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Estimate of the solution of the Dirichlet problem for parabolic equations and applications

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Abstract

In the present paper we consider the Dirichlet problem for quasilinear nonuniformly parabolic equations. A new sufficient condition which guarantees the a priori estimate of the maximum of the modulus of the solution is formulated. A several applications of this estimate are given.

1. Introduction and main result

Consider the following problem:

$$u_t - a_{ij}(t, \mathbf{x}, u, \nabla u)u_{x_i x_j} = F(t, \mathbf{x}, u, \nabla u) \quad \text{in } Q_T = \Omega \times (0, T) \tag{1.1}$$

(we assume the usual summation convention),

$$u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad \text{in } \Omega \quad \text{and} \quad u = \chi(t, \mathbf{x}) \quad \text{on } S_T = \partial\Omega \times [0, T], \tag{1.2}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\nabla u = (u_{x_1}, \ldots, u_{x_n})$, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$. Without loss of generality suppose that Ω lies in the strip $-l_1 < x_1 < l_1$. Assume that the functions $a_{ij}(t, \mathbf{x}, u, \mathbf{p})$, $F(t, \mathbf{x}, u, \mathbf{p})$ are

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defined on the set $\overline{Q}_T \times \mathbf{R} \times \mathbf{R}^n$ take finite values for $(t, \mathbf{x}) \in Q_T$ and finite $u, \mathbf{p}$ and

$$a_{ij}(t, \mathbf{x}, u, \mathbf{p})\xi_i\xi_j \geqslant 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbf{R}^n,\ (t, \mathbf{x}, u, \mathbf{p}) \in \overline{Q}_T \times \mathbf{R} \times \mathbf{R}^n. \tag{1.3}$$

There are several sufficient conditions which guarantee the boundedness of a classical solution of problem (1.1), (1.2) (see [1–5]). Remind that a classical solution is a solution belonging to $C^0(\overline{Q}_T) \cap C^{1,2}_{t,\mathbf{x}}(Q_T)$. Here $C^{1,2}_{t,\mathbf{x}}(Q_T)$ is the set of functions having the first derivative with respect to t and the second derivatives with respect to $\mathbf{x}$ continuous in Q_T; $C^0(\overline{Q}_T)$ is the set of continuous in $\overline{Q}_T$ functions. In the present paper we give a new sufficient condition guaranteeing the a priori estimate of $|u|$.

Suppose that the right side of the equation can be represented in the form

$$F(t, \mathbf{x}, u, \mathbf{p}) = f_1(t, \mathbf{x}, u, \mathbf{p}) + f_2(t, \mathbf{x}, u, \mathbf{p}), \tag{1.4}$$

where f_1 and f_2 have different properties. Suppose that $f_1(t, \mathbf{x}, u, \mathbf{p})$ for $(t, \mathbf{x}) \in Q_T$ and any u, p_1 satisfies the following restriction:

$$\bigl|f_1(t, \mathbf{x}, u, p_1, 0, \dots, 0)\bigr| \leqslant a_{11}(t, \mathbf{x}, u, p_1, 0, \dots, 0)\psi\bigl(|p_1|\bigr). \tag{1.5}$$

Here $\psi(\rho)$ is a continuously differentiable function, $\psi(\rho) > 0$ for $\rho > 0$, $\psi(0) \geqslant 0$ and we assume that

$$\int_0^{+\infty} \frac{d\rho}{\psi(\rho)} > 2l_1. \tag{1.6}$$

In order to formulate the conditions on $f_2(t, \mathbf{x}, u, \mathbf{p})$ let us introduce the function $h(x_1)$ as a solution of the following problem:

$$h'' + \psi\bigl(|h'|\bigr) = 0, \qquad h(-l_1) = M, \qquad h(l_1) = H. \tag{1.7}$$

Here $M \geqslant m \equiv \max\{\sup_\Omega |\phi|, \sup_{S_T} |\chi|\}$, the constant H will be defined below. Represent the solution of problem (1.7) in parametric form using the substitution $q(h) = h'(x_1)$, $q'_{x_1}(h(x_1)) = q(h)q'(h)$:

$$h(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} + M, \qquad x_1(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)} - l_1,$$

where $q \in [q_0, q_1]$ and q_0, q_1 are chosen such that $0 < q_0 < q_1 < +\infty$ and

$$x_1(q) = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)} = 2l_1.$$

This is possible due to (1.6). Put

$$H = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} + M.$$

Assume that f_2 satisfies the following conditions:

$$f_2(t, \mathbf{x}, u, p_1, 0, \dots, 0) \leqslant 0 \quad \text{for } u \geqslant M,\ \ p_1 \in [q_0, q_1], \tag{1.8$_1$}$$

$$f_2(t, \mathbf{x}, u, -p_1, 0, \dots, 0) \geqslant 0 \quad \text{for } u \leqslant -M,\ \ p_1 \in [q_0, q_1]. \tag{1.8$_2$}$$

Let us formulate now the main result.

Theorem. *Let $u(t, \mathbf{x})$ be a classical solution of problem* (1.1), (1.2). *Suppose that conditions* (1.3)–(1.6), (1.8) *are fulfilled*; *then*

$$\sup_{Q_T} |u| \leqslant h(x_1) \leqslant H.$$

The proof will be given in the second section.

In the third section we give examples of applications of the theorem. In particular, from the theorem one can obtain the following fact. Consider the linear heat equation for the anisotropic media, i.e. we suppose that in different directions the heat conductivity is different. In x_i direction the heat conductivity coefficient is $a_{ii}(t, \mathbf{x}) \geqslant 0$:

$$u_t = \operatorname{div}(a_{11} u_{x_1}, \dots, a_{nn} u_{x_n}) + f(t, \mathbf{x}), \tag{1.9}$$

here $u(t, \mathbf{x}) \geqslant 0$ is an absolute temperature, $f(t, \mathbf{x}) \geqslant 0$ is a source. Consider problem (1.9), (1.2) and assume that $\phi(\mathbf{x}) \equiv \chi(t, \mathbf{x}) \equiv 0$. Suppose for simplicity that a_{11} is constant. From the theorem it follows that

$$\sup_{Q_T} |u| \leqslant \frac{3}{2} l_1^2 \frac{\sup f}{a_{11}}.$$

We observe here the phenomenon of cooling of a body for fixed l_1 and $f(t, \mathbf{x})$ when the heat conductivity increases in one direction. In fact, $u(t, \mathbf{x}) \to 0$ as $a_{11} \to +\infty$. Similar effect we have in the nonlinear case. This phenomenon has a simple physical interpretation (see Section 3).

Moreover, from the theorem we can easily obtain the standard a priori estimate of $\sup |u|$ for linear equation (see Section 3, estimate (3.4)) as well as for the nonlinear one (see Section 2, Remark 4).

2. Proof of the theorem

Let $u(t, \mathbf{x})$ be a classical solution of problem (1.1), (1.2). Define the operator L by the following:

$$L(u) \equiv u_t - A_{11}\bigl(u_{x_1x_1} + \psi\bigl(|u_{x_1}|\bigr)\bigr) - \sum_{i+j>2} A_{ij}u_{x_ix_j},$$

where $A_{ij}(t,\mathbf{x}) \equiv a_{ij}(t,\mathbf{x},u,\nabla u)$. Obviously $h'(x_1) = q \geqslant 0$ (see (1.7)) and hence $u - h$ is nonpositive on $\Omega \cup S_T$. It is clear that $L(h) = 0$ and thus for $w = u - h$ we have

$$\begin{aligned} L(u) - L(h) \equiv L_0(w) &\equiv w_t - A_{11}(w_{x_1x_1} + \beta w_{x_1}) - \sum_{i+j>2} A_{ij}w_{x_ix_j} \\ &= F(t,\mathbf{x},u,\nabla u) - a_{11}(t,\mathbf{x},u,\nabla u)\psi\bigl(|u_{x_1}|\bigr), \end{aligned}$$

where (from the mean value theorem)

$$\beta = \psi'(\rho^*)\frac{|u_{x_1}| - |h_{x_1}|}{u_{x_1} - h_{x_1}}.$$

Due to the fact that $\psi(\rho)$ is a continuously differentiable function and that $u(t,\mathbf{x})$ is a classical solution we conclude that $|\beta| \leqslant |\psi'(\rho^*)| < +\infty$ in $\overline{Q}_T \setminus (S_T \cup \Omega)$.

Consider the function $\widetilde{w} = we^{-t}$. One can easily see that

$$\begin{aligned} &\widetilde{w}_t + \widetilde{w} - A_{11}(\widetilde{w}_{x_1x_1} + \beta\widetilde{w}_{x_1}) - \sum_{i+j>2} A_{ij}\widetilde{w}_{x_ix_j} \\ &\quad = e^{-t}\bigl(F(t,\mathbf{x},u,\nabla u) - a_{11}(t,\mathbf{x},u,\nabla u)\psi\bigl(|u_{x_1}|\bigr)\bigr). \end{aligned}$$

Suppose that $\widetilde{w}$ achieves positive maximum at the point $N \in \overline{Q}_T \setminus (S_T \cup \Omega)$. Then at this point $\widetilde{w} > 0$ and $\nabla\widetilde{w} = 0$, i.e. $u > h \geqslant M$ and $u_{x_1} = h' > 0$, $u_{x_i} = 0$ for $i = 2,3,\ldots,n$. Due to (1.4), (1.5), (1.8_1) we have

$$\begin{aligned} &\widetilde{w}_t + \widetilde{w} - A_{11}(\widetilde{w}_{x_1x_1} + \beta\widetilde{w}_{x_1}) - \sum_{i+j>2} A_{ij}\widetilde{w}_{x_ix_j}|_N \\ &\quad = e^{-t}\bigl(f_1(t,\mathbf{x},u,u_{x_1},0,\ldots,0) - a_{11}(t,\mathbf{x},u,u_{x_1},0,\ldots,0)\psi\bigl(|u_{x_1}|\bigr) \\ &\qquad + f_2(t,\mathbf{x},u,h',0,\ldots,0)\bigr)\big|_N \leqslant 0. \end{aligned}$$

This contradicts the assumption that $\widetilde{w}$ attains positive maximum in $\overline{Q}_T \setminus (S_T \cup \Omega)$. From the nonpositivity of $\widetilde{w}$ on $S_T \cup \Omega$ we conclude that $\widetilde{w} \leqslant 0$ in $\overline{Q}_T$ and hence

$$w = u(t,\mathbf{x}) - h(x_1) \leqslant 0 \quad \text{in } \overline{Q}_T.$$

Now consider the function $v \equiv u + h$. Obviously v is nonnegative on $S_T \cup \Omega$, because $h' \geqslant 0$. Define operator L_1:

$$L_1(u) \equiv u_t - A_{ij}u_{x_ix_j}.$$

It is clear that

$$L_1(u) = F(t,\mathbf{x},u,\nabla u) \quad \text{and} \quad L_1(h) = a_{11}(t,\mathbf{x},u,\nabla u)\psi\bigl(|h'|\bigr).$$

For $\tilde{v} = ve^{-t}$ we have

$$\tilde{v}_t + \tilde{v} - A_{ij}\tilde{v}_{x_i x_j} = e^{-t}\big(f(t,\mathbf{x},u,\nabla u) + a_{11}(t,\mathbf{x},u,\nabla u)\psi\big(|h'|\big)\big).$$

Suppose that function $\tilde{v}$ attains negative minimum at the point $N \in \overline{Q}_T \setminus (S_T \cup \Omega)$. At this point $\tilde{v} < 0$ and $\nabla\tilde{v} = 0$, i.e. $u < -h \leqslant -M$ and $u_{x_1} = -h' < 0$, $u_{x_i} = 0$ for $i = 2, 3, \dots, n$. Due to (1.4), (1.5), (1.8_2) we obtain

$$\begin{aligned}
&\tilde{v}_t + \tilde{v} - A_{ij}\tilde{v}_{x_i x_j}|_N \\
&\quad = e^{-t}\big(f_1(t,\mathbf{x},u,-h',0,\dots,0) + a_{11}(t,\mathbf{x},u,-h',0,\dots,0)\psi\big(|h'|\big) \\
&\qquad\quad + f_2(t,\mathbf{x},u,-h',0,\dots,0)\big)\big|_N \geqslant 0.
\end{aligned}$$

This contradicts the assumption that $\tilde{v}$ attains negative minimum at N. Taking into account that $\tilde{v} = (u+h)e^{-t} \geqslant 0$ on $S_T \cup \Omega$, we conclude that $\tilde{v} \geqslant 0$ in $\overline{Q}_T$ and hence

$$v = u(t,\mathbf{x}) + h(x_1) \geqslant 0 \quad \text{in } \overline{Q}_T.$$

Thus we obtain that $|u(t,\mathbf{x})| \leqslant h(x_1) \leqslant h(l_1) \equiv H$. The theorem is proved.

Let us formulate several remarks.

Remark 1. Instead of conditions (1.8) we can take the following ones:

$$f_2(t,\mathbf{x},u,-p_1,0,\dots,0) \leqslant 0 \quad \text{for } u \geqslant M,\ \ p_1 \in [q_0,q_1], \tag{2.1_1}$$

$$f_2(t,\mathbf{x},u,p_1,0,\dots,0) \geqslant 0 \quad \text{for } u \leqslant -M,\ \ p_1 \in [q_0,q_1]. \tag{2.1_2}$$

In this case the barrier is a solution of the same equation as in (1.7) but with the other boundary conditions, namely $h(-l_1) = H$, $h(l_1) = M$. The estimate here is $\sup|u(t,\mathbf{x})| \leqslant h(x_1) \leqslant h(-l_1) \equiv H$.

Remark 2. If conditions (1.8) and (2.1) are fulfilled then $\sup|u| \leqslant h(0)$.

Remark 3. The choice of the quantity H in (1.7) actually results from the necessity of the fulfillment of condition $h'(x_1) > 0$ for $|x_1| < l_1$.

Consider the following problem. Let $f_1 = f_1(t,\mathbf{x})$ and $a_{11} \equiv 1$. Denote by f_0 the $\sup|f_1(t,\mathbf{x})|$. As a barrier $h(x_1)$ we take the solution of the equation $h'' = -f_0$. The first boundary condition is $h(-l_1) = M$, instead of the second one we take the condition $h'(x_1) > 0$ for $|x_1| \leqslant l_1$. We obtain

$$h(x_1) = -\frac{x_1^2}{2}f_0 + l_1 f_0 x_1 + M + \frac{3l_1^2}{2}f_0.$$

The estimate in this case takes the following form

$$\sup|u| \leqslant h(0) = M + 2l_1^2 f_0.$$

Remark 4. Assume that $f_1(t, \mathbf{x}, u, \mathbf{0}) \equiv 0$ and conditions (1.3), (1.4) are fulfilled. Suppose that for some constant $M \geqslant m$

$$uf_2(t, \mathbf{x}, u, 0, 0, \ldots, 0) \leqslant 0 \quad \text{for } |u| > M.$$

Then

$$\sup_{Q_T} |u| \leqslant M.$$

In fact, here as a barrier we can take $h \equiv M$. Obviously $w^0 \equiv (u - M)e^{-t} \leqslant 0$ on $S_T \cup \Omega$. Moreover

$$L_2(w^0) \equiv w_t^0 + w^0 - A_{ij} w_{x_i x_j}^0 = e^{-t} f(t, \mathbf{x}, u, \nabla u).$$

Suppose that w^0 achieves positive maximum at the point $N \in \overline{Q}_T \setminus (S \cup \Omega)$. Then at this point $w^0 > 0$ and $\nabla w^0 = 0$, i.e. $u > M \geqslant 0$ and $\nabla u = 0$. Due to (1.4), (1.5), (1.8_1) we have

$$w_t^0 + w^0 - A_{ij} w_{x_i x_j}^0 \big|_N = e^{-t} f_2(t, \mathbf{x}, u, 0, 0, \ldots, 0)\big|_N \leqslant 0.$$

This contradicts the assumption that w^0 attains positive maximum in $\overline{Q}_T \setminus (S_T \cup \Omega)$. From the nonpositivity of w^0 on $S_T \cup \Omega$ we conclude that $w^0 \leqslant 0$ in $\overline{Q}_T$ and hence $u - h \leqslant 0$ in $\overline{Q}_T$.

Function $v^0 \equiv (u + M)e^{-t}$ is nonnegative on $S_T \cup \Omega$. It is clear that

$$v_t^0 + v^0 - A_{ij} v_{x_i x_j}^0 = e^{-t} F(t, \mathbf{x}, u, \nabla u).$$

Suppose that function v^0 attains negative minimum at the point $N \in \overline{Q}_T \setminus (S_T \cup \Omega)$. At this point $v^0 < 0$ and $\nabla v^0 = 0$, i.e. $u < -M \leqslant 0$ and $\nabla u = 0$. Due to (1.4), (1.5), (1.8_2) we obtain

$$v_t^0 + v^0 - A_{ij} v_{x_i x_j}^0 \big|_N = e^{-t} f_2(t, \mathbf{x}, u, \mathbf{0})|_N \geqslant 0.$$

This contradicts the assumption that v^0 attains negative minimum at N. Taking into account that $v^0 \geqslant 0$ on $S_T \cup \Omega$, we conclude that $v^0 \geqslant 0$ in $\overline{Q}_T$ and hence $u \geqslant -M$ in $\overline{Q}_T$.

So we obtain that $|u(t, \mathbf{x})| \leqslant M$.

3. Examples

Let us first consider the linear equation

$$u_t - a_{ij}(t, \mathbf{x}) u_{x_i x_j} = f(t, \mathbf{x}) + b_i(t, \mathbf{x}) u_{x_i} + c(t, \mathbf{x}) u \quad \text{in } Q_T = \Omega \times (0, T). \tag{3.1}$$

Denote

$$f_1 \equiv f(t,\mathbf{x}) + b_i(t,\mathbf{x})p_i, \qquad f_2 \equiv c(t,\mathbf{x})u,$$

assume that

$$a_{11}(t,\mathbf{x}) \geqslant a_0 > 0 \quad \text{and} \quad c(t,\mathbf{x}) \leqslant 0,$$

where a_0 is some positive constant. One can easily see that conditions (1.8) as well as (2.1) are fulfilled. Condition (1.5) is satisfied with $\psi(|p_1|) \equiv K(1+|p_1|)$, where $K = \max\{\sup|f|, \sup|b_1|\}a_0^{-1}$. In this case we can easily construct the barrier in the explicit form

$$h(x_1) = M - l_1 + e^{2Kl_1}K^{-1} - e^{K(l_1-x_1)}K^{-1} - x_1$$

as a solution of the problem

$$h''(x_1) + K\left|h'(x_1)\right| = -K, \qquad h(-l_1) = M,$$
$$h(l_1) = M - 2l_1 + K^{-1}\left(e^{2Kl_1} - 1\right).$$

Instead of the boundary condition at l_1 we can take condition $h'(x_1) > 0$ for $|x_1| < l_1$ (see Remark 3). Thus for the solution of problem (3.1), (1.2) from Remark 2 we obtain that

$$\sup_{Q_T}|u| \leqslant h(0) = M - l_1 + e^{Kl_1}K^{-1}\left(e^{Kl_1} - 1\right). \tag{3.2}$$

If in Eq. (3.1) the coefficient $b_1 \equiv 0$ then as a function ψ we can take the constant $K_1 = \sup|f|a_0^{-1}$, i.e. $h''(x_1) = -K_1$ and hence $h(x_1) = M - x_1^2K_1/2 + K_1l_1x_1 + 3K_1l_1^2/2$. The estimate has the following form:

$$\sup_{Q_T}|u| \leqslant h(0) = \frac{3}{2}K_1l_1^2 + M. \tag{3.2$_1$}$$

If $c(t,\mathbf{x}) \leqslant \lambda$, where λ is some positive constant then $|u| \leqslant h(0)e^{\lambda T}$.

Consider the heat equation under the assumption that the coefficient of heat conductivity is different in different directions

$$u_t - \operatorname{div}(a_{11}u_{x_1}, \ldots, a_{nn}u_{x_n}) = f(t,\mathbf{x}). \tag{3.3}$$

Here $u(t,\mathbf{x})$ is temperature. For the solution of problem (3.3), (1.2) the estimate (3.2) holds, here $b_1 = a_{11x_1}$. If $a_{11x_1} = 0$ then estimate (3.2$_1$) is valid.

Let us mention here that estimate (3.2) ((3.2$_1$)) depends not on the intensity of the source $f(t,\mathbf{x})$ but on the ratio

$$\max\left\{\sup|f|, \sup|b_1|\right\} \Big/ \inf a_{11} \quad \left(\sup|f| / \inf a_{11}\right),$$

where a_{11} is a coefficient of the heat conductivity in the x_1 direction.

Let $m = 0$. For the arbitrary fixed source f by choosing the constant a_0 to be sufficiently big we can make $|u|$ arbitrary small. As it has been already mentioned in the first section this fact has a simple physical explanation. The heat flow through the boundary in the x_1 direction increases as $a_0 = \inf a_{11}$ increases assisting to the cooling of the body. Remind that according to the Fourier law the heat flow through the boundary $S_T \equiv \partial\Omega \times (0,T)$ is given by the integral

$$\int_{S_T} (a_{11}u_{x_1}, \dots, a_{nn}u_{x_n}) \cdot \mathbf{n}\, d\mathbf{x}\, dt,$$

where $\mathbf{n}$ is an external normal vector to the boundary.

If on the boundary we require the absence of the heat flow $\partial u/\partial \mathbf{n} = 0$ (homogeneous Neumann problem), then the value of u is determined only by the intensity of the sources and by the initial data and does not depend on the conductivity of the media.

Using the theorem we can obtain an estimate independent of the coefficients of the principal part of the equation. Consider Eq. (3.1), let

$$a_{ij}\xi_i\xi_j \geqslant 0 \quad \text{and} \quad c(t,\mathbf{x}) \leqslant -c_0 < 0.$$

Put

$$f_1 \equiv 0, \qquad f_2 \equiv f(t,\mathbf{x}) + b_i(t,\mathbf{x})p_i + c(t,\mathbf{x})u.$$

As a barrier h here we take the constant $M = \max\{m, \sup|f|/c_0\}$. In that case obviously

$$f_2(t,\mathbf{x},u,h',0,\dots,0) \equiv f_2(t,\mathbf{x},u,\mathbf{0}) \leqslant 0 \quad \text{for } u > M,$$
$$f_2(t,\mathbf{x},u,-h',0,\dots,0) \equiv f_2(t,\mathbf{x},u,\mathbf{0}) \geqslant 0 \quad \text{for } u < -M.$$

Hence we conclude that for the solution of problem (3.1), (1.2) the following estimate takes place:

$$\sup_{Q_T}|u| \leqslant \max\left\{m, \frac{\sup|f|}{c_0}\right\}.$$

If $c(t,\mathbf{x}) < \lambda$, where λ is a positive constant, then

$$\sup_{Q_T}|u| \leqslant \inf_{\lambda > c_0}\left(e^{\lambda T}\max\left\{m, \frac{\sup|f|}{\lambda - c_0}\right\}\right), \tag{3.4}$$

where $c_0 = \sup c(t,\mathbf{x})$. This is a standard a priori estimate for the solution of problem (3.1), (3.2).

Let us pass to the nonlinear case. Consider the following semilinear equation

$$u_t - k(u)\Delta u = Q(u), \quad k(u) > 0. \tag{3.5}$$

We suppose that $Q(u)$ does not satisfy condition $uQ(u) \leqslant 0$ for $|u| > M$. It is well known that generally speaking the solution of the Dirichlet problem for that equation blows-up, i.e. there exists $t^* < +\infty$ such that $\sup_{\mathbf{x}} |u(t,\mathbf{x})| \to +\infty$ when $t \to t^*$.

From the theorem it follows that if in (3.5) function $Q(u)$ satisfies the inequality $|Q(u)| \leqslant C_0 k(u)$, where C_0 is some positive constant, then the solution is bounded for all $t > 0$. If instead of (3.5) we consider equation

$$u_t - k_i(u) u_{x_i x_i} = Q(u), \quad k_i(u) > 0, \quad i = 1, \dots, n,$$

then for the boundedness of the solution it is sufficient to require the fulfillment of the inequality $|Q(u)| \leqslant C_0 k_i(u)$ only for one value of index i.

Consider the nonlinear heat equation

$$u_t - \operatorname{div}\bigl(k(u)\nabla u\bigr) = Q(u), \tag{3.6}$$

where $k(u) > 0$ is continuously differentiable function.

Write this equation in the following form:

$$u_t - k(u)\Delta u = f_1,$$

where $f_1 \equiv Q(u) + k'(u)|\nabla u|^2$.

Suppose that $|Q(u)| \leqslant C_0 k(u)$ and $|k'(u)| \leqslant C_0 k(u)$. In that case condition (1.5) is fulfilled with $\psi(|p_1|) = C_0(1 + p_1^2)$. The integral in (1.6) is equal to $\pi/2C_0$. Thus in order to obtain the estimate of $|u|$ we require the constant C_0 to be less than $\pi/4l_1$.

If $uk'(u) \leqslant 0$ then we write Eq. (3.6) in the form

$$u_t - k(u)\Delta u = Q(u) + f_2,$$

where $f_2 \equiv k'(u)|\nabla u|^2$. In that case we obtain the estimate of $|u|$ without supplementary assumptions on C_0.

Suppose that $Q(z) \geqslant 0$ when $z \leqslant 1$ and $u \geqslant 1$ on the parabolic boundary of the domain. One can easily see that in this case $u(t,\mathbf{x}) \geqslant 1$. In fact, let

$$Lu \equiv u_t - \tilde{k}\Delta u - \tilde{k}_1|\nabla u|^2,$$

where $\tilde{k} = \tilde{k}(t,\mathbf{x}) = k(u)$, $\tilde{k}_1 = \tilde{k}_1(t,\mathbf{x}) = k'(u)$. Obviously $Lu = Q(u)$. For $v \equiv u - 1$ we obtain

$$Lu \equiv v_t - \tilde{k}\Delta v - \tilde{k}_1|\nabla v|^2 = Q(u).$$

We conclude that v cannot attain negative values at the internal points of the domain. Due to the fact that $v \geqslant 0$ on the parabolic boundary of the domain we conclude that $v \geqslant 0$ in the whole domain.

Consider the case when the coefficient of the heat conductivity has the form $k(u) = k_0 u^\alpha$, where $k_0 > 0$ and $\alpha > 0$ are some constants. Suppose that $Q(z) \geqslant 0$ for $z \leqslant 1$ and $u \geqslant 1$ on the parabolic boundary of the domain. Obviously $k'(u) \leqslant \alpha k(u)$. Let $|Q(u)| \leqslant \alpha k(u)$. We put $\psi(|p_1|) = \alpha(1 + p_1^2)$, to fulfill the conditions of the theorem it is necessary for α to be less than $\pi/4l_1$.

Similarly we can investigate the following equation (anisotropic case):

$$u_t - \operatorname{div}\big(k_1(u)u_{x_1}, \dots, k_n(u)u_{x_n}\big) = Q(u).$$

Let us demonstrate the application of the theorem on one more example. Consider the following problem (for simplicity we restrict ourselves by one-dimensional case):

$$u_t - u_{xx} = l^2 u_x^2 - u^2 \quad \text{in } Q_T = (-l, l) \times (0, T), \tag{3.7}$$

$$u(0, x) = u_0(x), \qquad u(t, \pm l) = 0, \qquad u_0(\pm l) = 0. \tag{3.8}$$

Suppose that $|u_{0x}(x)| \leqslant K$ and consequently $|u_0(x)| \leqslant K(l - |x|)$. Consider the auxiliary equation

$$u_t - u_{xx} = l^2 u_x^2 - f(u) \quad \text{in } Q_T \tag{3.9}$$

with conditions (3.8), where

$$f(u) = \begin{cases} u^2, & \text{for } |u| \leqslant Kl, \\ K^2 l^2, & \text{for } |u| > Kl. \end{cases}$$

Let us obtain the estimate $|u(t, x)| \leqslant Kl$ for the solution of auxiliary problem (3.9), (3.8). Consider the function $v(t, x) \equiv u(t, x) + h(x)$, where $h(x) = K \cdot (l + x)$. It is clear that

$$v_t - v_{xx} = l^2 u_x^2 - f(u).$$

For $\tilde{v}(t, x) = v(t, x)e^{-t}$ we obtain

$$\tilde{v}_t + \tilde{v} - \tilde{v}_{xx} = \big(l^2 u_x^2 - f(u)\big)e^{-t}.$$

If the function $\tilde{v}$ attains negative minimum at the point $N \in \overline{Q}_T \setminus \Gamma$ (Γ is parabolic boundary of the domain Q_T), then at this point $\tilde{v}_x = 0$, i.e. $u_x = -h' = -K$ and hence

$$\tilde{v}_t + \tilde{v} - \tilde{v}_{xx}|_N = \big(l^2 K^2 - f(u)\big)e^{-t}\big|_N \geqslant 0.$$

This contradicts the assumption that at the point N we have negative minimum. It is clear that on Γ the function $\tilde{v}$ is nonnegative, hence $\tilde{v} \geqslant 0$ in $\overline{Q}_T$. This gives us the estimate

$$u(t, x) \geqslant -K(l + x). \tag{3.10}$$

Now consider the function $w(t, x) \equiv u(t, x) + h_1(x)$ where $h_1(x) = K(l - x)$. For $\widetilde{w}(t, x) = w(t, x)e^{-t}$ we obtain

$$\widetilde{w}_t + \widetilde{w} - \widetilde{w}_{xx} = \big(l^2 u_x^2 - f(u)\big)e^{-t}.$$

At the point $N_1 \in \overline{Q}_T \setminus \Gamma$ of the negative minimum of function $\widetilde{w}$ we have $u_x = h' = K$ and

$$\widetilde{w}_t + \widetilde{w} - \widetilde{w}_{xx}|_{N_1} = \big(l^2K^2 - f(u)\big)e^{-t}\big|_{N_1} \geqslant 0.$$

From this contradiction and from the fact that $\widetilde{w} \geqslant 0$ on the parabolic boundary of the domain Q_T we conclude that $\widetilde{w} \geqslant 0$ in $\overline{Q}_T$. Hence

$$u(t,x) \geqslant -K(l-x). \tag{3.11}$$

From (3.10) and (3.11) we obtain

$$u(t,x) \geqslant -K\big(l-|x|\big) \geqslant -Kl.$$

Similarly we can obtain the estimate $u \leqslant Kl$. As a consequence we conclude that Eqs. (3.9) and (3.7) coincide and the estimate $|u(t,x)| \leqslant M = Kl$ holds as well for the solution of problem (3.7), (3.8).

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