4G_1} (\sigma^2 + \rho^2) \]

\[ -\frac{6}{\pi^2} \int_0^\infty dp p^2 [E_{\pm} + 2T \log(1 + e^{-E_{\pm}/T})]. \]

(6)

This is exactly the same free energy as the one that has been studied for the phase diagram of QCD with two colors in [26]. It leads to a Bose-Einstein condensation phase where \( \rho \neq 0 \) when \( \mu_I \) is larger than half the pion mass [26]. Therefore at \( \mu_B = 0 \), the Nambu–Jona-Lasinio model agrees with the results obtained in lattice simulations and in effective theories [18,20,21,24].

We then study \( \mu_B \neq 0 \) and \( \mu_I < \mu_I^{\text{crit}} \), which corresponds to the most relevant phenomenological situations, i.e., when the pion condensate vanishes. The free energy (2) then separates into a sum over each flavor

\[ \Omega = \sum_{f=u,d} \left( \frac{1}{8G_1} \sigma_f^2 - \frac{3}{\pi^2} \int_0^\infty dp p^2 \right) \]

\[ \times \left[ E_f^4 + 2T \log(1 + e^{-E_f^4/T}) \right], \]

(7)

where

\[ E_f^4 = \sqrt{\bar{p}^2 + (m + F^2(\bar{p})\sigma_f)^2} \pm \mu_f. \]

(8)

The free energy (7) has two remarkable properties. It is even in \( \mu_u \) and \( \mu_d \) separately and it can be expressed as a sum over the different quark flavors. Both of these properties are also found in the random matrix model studied in [24] and are essentially responsible for the striking changes in the phase diagram.

The evenness of the free energy in \( \mu_u \) and \( \mu_d \) implies that the crossover line that separates the hadronic phase from the quark–gluon-plasma phase in the \( \mu_B-T \) plane at \( \mu_I = 0 \) coincides with the corresponding crossover line in the \( \mu_I-T \) plane at \( \mu_B = 0 \). This property was indeed found in numerical lattice simulations [15,18].

Since \( \Omega = \sum_{f=u,d} \Omega_f (\mu_f) \), the free energy is minimized by minimizing each \( \Omega_f \) separately. But each \( \Omega_f \) is equal, up to a factor two, to the free energy studied by Berges and Rajagopal for zero diquark condensate in [5]. Therefore, for each flavor separately, the domains where \( \sigma_f \sim 0.4 \) GeV and \( \sigma_f \ll 0.4 \) GeV are separated by a first order line that ends in a critical endpoint where a crossover line starts. Now since \( \Omega_u(\mu_u) = \Omega_d(\mu_B + \mu_I) \) and \( \Omega_u(\mu_d) = \Omega_d(\mu_B - \mu_I) \), the whole phase diagram for QCD with two flavors in the \( \mu_B-T \) plane at nonzero \( \mu_I \) corresponds to a superposition of two of the usual phase diagrams at \( \mu_I = 0 \) shifted by \( 2\mu_I \). This is illustrated in Fig. 1 for \( m = 10 \) MeV and for both \( \mu_I = 0 \) and \( \mu_I = 30 \) MeV. There are now two first order lines that start at \( T = 0 \) and which end in a critical endpoint at \( T \sim 65 \) MeV. The temperature of the critical endpoint is not affected by the isospin chemical potential. Two crossovers emerge from the critical endpoints and intersect on the \( \mu_B = 0 \) axis at \( T \sim 200 \) MeV. Notice that the very low temperature part of this phase diagram cannot be trusted since we did not consider the possibility of a crystalline LOFF phase or a color superconductor. Nevertheless
Fig. 1. Phase diagram in the $\mu_B-T$ plane for a quark mass $m = 10$ MeV and an isospin chemical potential $\mu_I = 0$, and $\mu_I = 30$ MeV, respectively. At low temperature a first order phase transition takes place at the full line that ends in the critical endpoint. The dotted curves depict the crossover behavior. The condensates that are not displayed are of the order of the quark mass. This phase diagram can be trusted only above temperatures of a few tens of MeV since we did not consider the possibility of a crystalline LOFF phase or a color superconductor. The temperature of the critical endpoint is not affected by the isospin chemical potential.

3. Conclusions and discussion

In this Letter we have used a Nambu–Jona-Lasinio model to show that the introduction of an isospin chemical potential leads to qualitative changes in the QCD phase diagram at nonzero temperature and baryon chemical potential.

First, in agreement with lattice simulations [15,18] and the random matrix model analyzed in [24], we find that the crossover line at low $\mu_B$ and $\mu_I = 0$ is identical to the crossover line at low $\mu_I$ and $\mu_B = 0$.

Second, at low temperature, there are two first order phase transitions that end in two critical endpoints, and there are two crossovers at low baryon chemical potential. All these results are in complete agreement with the random matrix model studied in [24]. The new lattice techniques developed to study the small $\mu_B$ behavior at $\mu_I = 0$ can be used to study this pair of crossover lines.

The existence of this pair of crossover lines is important for heavy ion collision physics. First, as was shown in [27–29], the critical endpoints have definite signatures that can be observed in these experiments. The effect of the isospin chemical potential is twofold: it doubles the number of critical endpoints and pushes one of them to lower $\mu_B$. The second effect makes the presence of the critical endpoint easier to see in heavy ion collision experiments. Second, the transition from the quark–gluon-plasma to the hadronic phase will be softer at nonzero $\mu_I$ than at $\mu_I = 0$, since the system has to go through two crossover lines instead of one.

Finally if the strange quark is included, the hadronic phase and the quark–gluon-plasma phase are expected to be separated either by a crossover or by a first order phase transition at zero $\mu_I$, depending on the precise value of the strange quark mass [30]. The effect of a small strange quark mass is to push the critical endpoint towards higher $T$. We expect that the introduction of a nonzero $\mu_I$ at a physical value of the strange quark mass will generate two of these separation lines: either two crossovers or two first order phase transitions will be present at high temperature and low baryon chemical potential.

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