An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 42 (2001) 1431–1437
www.elsevier.nl/locate/camwa

Contribution to van der Waerden's Conjecture

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(Received September 2000; accepted October 2000)

Abstract—In this paper, we give two different elementary proofs for the inequality which states that the permanent of doubly stochastic matrices is greater than or equal to $(n!/n^n)$. This inequality was proved earlier by the author, and independently by Egorychev and Falikman. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Permanents, Stochastic matrices, Doubly stochastic matrices, van der Waerden's inequality.

1. INTRODUCTION

In this paper, we use the following notations. Let $\mathcal{M}$ be the set of the $n \times n$ matrices with real entries, where $n \geq 2$ is a fixed integer. Let $\mathcal{K}$ be the set of the column stochastic matrices. Let $\mathcal{H} \subset \mathcal{K}$ be the set of the doubly stochastic matrices. $A^* \in \mathcal{M}$ denotes the transpose of $A \in \mathcal{M}$.

Let the matrix

$$A := (a_{jk}) \in \mathcal{M}$$

be given. Then,

$$A_{i_1 \cdots i_k}^{j_1 \cdots j_l} := \begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_l} \\ \cdots & \cdots & \cdots \\ a_{i_k j_1} & \cdots & a_{i_k j_l} \end{pmatrix},$$

where

$$i_\alpha \ (\alpha = 1, \dots, k), \qquad j_\beta \ (\beta = 1, \dots, l),$$

are positive integers satisfying

$$1 \leq i_1 < \cdots < i_k \leq n, \qquad 1 \leq j_1 < \cdots < j_l \leq n,$$

and k and l run over the set $\{1, \dots, n\}$. Moreover, let

$$\operatorname{Per} A := \sum a_{1i_1} \cdots a_{ni_n}$$

be the permanent of A, where the summation is extended over all permutations. The properties of the permanents used in this paper can be found in the monograph [1].

Research was supported by the Hungarian Foundation for Scientific Research under Grant No. OTKA-1650/1991.

PII: S0898-1221(01)00252-8

The following conjecture was formulated by van der Waerden in 1926 [2]. If $A \in \mathcal{H}$, then

$$\operatorname{Per} A \geq \frac{n!}{n^n},$$

with equality if and only if $A = A_0$, where $A_0 \in \mathcal{H}$ is the matrix with all entries $a_{ij} = 1/n$.

The first complete proof of this conjecture was given by Egorychev in [3], and independently by Falikman in [4]. Their proof is based on a theorem of Alexandrov [5] which was generalised by London [6]. In the following, we shall refer to it as the van der Waerden-Egoroychev-Falikman (WEF) theorem.

In the following, we give two elementary proofs of the WEF theorem. These two proofs are quite different from that given in [3] and [4]. We shall not even use theorems of London and Alexandrov.

In the first proof, we shall use the following lemma [7, Lemma 4.1].

LEMMA 1.1. *Let us suppose that the real numbers a_j, b_j $(j = 1, \dots, n)$ satisfy the following monotonicity conditions:*

$$a_1 \geq \cdots \geq a_n, \qquad b_1 \geq \cdots \geq b_n.$$

Then, the inequality

$$\frac{1}{n}\sum_{j=1}^{n} a_j b_j \geq \frac{1}{n}\sum_{j=1}^{n} a_j \frac{1}{n}\sum_{j=1}^{n} b_j \geq \frac{1}{n}\sum_{j=1}^{n} a_j b_{n-j+1} \tag{1.1}$$

holds, with equality if and only if either $a_1 = a_n$, or $b_1 = b_n$.

One can explain inequality (1.1) saying that the maximum corresponds to "similar ordering" of $a_1, \dots, a_n$, and $b_1, \dots, b_n$, and the minimum to "opposite ordering" of them; some remarks with respect to this lemma can be found in the paper [8]. The second proof of WEF's theorem is based on its equivalency to the theorem of the author [9]. This remark shows that the proof of WEF's theorem was ready in 1977.

The structure of the paper is the following. Section 2 gives the first proof of WEF's theorem, and Section 3 the second one, respectively.

2. THE FIRST PROOF

By Theorem 3.1 of the paper [7], it is enough to prove the following.

THEOREM 2.1. *Equation*

$$\operatorname{Per} A = \frac{n!}{n^n}$$

has only one solution $A = A_0$ over $\mathcal{H}$.

PROOF. Let $A := (a_{jk}) \in \mathcal{H}$. Then,

$$\operatorname{Per} A = \sum_{k=1}^{n} a_{jk} \operatorname{Per} A_j^k,$$

where $1 \leq j \leq n$ is a fixed integer, and A_j^k is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the j^{th} row, and k^{th} column.

Let us denote now by permutations $\overline{a}_{j_1}, \dots, \overline{a}_{j_n}$ and $\underline{a}_{j_1}, \dots, \underline{a}_{j_n}$ of elements $a_{j_1}, \dots, a_{j_n}$ the similar ordering, and opposite ordering, respectively, with respect to the elements of the sequence

$$\operatorname{Per} A_j^1, \dots, \operatorname{Per} A_j^n.$$

Then,

$$\sum_{k=1}^{n} \overline{a}_{jk} \operatorname{Per} A_j^k \geq \operatorname{Per} A \geq \sum_{k=1}^{n} \underline{a}_{jk} \operatorname{Per} A_j^k$$

(see [10, Theorem 368]), and consequently by Lemma 1.1, the equality

$$\operatorname{Per} A = \frac{1}{n} \sum_{k=1}^{n} a_{jk} \sum_{k=1}^{n} \operatorname{Per} A_j^k \tag{2.1}$$

holds if and only if either

$$a_{j_1} = \cdots = a_{jn} = \frac{1}{n}, \tag{2.2}$$

or

$$\operatorname{Per} A_j^1 = \cdots = \operatorname{Per} A_j^n. \tag{2.3}$$

Applying this procedure for $j = 1, \ldots, n$, we get that equalities (2.2) and (2.3) must hold for $j = 1, \ldots, n$.

Let denote now the permutations $\overline{a}_{1k}, \ldots, \overline{a}_{nk}$ and $\underline{a}_{1k}, \ldots, \underline{a}_{nk}$ of the elements $a_{1k}, \ldots, a_{nk}$ by similar ordering, and opposite ordering, respectively, with respect to the elements of the sequence

$$\operatorname{Per} A_1^k, \ldots, \operatorname{Per} A_n^k.$$

Then, in the same way as above,

$$\sum_{j=1}^{n} \overline{a}_{jk} \operatorname{Per} A_j^k \geq \operatorname{Per} A \geq \sum_{j=1}^{n} \underline{a}_{jk} \operatorname{Per} A_j^k.$$

Consequently, equality

$$\operatorname{Per} A = \frac{1}{n} \sum_{j=1}^{n} a_{jk} \sum_{j=1}^{n} \operatorname{Per} A_j^k \tag{2.4}$$

holds if and only if either

$$a_{1k} = \cdots = a_{nk} = \frac{1}{n}, \tag{2.5}$$

or

$$\operatorname{Per} A_1^k = \cdots = \operatorname{Per} A_n^k \tag{2.6}$$

holds. Applying this procedure for $k = 1, \ldots, n$, we get that equalities (2.5) and (2.6) are satisfied for $k = 1, \ldots, n$.

Summing up, we get that equalities (2.1) and (2.4) are satisfied in the cases $j, k = 1, \ldots, n$ if and only if either $A = A_0$, or all the quantities

$$\operatorname{Per} A_j^k, \qquad j, k = 1, \ldots, n, \tag{2.7}$$

are equal.

From here, if all the quantities (2.7) are equal, then

$$\operatorname{Per} A_j^k = \operatorname{Per} A, \qquad j, k = 1, \ldots, n. \tag{2.8}$$

Consequently, it remains to show if condition (2.8) is satisfied, then $A = A_0$. ∎

THEOREM 2.2. *Let $A \in \mathcal{H}$. If condition* (2.8) *is satisfied, then* $A = A_0$.

PROOF. Let us write matrix A in the form

$$A = (A_{n-2}a_{n-1}a_n),$$

where

$$A_{n-2} = A_{1\cdots n}^{1\cdots n-2}, \tag{2.9}$$

and a_{n-1} and a_n are the $(n-1)^{\text{th}}$ and n^{th} column vector of A, respectively.

Let

$$A_{jk}, \qquad 1 \le j < k \le n, \tag{2.10}$$

be the permanents of the $(n-2) \times (n-2)$ matrix, which one can obtain from matrix (2.9) by deleting the j^{th} and k^{th} rows. Thus,

$$\begin{aligned} \operatorname{Per}(A_{n-2}a_{n-1}a_n) &= \sum_{1\le j<k\le n} \operatorname{Per}\begin{pmatrix} a_{jn-1} & a_{jn} \\ a_{kn-1} & a_{kn} \end{pmatrix} A_{jk}, \\ \operatorname{Per}(A_{n-2}a_{n-1}a_{n-1}) &= \sum_{1\le j<k\le n} \operatorname{Per}\begin{pmatrix} a_{jn-1} & a_{jn-1} \\ a_{kn-1} & a_{kn-1} \end{pmatrix} A_{jk}, \\ \operatorname{Per}(A_{n-2}a_n a_n) &= \sum_{1\le j<k\le n} \operatorname{Per}\begin{pmatrix} a_{jn} & a_{jn} \\ a_{kn} & a_{kn} \end{pmatrix} A_{jk}. \end{aligned} \tag{2.11}$$

Using condition (2.8),

$$\operatorname{Per}(A_{n-2}a_{n-1}a_n) = \operatorname{Per}(A_{n-2}a_{n-1}a_{n-1}) = \operatorname{Per}(A_{n-2}a_n a_n) = \operatorname{Per} A,$$

and consequently,

$$\operatorname{Per}(A_{n-2}a_{n-1}a_n) = \frac{1}{2}\left[\operatorname{Per}(A_{n-2}a_{n-1}a_{n-1}) + \operatorname{Per}(A_{n-2}a_n a_n)\right];$$

i.e.,

$$\Delta := \sum_{1\le j<k\le n} A_{jk}(a_{kn} - a_{kn-1})(a_{jn-1} - a_{jn}) = 0.$$

Let

$$\overline{\Delta} := \sum_{1\le j<k\le n} \overline{A}_{jk}\,(a_{kn} - a_{kn-1})(a_{jn-1} - a_{jn})$$

be the upper sum, and

$$\underline{\Delta} := \sum_{1\le j<k\le n} \underline{A}_{jk}(a_{kn} - a_{kn-1})(a_{jn-1} - a_{jn})$$

the lower sum of elements (2.10), with respect to the sequence

$$(a_{kn} - a_{kn-1})(a_{jn-1} - a_{jn}), \qquad 1 \le j < k \le n. \tag{2.12}$$

Consequently,

$$\overline{\Delta} \ge \Delta = 0 \ge \underline{\Delta}.$$

Moreover,

$$\overline{\Delta} \le \frac{1}{\binom{n}{k}} \sum_{1\le j<k\le n} A_{jk} \sum_{1\le j<k\le n} (a_{kn} - a_{kn-1})(a_{jn-1} - a_{jn}) \ge \underline{\Delta},$$

by Lemma 1.1, with equality if and only if either all the quantities (2.8), or all the quantities (2.12) are equal. In this case,

$$\sum_{1\le j<k\le n} A_{jk} \sum_{1\le j<k\le n} (a_{kn}-a_{kn-1})(a_{jn-1}-a_{jn}) = 0, \tag{2.13}$$

by Lemma 1.1. Since the numbers (2.10) are positive, we get that necessarily

$$\sum_{1\le j<k\le n} (a_{kn}-a_{kn-1})(a_{jn-1}-a_{jn}) = 0, \tag{2.14}$$

by (2.13), and all the quantities (2.12) are equal. On the other hand,

$$\begin{aligned}\sum_{1\le j<k\le n} &(a_{kn}-a_{kn-1})(a_{jn-1}-a_{jn}) \\ &= \frac{1}{2}\left\{\sum_{k=1}^{n}(a_{kn}-a_{kn-1})\sum_{j=1}^{n}(a_{jn-1}-a_{jn}) - \sum_{j=1}^{n}(a_{jn}-a_{jn-1})(a_{jn-1}-a_{jn})\right\} \\ &= \sum_{j=1}^{n}(a_{jn}-a_{jn-1})^2;\end{aligned}$$

i.e., we obtained that (2.14) holds if and only if $a_n = a_{n-1}$. Since a_{n-1}, a_n may be two arbitrary columns of A, we obtained that two arbitrary columns of A are equal; i.e., all columns of A are equal; i.e., $A = A_0$ if $A \in \mathcal{H}$. This means the condition that all the elements of (2.14) are equal is satisfied.

This finishes the proof of Theorem 2.2, and of Theorem 2.1, too. ∎

3. THE SECOND PROOF

In his paper [9], the author proved the following theorem. (This paper has a Hungarian version [11].)

If the row sums and the column sums of $A \in \mathcal{M}$ are equal to one, moreover

$$x \in R, \quad y \in R, \qquad x + y = 1,$$

then

$$x^2 \operatorname{Per}(AA^*)^{1/2} + y^2 \operatorname{Per}(A^*A)^{1/2} + 2xy \operatorname{Per} A \ge \frac{n!}{n^n},$$

with equality if and only if $A = A_0$.

We need the following particular case of this theorem.

THEOREM 3.1. *If $A \in \mathcal{H}$ and*

$$x \ge 0, \quad y \ge 0, \quad x + y = 1,$$

then the permanental equation

$$x^2 \operatorname{Per}(AA^*)^{1/2} + y^2 \operatorname{Per}(A^*A)^{1/2} + 2xy \operatorname{Per} A = \frac{n!}{n^n} \tag{3.1}$$

has only one solution $A = A_0$ over $\mathcal{H}$.

The suitable particular case of the WEF's theorem is the following.

THEOREM 3.2. *If $A \in H$, then the equation*

$$\operatorname{Per} A = \frac{n!}{n^n}$$

has only one solution $A = A_0$ over $\mathcal{H}$.

The aim of this section is to show the following theorem.

THEOREM 3.3. *If $A \in \mathcal{H}$, then Theorems 3.1 and 3.2 are equivalent.*

PROOF. First we show if Theorem 3.2 holds, then Theorem 3.1 holds, too. Namely, if Theorem 3.2 holds, then the WEF's theorem is valid, too. Consequently,

$$\operatorname{Per} A \geq \frac{n!}{n^n}, \qquad \operatorname{Per}(AA^*)^{1/2} \geq \frac{n!}{n^n}, \qquad \operatorname{Per}(A^*A)^{1/2} \geq \frac{n!}{n^n},$$

with equality in the three inequalities if and only if

$$A = (AA^*)^{1/2} = (A^*A)^{1/2} = A_0.$$

Consequently,

$$x^2 \operatorname{Per}(AA^*)^{1/2} + y^2 \operatorname{Per}(A^*A)^{1/2} + 2xy \operatorname{Per} A \geq (x+y)^2 \frac{n!}{n^n} = \frac{n!}{n^n},$$

with equality if and only if

$$\operatorname{Per}(AA^*)^{1/2} = \operatorname{Per}(A^*A)^{1/2} = \operatorname{Per} A = \frac{n!}{n^n},$$

i.e., if $A = A_0$ by Theorem 3.2. This means Theorem 3.2 contains Theorem 3.1.

We show now that conversely, Theorem 3.1 contains Theorem 3.2. Namely, identity (3.1) can be written in the form

$$x^2 \left[\operatorname{Per}(AA^*)^{1/2} + \operatorname{Per}(A^*A)^{1/2} - 2 \operatorname{Per} A \right] - 2x \left[\operatorname{Per}(A^*A)^{1/2} - \operatorname{Per} A \right] + \left[\operatorname{Per}(A^*A)^{1/2} - \frac{n!}{n^n} \right] \equiv 0.$$

Thus, (3.1) holds if and only if

$$\operatorname{Per} A = \frac{1}{2} \left[\operatorname{Per}(AA^*)^{1/2} + \operatorname{Per}(A^*A)^{1/2} \right],$$
$$\operatorname{Per} A = \operatorname{Per}(A^*A)^{1/2},$$
$$\operatorname{Per}(A^*A)^{1/2} = \frac{n!}{n^n}.$$

From here,

$$\operatorname{Per} A = \operatorname{Per}(AA^*)^{1/2} = \operatorname{Per}(A^*A)^{1/2}.$$

Substituting

$$\operatorname{Per}(AA^*)^{1/2} = \operatorname{Per} A, \qquad \operatorname{Per}(A^*A)^{1/2} = \operatorname{Per} A,$$

into polynomial (3.1), we get

$$\operatorname{Per} A = \frac{n!}{n^n},$$

which has the only solution $A = A_0$ over $\mathcal{H}$ by Theorem 3.1. Comparing the two proved statements, we get Theorem 3.2.

Since Theorem 3.2 contains the full WEF's theorem, we obtained a new proof of this theorem by Theorem 3.3. ∎

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