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On Continuity of Metric Projections

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1. INTRODUCTION

In this note we collect a number of complements and extensions to some known results due to Panda and Kapoor [12], Oshman [10], and Holmes and Kripke [6], and also point out an error in the main result of [8]. These results are all concerned with the common theme of continuity of metric projections.

Let X be a (real or complex) normed linear space and M a subset of X. The *metric projection* onto M is the mapping $P_M : X \to 2^M$ which associates with each x in X its (possibly empty) set of nearest points in M. Thus

$$P_M(x) = \{m \in M: \| x - m \| = d(x, M)\},$$

where $d(x, M) = \inf\{\| x - y \|: y \in M\}$. M is called *proximinal* (resp., *Chebychev* if $P_M(x)$ contains at least (resp., exactly) one point for each x in X. A sequence (y_n) in M is called *minimizing* for x in X if $\lim \| x - y_n \| = d(x, M)$. Recall that a normed linear space X is rotund (R) or strictly convex provided the boundary of the unit ball in X contains no line segements.

Vlasov [16] introduced the concept of compact local uniform rotundity (CLUR). Oshman [11] and, independently, Panda and Kapoor [12] have shown that in CLUR space the class of Chebychev sets with continuous metric projections coincides with the class of approximatively compact Chebychev sets. In Section 2, we extend the results of [12] to a more general setting. For example (Theorem 2.6), in a CLUR space, the metric projection onto a proximinal set is compact-valued and upper semicontinuous on a dense set.

In Section 3, we consider a class of β-suns investigated by Oshman [10], enlarge this class somewhat, and show that each member of this class has a continuous metric projection irrespective of the gemometry of the Banach space.

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Section 4 contains a few complements to some results of Holmes and Kripke [6] concerning linear metric projections. For example (Proposition 4) for each $1 \leqslant r \leqslant 2$, we can construct a Banach space X and a subspace M of X so that M is Chebyshev, P_M is linear, $\| P_M \| = r$, and the kernel of P_M is also a Chebyshev subspace with a continuous metric projection.

In Section 5, we point out that the "Klee" space does not have the property attributed to it in [8]. Thus several questions and conjectures concerning the continuity of metric projections (which this space apparently settled) are still unresolved.

We will use the following notation throughout. If X is a normed linear space, then

$$S(X) = \{x \in X: \| x \| = 1\}.$$

A word about the organization of this paper. With the exception of some definitions given in early sections and used in later ones, each section is independent of the others, and thus can be read in any order.

2. Properties of CLUR Spaces

In this section we note that the main results of Panda and Kapoor [12] in CLUR spaces can be extended to the setting of proximinal (rather than Chebychev) sets, and outer radially lower semicontinuous (rather than continuous) metric projections.

A normed linear space X is called *compactly locally uniformly rotund* (CLUR) if x, x_n in $S(X)$ and $\| x_n + x \| \to 2$ imply that (x_n) has a convergent subsequence.

Clearly, every finite-dimensional space is CLUR, and CLUR generalizes the notion of a locally uniformly rotund (LUR) space. Indeed, it is easy to verify that X is LUR if and only if X is CLUR and (R). (In [12], CLUR was called property (M). However, we conform to the earlier designation [16, 17].)

Definition 2.1. Let K be a subset of X. The metric projection P_K is said to be outer radially lower semicontinuous (o.r.l.s.c.) at x_0 if for every y_0 in $P_K(x_0)$ and each open set W with $W \cap P_K(x_0) \neq \varnothing$ there exists a neighborhood U of x_0 such that $W \cap P_K(x) \neq \varnothing$ for every x in $U \cap \{x_0 + \lambda(x - y_0): \lambda \geqslant 0\}$. P_K is called o.r.l.s.c. if it is o.r.l.s.c. at each point of x.

This generalization of lower semicontinuity was introduced and studied in [3, 4] (where it was called "ORL continuity"). It was shown there, for example, that every sun has an o.r.l.s.c. metric projection; and in spaces of type $C(T)$ or $l_1(\Gamma)$, the converse is valid.

By making the appropriate modifications of the proof of Theorem 1 in [12]—using Blatter's generalization [2, Lemma 4] of a result of Vlasov [16, Lemma 1]—we can prove the following result.

THEOREM 2.2. *Let X have property CLUR and let K be a proximinal subset. If P_K is o.r.l.s.c. and P_K is compact-valued, then K is approximatively compact.*

Panda and Kapoor [12, Theorem 1] proved this in the special case when K is a Chebychev set with a continuous metric projection. Since every apprximatively compact Chebychev set has a continuous metric projection (Singer [13]), we obtain

COROLLARY 2.3. *Let K be a Chebychev subset of a space having property CLUR. The following-statements are equivalent.*

(1) *K is approximatively compact,*

(2) *P_K is continuous,*

(3) *P_K is-o.r.l.s.c.*

Oshman [11] and, independently, Panda and Kapoor [12] proved the implication (2) $\Rightarrow$ (1). We do not know whether the converse of Theorem 2.2 is valid. However, the above corollary shows that the converse *is* valid for Chebychev sets.

DEFINITION 2.4. Let K be a subset of X. P_K is upper semicontinuous (u.s.c.) at x_0 if for each open set W with $P_K(x_0) \subset W$, there exists a neighborhood U of x_0 such that $P_K(x) \subset W$ for every x in U. P_K is called u.s.c. if it is u.s.c. at each point of X.

For Chebychev sets, u.s.c. and continuity are obviously identical.

LEMMA 2.5. *Let K be a proximinal subset of X, $x \in X$, and suppose every minimizing sequence for x has a convergent subsequence. Then P_K is u.s.c. at x and $P_K(x)$ is compact.*

Proof. That $P_K(x)$ is compact is obvious since every sequence in $P_K(x)$ is minimizing. If P_K were not u.s.c. at x, there exist an open set $W \supset P_K(x)$ and a sequence $x_n \to x$ such that $P_K(x_n)\backslash W = \varnothing$. Choose $y_n \in P_K(x_n)\backslash W$. Then

$$\| x - y_n \| \leqslant \| x - x_n \| + \| x_n - y_n \| = \| x - x_n \| + d(x_n, K) \to d(x, K)$$

so (y_n) is minimizing for x. Let (y_{n_k}) be a subsequence with $y_{n_k} \to y_0$. Then $y_0 \in K$ and, from the above inequality, $\| x - y_0 \| \leqslant d(x, K)$. Hence $y_0 \in P_K(x) \subset W$, so $y_{n_k} \in W$ eventually, a contradiction. ∎

In particular (Singer [13]): Every approximatively compact set has a u.s.c. metric projection. The main result of this section is

THEOREM 2.6. *Let X be a CLUR space, K a proximinal subset, $x \in X$, $y \in P_K(x)$, $0 < \lambda < 1$, and $z = \lambda x + (1 - \lambda) y$. Then:*

(1) *every minimizing sequence for z has a convergent subsequence,*

(2) *P_K is u.s.c. at z,*

(3) *$P_K(z)$ is compact.*

In particular, there is a dense subset of X on which P_K is u.s.c. and compact-valued.

Proof. In view of Lemma 2.5, it suffices to prove (1). We may assume $x \in K$. Note that $y \notin P_K(z)$. For each n, let

$$K_n = \{u \in X: \| u - z \| \leqslant \| z - y \| + 1/n, \| u - x \| \geqslant \| x - y \|\}.$$

Let (y_n) in K be a minimizing sequence for z. Then there is a subsequence $(y_{m(n)})$ such that $y_{m(n)} \in K_n$ for each n. By Lemma 2 of [12], $(y_{m(n)})$ (and hence (y_n) also) has a convergent subsequence. ∎

COROLLARY 2.7 (Panda and Kapoor [12]). *Let X be a CLUR space. Then the metric projection onto each Chebychev set is continuous on a dense subset of X.*

COROLLARY 2.8 (Panda and Kapoor [12]). *In a CLUR space, every proximinal β-sun is approximatively compact.*

Proof. Let K be a proximinal β-sun in X and $z \in X \backslash K$. Choose $y \in P_K(z)$ so that $y \in P_K(z + \lambda(z - y))$ for all $\lambda \geqslant 0$. Given $\lambda_0 > 0$, let $x = z + \lambda_0 \times (z - y)$. Then z is a convex combination of x and y and Theorem 2.6 implies every minimizing sequence for z has a convergent subsequence. Thus K is approximatively compact. ∎

It is natural to ask whether an approximation theoretic characterization of CLUR spaces exists. For example, are these spaces characterized by the property that the class of approximatively compact sets coincides with the class of proximinal sets having u.s.c. (os o.r.l.s.c.) and compact-valued metric projections?

3. A CLASS OF SUNS

A normed linear space X is called NLUR (Oshman [9]) if (x_n) in $S(X)$, (f_n), $(f_0^{(n)})$ in $S(X^*)$, $f_n(x_n) = 1 = f_0^{(n)}(x_0)$ $(n = 0, 1, 2, \ldots)$, $d(x_0, H_0^{(n)} \cap H_n) \to 0$, and $d(x_n, H_0^{(n)} \cap H_n) \to 0$ imply $x_n \to x_0$, where

$$H_0^{(n)} = \{x \in X: f_0^{(n)}(x) = 1\} \qquad \text{and} \qquad H_n = \{x \in X: f_n(x) = 1\}.$$

Oshman [9] essentially showed that in a reflexive rotund space with NLUR, the metric projection onto every Chebychev β-sun is continuous. It is not known whether the converse is true. (The converse is false without the assumption of rotundity. This follows by observing that each non-rotund fiinite-dimensional space is CLUR but not NLUR and then applying Corollary 2.8, above.) As an initial attempt to obtain a (negative) answer to this question we investigate the class of β-suns studied by Oshman [10, Lemma 2]. We show that these β-suns—indeed, an even larger class of β-suns—always have continuous metric projections. The essential ideas of the proof of Lemma 3.1 can be found in [10, Lemma 2]. Since the proof is short, we have included it here for completeness.

For the remainder of this section, unless otherwise stated, we fix a normed linear space X, $x_0 \in S(X)$, and a weak* compact subset $\{f_i : i \in I\}$ of the set $\{f \in S(X^*): f(x_0) = 1\}$. Let

$$V = \bigcup_{i \in I} \{y \in X: f_i(y) \geqslant 1\}.$$

LEMMA 3.1. *V is a proximinal β-sun. More explicitly, $x + d(x)\, x_0 \in P_V(x)$ for each $x \in X \backslash V$, where $d(x) = inf_i[1 - f_i(x)]$.*

Proof. Since V is the union of the closed half spaces $H_i = (y \in X: f_i(y) \geqslant 1\}$ $(i \in I)$, it follows that for each $x \in X \backslash V$, $d(x, V) = \inf_i d(x, H_i) = \inf_i[1 - f_i(x)] = d(x)$.

Let $y = x + d(x)\, x_0$. Then $\| y - x \| = d(x) = d(x, V)$. Thus to show $y \in P_V(x)$, we need only show $y \in V$. By weak* compactness we can choose an index $i_0 \in I$ so that $d(x) = 1 - f_{i_0}(x)$. Then

$$f_{i_0}(y) = f_{i_0}(x) + d(x) = 1,$$

so $y \in V$. In particular, V is proximinal.

It remains to show that V is a β-sun. Let $x \in X \backslash V$ and $x' = x + d(x)\, x_0$. By the preceeding argument, $x' \in P_V(x)$. Let $\lambda = x' + \lambda(x - x')$. We will show that $x' \in P_V(x_\lambda)$. For each $v \in V$, choose $i \in I$ so that $f_i(v) \geqslant 1$. Then

$$\begin{aligned} \| x_\lambda - x' \| &= \lambda \| x - x' \| = \lambda\, d(x) = d(x) - (1 - \lambda)\, d(x) \\ &\leqslant 1 - f_i(x) - (1 - \lambda)\, d(x) \leqslant f_i(v - x) - (1 - \lambda)\, d(x) \\ &= f_i(v - x_\lambda) \leqslant \| v - x_\lambda \|. \end{aligned}$$

Thus $x' \in P_V(x)$ and V is a β-sun. ∎

Oshman [10] established Lemma 3.1 in the particular case when

$$V = \{y \in X: f(y) \geqslant 1 \text{ for some } f \in S(X^*) \text{ with } f(x_0) = 1\}.$$

THEOREM 3.2. *If V is a Chebychev set—which will be the case when X is rotund—then P_V is continuous.*

Proof. Assume V is Chebychev. By Lemma 3.1, $P_V(x) = x + d(x)\, x_0$ for each x in $X \backslash V$. Since P_V is continuous on V, to show continuity of P_V, it obviously suffices to show that the function $x \to d(x)$ is continuous on $X \backslash V$. Let $x \in X \backslash V$ and $x_n \to x$. Since $d(x_n) \leqslant 1 - f_i(x_n)$ for each $i \in I$, we have $\limsup d(x_n) \leqslant d(x)$. Next choose $i_n \in I$ so that $d(x_n) = 1 - f_{i_n}(x_n)$. Let $f \in X^*$ be a weak* cluster point of (f_{i_n}). Then $f = f_{i_0}$ for some $i_0 \in I$. Since

$$\begin{aligned} |f_{i_n}(x_n) - f(x)| &\leqslant f_{i_n}(x_n) - f_{i_n}(x)| + |f_{i_n}(x) - f(x)| \\ &\leqslant |\, x_n - n\,| + |f_{i_n}(x) - f(x)|, \end{aligned}$$

we have (by passing to a subsequence if necessary) that $f_{i_n}(x_n) \to f(x)$. Hence

$$\liminf d(x_n) = 1 - f(x) \geqslant d(x).$$

Thus

$$d(x) \leqslant \liminf d(x_n) \leqslant \limsup d(x_n) \leqslant d(x)$$

and so $d(x) = \lim d(x_n)$. This shows that d is continuous on $X \backslash V$.

It remains to show that if X is rotund, then V is a Chebychev set. In view of Lemma 3.1, it suffices to show that *any* proximinal β-sun K in a rotund space X is Chebychev. If this result were false, there would be an $x \in X \backslash K$ and distict points x', x'' in $P_K(x)$. We may assume $x = 0$. Choose $y' \in P_K(0)$ so that $y' \in P_K[y' + \lambda(0 - y')]$ whenever $\lambda \geqslant 0$. In particular, $y' \in P_K(-y')$. Now either $x'' \neq y'$ or $x' \neq y'$, say, the former. Then since equality holds in the triangle inequality, we obtain $y' = x'$, a contradiction. ∎

4. LINEAR METRIC PROJECTIONS

If M is a Chebyshev subspace of the normed linear space X, the *kernel* of th metric projection P_M will be denoted by M^0. Thus

$$M^0 = \{x \in X : P_M(x) = 0\} = \{x \in X : \|x\| = d(x, M)\}.$$

It is well known (Holmes and Kripke [6]) that P_M is linear if and only if M^0 is a linear subspace. In the same paper, they proved the following results

THEOREM (Holmes and Kripke [6]). *Let M be a Chebychev subspace of X.*

(1) *If P_M is continuous, then $I + P_M$ is a homeomorphism of X with itself.*

(2) *If P_M is linear, then $\| P_M \| = 1$ if and only if M^0 is a proximinal subspace and $x - P_M(x) \in P_M(x) \in P_{M^0}(x)$ for each $x \in X$.*

(3) *If P_M is linear and M^0 is a Chebychev subspace, then the following statements are equivalent:*

(a) $\| P_M \| = 1$,

(b) $P_{M^0} = I - P_M$,

(c) $(M^0)^0 = M$.

Our first observation is that to compute the norm of certain linear metric projections, it suffices to consider only the unit ball of a rather small subset.

LEMMA 4.1. *If M is a Chebychev subspace of the normed linear space X, P_M is linear, and M^0 is proximinal, then*

$$\| P_M \| = \sup\{\| P_M(x)\|: x \in (M^0)^0, \| x \| \leqslant 1\}.$$

Proof. We first note that since M is Chebychev, each $x \in X$ has a unique representation in the form $x = m + m^0$, where $m \in M$ and $m^0 \in M^0$. In fact, $m = P_M(x)$ and $m^0 = x - P_M(x)$. Thus $X = M \oplus M^0$. Since M^0 is proximinal, we also have $X = M^0 + (M^0)^0$ (i.e., $X = \{m^0 + m^{00}: m^0 \in M^0, m^{00} \in (M^0)^0\}$. Finally, recall that since M^0 is a subspace, P_{M^0} is "additive modulo M^0": $P_{M^0}(u + v) = P_{M^0}(u) + P_{M^0}(v)$ for any $u \in X$ and $v \in M^0$. Thus we have $x = m + m^0$ and $m = m_1{}^0 + m^{00}$, where $m \in M$, $m^0, m_1{}^0 \in M^0$, and $m^{00} \in (M^0)^0$. Since M^0 is a subspace $m_1{}^0 + m^0 \in M^0$ and

$$x = (m_1{}^0 + m^0) + m^{00},$$

so

$$m^0 + m_1{}^0 \in m^0 + m_1{}^0 + P_{M^0}(m^{00}) = P_{M^0}(m^0 + m_1{}^0 + m^{00}) = P_{M^0}(x).$$

In particular, $\| m^{00} \| = \| x - (m^0 + m_1{}^0)\| \leqslant \| x \|$. Also, by linearity of P_M,

$$P_M(x) = P_M(m) = P_M(m_1{}^0 + m^{00}) = P_M(m_1{}^0) + P_M(m^{00}) = P_M(m^{00}).$$

This shows that for each $x \in X$ with $\| x \| < 1$, there is an $m^0 \in (M^0)^0$ with $\| m^{00} \| \leqslant 1$ and $P_M(m^{00}) = P_M(x)$. Thus

$$\begin{aligned} \| P_M \| &= \sup\{\| P_M(x)\|: x \in X, \| x \| \leqslant 1\} \\ &\leqslant \sup\{\| P_M(x)\|: x \in (M^0)^0, \| x \| \leqslant 1\} \leqslant \| P_M \| \end{aligned}$$

and the result follows. ∎

If M is a Chebychev subspace of X, the *norm* of P_M, whether P_M is linear or not, is defined by

$$\| P_M \| = \sup\{\| P_M(x)\| : \| x \| \leqslant 1\}.$$

Since P_M is the identity on M, $\| P_M \| \geqslant 1$. On the other hand, since

$$\| P_M(x) \| \leqslant \| P_M(x) - x \| + \| x \| \leqslant 2 \| x \|$$

for every x, it follows that $\| P_M \| \leqslant 2$. Thus $1 \leqslant \| P_M \| \leqslant 2$. In a Hilbert space it is well known that the metric projection onto a closed subspace is always linear and has norm one (viz., it is the orthogonal projection). In general normed spaces however, with the exception of Chebychev subspaces of codimension one, Chebychev subspaces having linear metric projections are relatively scarce. (For example, the space $C[0, 1]$ has none with finite dimension [5].) The next two results state, in particular, that linear metric projections exist with every norm size possible. (We use the notation $l_1(2)$ to denote the space $\mathbb{R}^2$ with the norm $\|(\alpha, \beta)\| = |\alpha| + |\beta|$.)

PROPOSITION 4.2. *For each real number r with $1 \leqslant r \leqslant 2$, there is a one dimensional subspace $M = M_r$ of $l_1(2)$ with the following properties*:

(1) *M is Chebychev,*

(2) *P_M is linear,*

(3) $\| P_M \| = r$,

(4) *M^0 is Chebychev,*

(5) *P_{M^0} is linear.*

Proof. Choose an angle $\theta \in [0, \pi/4)$ so that $\tan\theta = r - 1$. Define the subspace $M = M_r$ by

$$M = \{x = (\gamma, \gamma \tan\theta) : \gamma \in \mathbb{R}\}.$$

For any $x = (\alpha, \beta)$ in $l_1(\mathbb{R})$ and $y = (\gamma, \gamma \tan\theta)$ in M, we have

$$\begin{aligned} \| x - y \| &= |\alpha - \gamma| + |\beta - \gamma \tan\theta| \\ &\geqslant |\alpha - \gamma| \tan\theta + |\beta - \gamma\tan\theta| \\ &\geqslant |\alpha\tan\theta - \beta| \end{aligned}$$

and equality holds only if $\alpha = \gamma$ (since $0 \leqslant \tan\theta < 1$). But $y_0 = (\alpha, \alpha\tan\theta)$ satisfies $\| x - y_0 \| = |\beta - \alpha\tan\theta|$. Thus M is Chebychev and

$$P_M(\alpha, \beta) = (\alpha, \alpha\tan\theta) \qquad \forall(\alpha, \beta) \text{ in } l_1(2). \tag{*}$$

In particular, (1) is verified, and (2) follows from the relation (*).

(*) If $\|(\alpha, \beta)\| \leqslant 1$, then

$$\| P_M(\alpha, \beta)\| = \|(\alpha, \alpha \tan \theta)\| = |\alpha| (1 + \tan \theta) = |\alpha| r = r$$

and equality holds for $(\alpha, \beta) = (1, 0)$. Thus $\| P_M \| = r$.

(4) From (*) it follows that $(\alpha, \beta) \in M^0 \Leftrightarrow \alpha = 0$. That is,

$$M^0 = \{(0, \beta): \beta \in \mathbb{R}\}.$$

If $x = (\alpha, \beta)$ and $y = y = (0, \gamma) \in M^0$, then

$$\| x - y \| = |\alpha| + |\beta - \gamma| \geqslant |\alpha|$$

and equality holds-only if $\beta = \gamma$. This shows that M^0 is Chebychev and

$$P_{M^0}(\alpha, \beta) = (0, \beta). \tag{**}$$

(5) follows from Eq. (**).

(6) For each (α, β) with $\|(\alpha, \beta)\| \leqslant 1$, we have

$$\| P_{M^0}(\alpha, \beta)\| = |\beta| \leqslant \|(\alpha, \beta)\| \leqslant 1,$$

so $\| P_{M^0} \| = 1$.

(7) $$\begin{aligned}(M^0)^0 &= \{(\alpha, \beta): P_{M^0}(\alpha, \beta) = 0\} \\ &= \{(\alpha, 0): \alpha \in \mathbb{R}\}\end{aligned}$$

and $(M^0)^0 \neq M$ if $\tan \theta \neq 0$, i.e., if $r \neq 1$. ∎

Remark. Using Lemma 4.1, it is not hard to see that there is *no* Chebychev subspace M of $l_1(2)$ with (P_M linear and) $\| P_M \| = 2$. More generally, we prove

LEMMA 4.3. *If X is a finite-dimensional normed linear space and M a Chebychev subspace, then $\| P_M \| < 2$.*

Proof. If $\| P_M \| = 2$, then by compactness of the unit ball in X and the continuity of P_M, there would exis- $x \in X$ with $\| x \| = 1$ and $\| P_M(x)\| = 4$. Hence

$$2 = \| P_M(x)\| \leqslant \| P_M(x) - x \| + \| x \| \leqslant 2 \| x \| = 2.$$

Thus equality must hold throughout so $\| P_M(x) - x \| = \| x \|$. This implies that $P_M(x) = 0$ which contradicts $\| P_M(x)\| = 2$. Thus $\| P_M \| < 2$. ∎

Thus to find a Chebychev subspace with (linear) metric projection having norm 2, we are forced to consider infinite-dimensional spaces.

PROPOSITION 4.4. *There is a subspace M of $C[0, 1]$ with the properties:*

(1) *M is Chebychev,*

(2) *P_M is linear,*

(3) $\| P_M \| = 2$,

(4) *M^0 is Chebychev,*

(5) *P_{M^0} is not linear,*

(6) $\| P_{M^0} \| = 1$,

(7) $(M^0)^0 \neq M$.

Proof. Let $M = \{x \in C[0, 1]: \int_0^1 x(t)\, dt = 0\}$. For each $x \in C[0, 1]$, let $\int_0^1 x(t)\, dt = 0\}$. For each $x \in C[0, 1]$, let $\tilde{x} = x - \int_0^1 x(t)\, dt$. Then $\tilde{x} \in M$ and $\| x - \tilde{x} \| = | \int_0^1 x(t)\, dt |$. If $y \in M$, then

$$\| x - y \| \geqslant \int_0^1 | x(t) - y(t) |\, dt \geqslant \left| \int_0^1 [x(t)\, dt \right| = \left| \int_0^1 x(t)\, dt \right| = \| x - \tilde{x} \|.$$

This shows that $\tilde{x} \in P_M(x)$. Now equality holds in the above inequality if and only if $x - y = c$, a constant. Since $y \in M$, $c = \int_0^1 x(t)\, dt$. That is, $y = \tilde{x}$. This proves that M is Chebychev and

$$P_M(x) = x - \int_0^1 x(t)\, dt. \tag{*}$$

(2) The linearity of P_M follows immediately from (*).

(3) Given $0 < \epsilon < 1$ define

$$\begin{aligned} x(t) &= 1 - \frac{2}{\epsilon} t \qquad \text{if} \quad 0 \leqslant t \leqslant \epsilon, \\ &= -1 \qquad\quad\ \text{if} \quad \epsilon \leqslant t \leqslant 1. \end{aligned}$$

Then $x \in C[0, 1]$, $\| x \| = 1$, and $P_M(x) = x + 1 - \epsilon$. Since $x(0) = 1$, we have that $\| P_M(x) \| \geqslant 2 - \epsilon$. Since ϵ was arbitrary, $\| P_M \| \geqslant 2$ and hence $\| P_M \| = 2$.

(4) From (*) we obtain

$$M^0 = \{x \in C[0, 1]: P_M(x) = 0\} = \{x: x \text{ a constant}\}.$$

It is well known that the constant functions form a one-dimensional Chebychev subspace of $C[0, 1]$. Indeed, if $x \in C[0, 1]$, then

$$P_{M^0}(x) = \tfrac{1}{2} [\max_{0 \leqslant t \leqslant 1} x(t) + \min_{0 \leqslant t \leqslant 1} x(t)].$$

(5) Define x and y on $[0, 1]$ by

$$\begin{aligned} x(t) &= 0 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{4}, \\ &= 4t - 1 && \text{if } \tfrac{1}{4} \leqslant t \leqslant \tfrac{1}{2}, \\ &= 1 && \text{if } \tfrac{1}{2} \leqslant t \leqslant 1; \end{aligned}$$

$$\begin{aligned} y(t) &= 1 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= -4t + 3 && \text{if } \tfrac{1}{2} \leqslant t \leqslant \tfrac{3}{4}, \\ &= 0 && \text{if } \tfrac{3}{4} \leqslant t \leqslant 1. \end{aligned}$$

Then $x, y \in C[0, 1]$ and $P_{M^0}(x) = \frac{1}{2} = P_{M^0}(y)$. But

$$\begin{aligned} (x + y)(t) &= 1 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{4}, \\ &= 4t && \text{if } \tfrac{1}{4} \leqslant t \leqslant \tfrac{1}{2}, \\ &= 4(1 - t) && \text{if } \tfrac{1}{2} \leqslant t \leqslant \tfrac{3}{4}, \\ &= 1 && \text{if } \tfrac{3}{4} \leqslant t \leqslant 1, \end{aligned}$$

so that $P_{M^0}(x + y) = \frac{3}{2} \neq P_{M^0}(x) + P_{M^0}(y)$. Thus P_{M^0} is not linear.

(6) If $\| x \| \leqslant 1$, then $-1 \leqslant x(t) \leqslant 1$ for all t implies

$$-1 \leqslant \tfrac{1}{2}[\max_t x(t) + \min_t x(t)] \leqslant 1,$$

i.e., $\| P_{M^0}(x)\| \leqslant 1$. Thus $\| P_{M^0}(x)\| \leqslant 1$. Thus $\| P_{M^0} \| = 1$.

(7) From (5) and the previously mentioned Holmes–Kripke result, it follows that $(M^0)^0$ is not linear and hence $(M^0)^0 \neq M$. A simple direct proof is also available. Define x on $[0, 1]$ by

$$\begin{aligned} x(t) &= -4t + 1 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= -1 && \text{if } \tfrac{1}{2} \leqslant t \leqslant 1, \end{aligned}$$

Then $P_{M^0}(x) = 0$, i.e., $x \in (M^0)^0$, but $\int_0^1 x(t)\, dt = -\frac{1}{2}$ so $x \notin M$. ∎

Remark. It would be interesting to know exactly which Banach spaces contain Chebychev subspaces having linear metric projections with norm 2. (The above shows that $C[0, 1]$ is such a space, but finite-dimensional spaces and Hilbert spaces are not.)

5. The Klee Space and Continuous Metric Projections

This section concerns itself with the necessaty and sufficient conditions for a Banach space to have a continuous metric projection onto every closed convex set. Oshman [11] has given a geometrical characterization of such spaces, namely, that it have property (0). A Banach space X is said to possess property (0) if X is reflexive, and if (x_n) in $S(X)$, (f_n) in $S(X^*)$, $f_n(x_n) = 1$ $(n = 0, 1, 2)$ $f_n \to f_0$ weakly, $d(x_0, H_0 \cap H_n) \to 0$, $d(x_n, H_0 \cap H_n) \to 0$ imply $x_n \to x_0$, where $H_n = \{x \in X \mid f_n(x) = 1\}$.

One notes that this characterization is extremely complicated and difficult to apply to specific Banach spaces. It should be noted that if X is reflexive and X^* is Frechet smooth, then X has property (0). Indeed if $(x_n) \subset S(X)$, $f_n \in S(X^*)$, $f_n(x_n) = 1$ $(n = 0, 1, \ldots)$, and $d(x_n, H_0 \cap H_n) \to 0$, choose any $y_n \in H_0 \cap H_n$ such that $\| x_n - y_n \| \to 0$. Then

$$\| f_0(x_n) - 1 \| = \| f_0(x_n) - f_0(y_n) \| \leqslant \| x_n - y_n \|.$$

By Smulian's characterization [14] of Frechet smoothness of X^*, it follows that (x_n) converges. By strict convexity of X, the limit is x.

Both Oshman [11] and Vlasov [17] have conjectured that X possessing property (0) is equivalent to X^* being Frechet smooth.

In [17], Vlasov has shown that they are indeed equivalent if X is a smooth Banach space. Previous work [8] by the second author of this paper claimed that the dual of the Klee space [7], was an example of a Banach space such that the metric projection onto every closed subspace was continuous and yet the dual of that space was not Frechet smooth. A mistake in the proof of [8, Lemma 2.1] renders all claims in that paper invalid. We show in this section that the dual space of the Klee space does not possess property (0). In particular, this shows that there exists a reflexive, rotund Banach space, whose dual space has a norm that is Frechet differentiable at all points of its unit ball except for two antipodal points and yet this space fails to possess property (0). This seems to lend considerable support to a positive answer to Oshman and Vlasovs conjecture. It is evident why such a space should have been considered as a possible counterexample to their conjecture. One needs only to verify the defining property of property (0) for those sequences (f_n) in $S(X^*)$ which converge weakly to either of the two antipodal points of this unit ball of X^* where the norm is not Frechet differentiable. Symmetry reduces the problem and looking at only one of the points. Unfortunately the defining condition fails at these points as we show.

Recall that l_2 is the Banach space of square summable sequences of real numbers $x = (x_0, x_1, x_2, \ldots)$ with norm $\| x \|_2 = (\sum_0^\infty x_n^2)^{1/2}$. The vector δ_n will denote the sequence whose nth coordinate is one and all others are zero. V denotes the subspace $\{x \in l_2 \mid x_0 = 0\}$ and the norm is denoted $\| x \|_V = (\sum_1^\infty x_n^2)^{1/2}$.

Klee [7] exhibited an equivalent renorming of l_2—call the space X—such that X is smooth and that the norm of X is Frechet differentiable at all points of $S(X)$ except $\pm\delta_0$. If we let $Y = X^*$ denote the dual space of X, it is clear that Y is a reflexive, rotund Banach space from elementary duality considerations. It is the space Y that we will show does not possess property (0).

We recall the renorming of l_2 given in [7]. Let

$$\mathscr{U}_\eta = \{\delta_0, -\delta_0\} \cup \left\{ x \in l_2 \mid |x_0| < 1, \sum_{i=1}^{\infty} \left| \frac{x_i}{\eta_i(x_0)} \right|^2 \leqslant 1 \right\}$$

where each η_i is an even function on $[-1, 1]$ to $[0, 1]$ with the following properties.

Given a sequence ϵ_i of positive real numbers decresing to zero one has

(1) η_i is the continuous and concave with $\eta_i(0) = 1$, $\eta_i(1 - \epsilon_i) = 2\epsilon_i$ and $\eta_i(1) = 0$ for all i,

(2) η_i is differentiable on $[0, 1)$ with $\eta_i(0) = 0$ and $\eta_i(1 - \epsilon_i) = -1$ for all i;

(3) η_i has a vertical tangent at 1 (i.e., $\lim_{\lambda \to 1^-} \eta_i(\lambda) = -\infty$ for all i).

To facilitate our computations, we shall work with the following particular functions having these properties:

$$\begin{aligned} \eta_i(\lambda) &= 1 - \frac{\lambda^2}{4\epsilon_i}, && 0 \leqslant \lambda \leqslant 2\epsilon_i, \\ &= 1 - \lambda + \epsilon_i, && 2\epsilon_i \leqslant \lambda \leqslant 1 - \epsilon_i, \\ &= \sqrt{4\epsilon_i(1 - \lambda)}, && 1 - \epsilon_i \leqslant \lambda \leqslant 1. \end{aligned}$$

In [7] Klee let K to be the closed convex hull of $\mathscr{U}_\eta : K = \overline{\mathrm{co}}(\mathscr{U}_\eta)$. The guage $\rho_K(\cdot)$ of K is taken to be the norm of the Klee space X. We set $Y = X^*$ with norm defined by the gauge $\rho_{K^0}(\cdot)$ where K^0 denotes the polar set of K in Y.

Given x in $S(X)$, it is necessary for us to find the norm duality mapping $T: S(X) \to S(Y)$ which has the property that $(x, Tx) = \rho_K(x)\, \rho_{K^0}(Tx) = 1$. One checks that $T\delta_0 = \delta_0$. Let $F = \{x \in l_2 \mid \sum_1^\infty |x_i/\eta_i(x_0)|^2 = 1\}$ denote a surface in l_2. By standard infinite-dimensional calculus techniques in l_2, one can determine the equation of a supporting hyperplane to F at any specific point $\bar{x}$ on F. Such an equation would have the form $(\varphi, x - \bar{x}) = 0$ where φ is in l_2. Since F is a symmetric set, φ would also determine a supporting hyperplane at $(-\bar{x})$. Any supporting hyperplane would also support the closed convex hull of $\{-\delta_0, +\delta_0\} \cup F$ and hence support K, the unit ball of the Klee space. Normalizing the linear functional φ will yield the norm duality element for an element in $\mathscr{U}_\eta$. In particular one finds that if f is in $S(X), f = (f_0, f_1, \ldots)$, then

$$Tf = \frac{1}{1 + A(f) \cdot f_0} \left(A(f), \frac{f_1}{\eta_1^2(f_0)}, \frac{f_2}{\eta_2^2(f_0)}, \ldots \right),$$

where $A(f) = \sum_{i=1}^\infty - (f_i^2/\eta_i^3(f_0))\, \eta_i(f_0)$.

To show that Y does not possess property (0) set

$$f^j = (1 - \epsilon_j)\,\delta_0 + \eta_1(1 - \epsilon_j) \times \delta_1 + \eta_j(1 - \epsilon_j)\,y\delta_j\,,$$

where $x^2 + y^2 = 1$, $xy \neq 0$, and the (ϵ_j) is the precise sequence determining the properties of the η_j. Clearly f^j is in $S(X)$. Tf^j is in $S(Y)$ and is given by

$$Tf^j = \frac{1}{1 + \epsilon_j}\,\delta_0 + \frac{x}{1 + \epsilon_j}\sqrt{\frac{\epsilon_j}{\epsilon_1}}\,\delta_1 + \frac{y}{1 + \epsilon_j}\,\delta_j\,.$$

As $j \to \infty$, $\epsilon_j \to 0$, $f^j \to \delta_0$ weakly and $Tf^j \to \delta_0$ weakly.

Since Y is a renorming of l_2 with an equivalent norm, it can be shown that $\| Tf^j - \delta_0 \|_2^2 = 1/(1 + \epsilon_j)^2(\epsilon_1^2 + (\epsilon_j/\epsilon_1)\,x^2 + y^2) \geqslant y^2/2 > 0$ for all j. Thus Tf^j does not converge strongly to δ_0. This would refute Y processing property (0) if we can show that the f^j and Tf^j satisfy these remaining hypothesis of property (0). In particular we must show that $d(Tf^j, H_0 \cap H_j) \to 0$ and To facilitate the computation we present another renorming of l_2. Let B denote the closed convex hull of $\{\delta_0 + S(V), -\delta_0 + S(V)\}$. K is contained in B and l_2 with guage ρ_B can be shown to be equivalent renorming of l_2. In fact, $\rho_B(x) = \max\{| x_0 |, \| x_v \|_v\}$, where $x = x_0 + x_v$, x_v in V. In the space Y, we also obtain an equivalent renorming using ρ_{B^0} as the norm. In particular $\rho_{B^0}(y) = | y_0 | + \| y_v \|_V$. Clearly $B^0 \subset K^0$ and $\rho_{K^0}(y) \leqslant \rho_{B^0}(y)$ for all y in Y. Thus $d_{K^0}(Tf^j, H_0 \cap H_j) \leqslant d_{B^0}(Tf^j, H_0 \cap H_j)$ and $d_{K^0}(\delta_0, H_0 \cap H_j) \leqslant d_{B^0}(\delta_0, H_0 \cap H_j)$. We will show that the larger distances approach zero insuring that the smaller ones do likewise. By definition

$$\begin{aligned} d_{B^0}(\alpha, H) = {} & \inf_{h \in H} \beta_{B^0}(\alpha - h) \\ & \inf_{h \in H}\{| \alpha_0 - h_0 | + \| \alpha_v - h_v \|_V\}. \end{aligned}$$

We note that in our specific case

$$H_0 = \{z \in l_2 \mid z_0 = 1\}$$

and

$$H_j = \{z \in l_2 \mid (1 - \epsilon_j)\,z_0 + x\eta_1(1 - \epsilon_j)\,z_1 + y\eta_j(1 - \epsilon_j)\,z_j = 1\}.$$

Thus

$$H_0 \cap H_j = \{z \in l_2 \mid z_0 = 1 \text{ and } x\eta_1(1 - \epsilon_j)\,z_1 + y\eta_j(1 - \epsilon_j)\,z_j = \epsilon_j\}.$$

Then

$$\begin{aligned} \rho_{B^0}(\alpha, H_0 \cap H_j) &= \inf_{z \in H_0 \cap H_j}\{| \alpha_0 - 1 | + \| \alpha_v - z_v \|_V\} \\ &= | \alpha_0 - 1 | + \inf_{z \in H_0 \cap H_j} \| \alpha_V - z_V \|_V \\ &= | \alpha_0 - 1 | + \inf_{z_v \in \mathscr{M}} \| \alpha_V - z_V \|_V\,, \end{aligned}$$

where $\mathscr{M} = \{z \in V \mid x\eta_1(1 - \epsilon_j) z_1 + y\eta_j(1 - \epsilon_j) z_j = \epsilon_j\}$ is a hyperplane in V.

Then using the formula for the distance from an element in a space to a hyperplane $\varphi(z) = c$ (i.e., $d(\alpha, H) = |\varphi(\alpha) - c|/\|\varphi\|$) one has $\rho_{B^0}(\delta_0, H_0 \cap H_j) = |1 - 1| + \inf_{z_\nu \in \mathscr{M}} \|z_v\|_V$

$$= 0 + \frac{|\epsilon_j|}{2|\epsilon_j|\sqrt{x^2(\epsilon_1/\epsilon_j) + y^2}} = \frac{1}{2\sqrt{x^2(\epsilon_1/\epsilon_j) + y^2}} \to 0$$

as $j \to \infty$.

Similarly $d_{B^0}(Tf^j, H_0 \cap H_j) = |1 - 1/(1 + \epsilon_j)| + \inf_{z_\nu \in \mathscr{M}} \|Tf^j - z_v\|_V$

$$= \left|1 - \frac{1}{1+\epsilon_j}\right| + \frac{|2\epsilon_j x^2/(1 + \epsilon_j) + 2\epsilon_1 y^2/(1 + \epsilon_j) - \epsilon_j|}{2\epsilon_j \text{—} x^2(\epsilon_1/\epsilon_j) + y^2}$$

$$= \left|1 - \frac{1}{1+\epsilon_j}\right| + \frac{|2\epsilon_j/(1 + \epsilon_j) - \epsilon_j|}{|2\epsilon_j| \text{—} x^2(\epsilon_2/\epsilon_j) + y^2}$$

$$= \left|1 - \frac{1}{1+\epsilon_j}\right| + \frac{[2/(1 + \epsilon_j) - 1]}{\text{—} x^2(\epsilon_1/\epsilon_j) + y^2} \to 0 \quad \text{as } j \to \infty.$$

This concludes the proof that the dual of the Klee space does not possess property (0).

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References

1. J. Blatter, Weiteste Punkte und nachste Punkte, *Rev. Roumaine Math. Pures Appl.* **14** (1969), 615–621.
2. B. Brosowski and F. Deutsch, On some geometrical properties of suns, *J. Approximation Theory* **10** (1974), 245–267.
3. B. Brosowski and F. Deutsch, Radial continuity of set-valued metric projections, *J. Approximation Theory* **11** (1974), 236–253.
4. B. Brosowski and R. Wegmann, Charakterisierung besten Approximationen in normierten Vektorraumen, *J. Approximation Theory* **3** (1970), 369–397.
5. E. W. Cheney and K. H. Price, Minimal projections, *in* "Approximation Theory" (A. Talbot, Ed.), pp. 261–289, Academic Press, New York, 1970.

6. R. B. HOLMES AND B. R. KRIPKE, Smoothness of approximation, *Michigan Math. J.* **15** (1968), 225–24.
7. V. KLEE, Two renorming constructions related to a question of Anselone, *Studia Math.* **33** (1969), 231–242.
8. J. M. LAMBERT, Continuous metric projections, *Proc. Amer. Math. Soc.* **48** (1975), 179–184.
9. E. V. OSHMAN, Chebychev sets, contnuity of the metiic projection, and some geometric properties of the unit sphere in a Banach space, *Izv. Vyssh. Uchebn. Zaved. Matematika* **4** (1969), 38–46. [Russian]
10. E. V. OSHMAN, Chebychev sets and continuous metric projections, *Izv. Vyssh. Uchebn. Zaved. Matematika* **9** (1970), 78–82. [Russian]
11. E. V. OSHMAN, Continuity criterion for metric projections in a Banach space, *Mat. Zametki* **10** (1971), 459–468. [Russian; English transl. in *Math. Notes* **10** (171), 697–701.]
12. B. B. PANDA AND O. P. KAPOOR, Approximative compactness and continuity of metric projections, *Bull. Austral. Math. Soc.* **11** (1974), 47–55; Corrigendum: *Bull. Austral. Math. Soc.* **12** (1975), 319–320.
13. I. SINGER, Some remarks on approximative compactness, *Rev. Roumaine Math. Pures Appl.* **9** (1964), 167–177.
14. V. SMULIAN, Sur la dérivabilité de la norme dans l'es de Banach, *C.R. Dokl. Acad. Sci. USSR* **27** (1940), 643–648.
15. L. P. VLASOV, On Chebychev sets, *Dokl. Akad. Nauk SSSR* **173** (1967), 491–494. [Russian; English transl. in *Sov. Math. Dokl.* **8** (1967), 401–404.]
16. L. P. VLASOV, Chebychev sets and approximatively convex sets, *Mat. Zametki* **2** (1967), 191–200. [Russian; English transl. in *Math. Notes* **2** (1967), 600–605.]
17. L. P. VLASOV, Approximative properties of sets in normed linear spaces, *Uspeki Mat. Nauk* **28** (1973), 1–66. [Russian; English transl. in *Russian Math. Surveys* **28** (1973), 1–66.]