Local PI Theory of Jordan Systems II

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Communicated by Georgia Benkart

Received June 23, 1998

DEDICATED TO THE MEMORY OF EULALIA GARCÍA RUS

We pursue the study, initiated in a previous paper, of Jordan systems having nonzero local algebras that satisfy a polynomial identity. We define the extended centroid of a nondegenerate Jordan system, the corresponding central extension, which we call the extended central closure, and prove a Jordan analogue of Martindale’s theorem on prime algebras having a generalized identity: If $J$ is a nondegenerate Jordan system with nonzero PI-elements, then the extended central closure of $J$ has nonzero socle, equal to its PI ideal.

INTRODUCTION

This paper is a continuation of [27], where we started the study of Jordan systems having nonzero local PI algebras. This condition has repeatedly arisen in several papers on Jordan theory in a more or less explicit form, beginning with Zelmanov’s work on triple systems and pairs, and on Goldie’s theorems for Jordan algebras. That condition also appears in Anquela, Cortés, and Montaner’s work on primitive Jordan algebras. More recently, D’Amour and McCrimmon’s extension [2] to quadratic systems of Zelmanov’s results revealed the central role of local algebras and, in particular, local PI algebras, in Jordan theory. This has been exploited by Anquela and Cortés [3] in their study of primitive Jordan systems, which is based on the local characterization of primitivity.

As commented in our previous paper [27], the present work was motivated by the on-going project of extending to quadratic Jordan algebras Zelmanov’s version for linear Jordan algebras of Goldie’s theory. This

1 Partially supported by the DGES, PB97-1069-C02-02.
requires the comparison of inner ideals of special Jordan algebras and one-sided ideals of their associative envelopes (to deal with notions like uniform dimension or singularity), and here the nonexistence of PI-elements is essential. Thus, a structural result was needed to deal with the presence of nonzero PI-elements.

A different motivation comes from the classification of strongly prime Jordan systems. There, as in Jordan algebras (see [32]), the presence of some particular classes of identities is of fundamental structural importance [2]. So that situation suggests the interest in the study of general PI–Jordan systems. There is, however, an important feature in the kind of identities that are significant in that classification: they are homotope polynomial identities, i.e., identities (of algebras) which hold in every homotope (hence in every local algebra) of the system. This, of course, leads again to the study of local PI algebras.

This twofold motivation suggests two sources of inspiration, associative PI theory for the latter, and associative GPI theory for the first. The relation with this second theory becomes apparent if one realizes that the presence of a GPI in a semiprime associative algebra is equivalent to the existence of a PI one-sided ideal, and this is equivalent to the existence of a nonzero local PI algebra (see [28]). Therefore, from a structural viewpoint, the results that should be extended to Jordan systems are Kaplansky’s and Amitsur’s theorems on primitive PI and GPI algebras, respectively, and Posner’s and Martindale’s theorems on prime PI and GPI algebras, respectively.

In our previous paper [27] we defined the set $\text{PI}(J)$ of PI-elements of $J$ (those which produce a PI local algebra), proved that it is an ideal if $J$ is nondegenerate, and studied primitive algebras with $\text{PI}(J) \neq 0$. The main result about these systems was an analogue of Amitsur’s theorem on GPIs, asserting that they have nonzero socle, equal to the set of PI-elements. From this, one gets an analogue of Kaplansky’s theorem saying that homotope–PI primitive Jordan systems are simple with capacity.

In the present paper we address the corresponding theorems in the prime case. Our main result is Theorem 5.1, which asserts that a strongly prime Jordan system with nonzero PI-elements has a primitive extended central closure.

The paper is organized as follows. After a section devoted to preliminaries, we define in Section 1 the extended centroid of a nondegenerate Jordan system. The construction is based on the non-associative extension of the extended centroid given in [9, 12], although we have to surmount the difficulties inherent in our quadrating setting. In Section 2 we define the scalar extension naturally attached to the extended centroid, which we call the extended central closure (to distinguish it from the ordinary central closure), and, in Section 3 we prove some of its properties.
results are inspired by the idea of relating “central quotients” to localizations in certain categories, as expressed in [31]. We point out, however, that the nonlinearity of Jordan systems only allows a rather rudimentary version of those ideas. We point out, also, that there are several precedents [10, 11] of the use of extended centroids in (linear) Jordan theory. Section 4 is devoted to the proof of our main theorem, which we base on an ultrafilter argument inspired by the one originally used by Martindale [21]. Finally, in Section 5 we specialize our result to homotope–PI algebras, to obtain an analogue of Posner–Rowen’s theorem for Jordan systems.

As we mentioned in our previous paper [27], our approach to the subject was motivated by D’Amour and McCrimmon’s work [1, 2] and, especially, by their strategy of combining the use of local algebras, Loos’ theory of the socle, and the theory of Jordan PI algebras. Thus, we have not made use of the general classification theorems [2, 33] of Jordan systems other than algebras (the latter, appearing implicitly in the PI theory for algebras) and, in particular, of the prime dichotomy for pairs and triples and the construction of hermitian polynomials. In addition to that, some of our results overlap with those of [2], sometimes giving stronger versions of them. As in [27], some arguments could have been simplified by using further results, such of those of [3–5], but they all rely on the classification theorems of [2, 33].

0. PRELIMINARIES

Throughout Φ will be a fixed unital commutative ring.

0.1

We will work with Jordan pairs, triple systems, and algebras over Φ. Our main sources of notation, terminology, and basic results are [15, 16, 26]. We record in this section some of those notations and results.

A Jordan algebra $J$ has products $x^2$ and $U_{x,y}$, quadratic in $x$ and linear in $y$, whose linearizations are

$$x \circ y = V_{x,y} = (x + y)^2 - x^2 - y^2,$$

$$U_{x,y} = V_{x,y} = \{x, z, y\} = U_{x,y} z - U_{x,z} - U_{y,z}.$$

A Jordan pair $V = (V^+, V^-)$ has products $Q_{x,y}$ for $x \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,y} = D_{x,y} z = \{x, y, z\} = Q_{x,z} y - Q_{y,z} y$. We also mention the important Bergmann operators given by $B_{x,y} z = z - D_{x,y} z + Q_{x,y} z$ for $x, z \in V^\sigma, y \in V^{-\sigma}$, which satisfies $Q_{B_{x,y} z} = B_{x,y} Q_{B_{x,y} z} B_{x,y}$. 
A Jordan triple system $T$ has product $P_{x,y}$ whose linearizations are $P_{x,y}z = L_{x,z}y = (x,z,y) = P_{x+y}z - P_xz - P_yz$. The Bergmann operator in triple systems is given by $B_{x,y}z = z - L_{x,y}z + P_xP_yz$.

A Jordan algebra gives rise to a Jordan triple system with $P = U$. If a Jordan triple system has an element $1$ with $P_1 = \text{Id}$, the identity, then it is a unital Jordan algebra with square $x^2 = P_1$.

We denote by $\Gamma(J)$ the centroid of a Jordan system $J$. This is a reduced commutative ring if $J$ is nondegenerate, a domain acting faithfully on $J$ if $J$ is strongly prime, and a field if $J$ is simple (see [25, 2.8] in addition to the general references).

0.2

Doubling a Jordan triple system $T$ produces a Jordan pair $V(T) = (T, T)$ with $Q_{x,y} = P_{x,y}$. Reciprocally, each Jordan pair $V = (V^+, V^-)$ gives rise to a polarized triple system $T(V) = V^+ \oplus V^-$ with product $P_{x^+ \circ y^+} y^- = Q_{x^+} y^- \oplus Q_{y^-} x^-$. Niceness conditions such as nondegeneracy, primeness, strong primeness, and others are inherited by the polarized triple system of a Jordan pair. However this no longer holds in the reverse direction, from Jordan triple systems to their double Jordan pairs. To remedy that situation, D’Amour and McCrimmon [1, p. 229] and Anquela and Cortés [3, p. 667] defined tight doubles.

Given a Jordan triple system $T$, a tight double of $T$ is a quotient pair $V(T)/I = (T/I^+, T/I^-)$ where $I$ is an ideal $(I^+, I^-)$ of $V(T)$ which is maximal with respect to $I^+ \cap I^- = 0$ (so that the $I^\sigma$ are semi-ideals of $T$, but they may not be ideals). These always exist and share niceness properties with $T$ (see 5.2 and 5.3 of [3]).

0.3

Since at some point we will make use of an ultraproduct argument involving systems with nonzero socle, it will be important for us to be able to express strong primeness by means of an elementary (or first-order) sentence, so that we can pass it from the factor to the ultraproduct. Such a characterization was found in [5] and was termed elemental primeness:

—A Jordan pair $V = (V^+, V^-)$ is elementally prime if for any pair of elements $x, y \in V^\sigma$, $\sigma = \pm, Q_x Q_y \sigma Q_y = 0$ implies $x = 0$ or $y = 0$.

—A Jordan algebra $J$ is elementally prime if for any $x, y \in J, U_i U_j U_k = 0$ implies $x = 0$ or $y = 0$.

—A Jordan triple system $T$ is elementally prime if for any $x, y \in T, P_x P_y P_y = 0$ and $P_x P_y P_y P_y = 0$ imply $x = 0$ or $y = 0$. 
By the results of [5], strong primeness is equivalent to elemental primeness. Although in its general form these results make use of the classification theorems of [2], of which we want to make our work independent, as explained in the Introduction, we will only need these facts for nondegenerate systems with nonzero socle. Since the results of [2] are only needed in [5] to reduce the problem to a separate study of hermitian systems and forms of systems with nonzero socle, we can make use of the results concerning the latter, which are independent of [2], to state:

0.4. Theorem. Let J be a nondegenerate Jordan system with nonzero socle. Then J is strongly prime if and only if it is elementally prime.

Proof. That any elementally prime prime Jordan system is strongly prime is straightforward (1.3 of [5]).

For the reciprocal, suppose first that our system J is a Jordan pair. If J is strongly prime, then Soc(J) is strongly prime by 2.5 of [22] and hence simple by Theorem 2 of [17]. Then, by 1.9 of [5], Soc(J) is elementally prime, and J itself is elementally prime by 1.6 of [5], and none of these results depends on [2].

Next, if J is a strongly prime Jordan algebra, then the pair (J, J) is strongly prime by [5, 1.12], and since Soc((J, J)) = (Soc(J), Soc(J)), the previous case applies, and (J, J) is elementally prime; hence J is elementally prime by [5, 1.11].

Finally, let J be a Jordan triple system, and let W be a tight double. Then W is strongly prime (by 5.2 of [3]) and has nonzero socle (by 4.3 of [27]); hence it is elementally prime.

If W = (J, J) is already tight, then J is elementally prime (and in fact, satisfies the stronger version: \( P_a P_b P_c = 0 \) if \( x = 0 \) or \( y = 0 \)). If \((J, J)\) is not tight, Soc(J) = Soc(T(W)) through the natural embedding \( J \subseteq T(W) \) (see 4.3(d) of [27]). Now, Soc(W) is elementally prime by 1.6(a) of [5], and the trivial equality Soc(T(W)) = T(Soc(W)) implies that Soc(T(W)) is elementally prime by 2.6 of [5]. Thus Soc(J) = Soc(T(W)) is elementally prime; hence J is elementally prime by 2.8 of [5]. Notice that this again is independent of the results of [2].  

0.5

Let \((V^+ V^-)\) be a Jordan pair and let \( a \in V^\sigma \), where \( \sigma = \pm \). The a-homotope of V, denoted by \((V^-)^{(a)}\), is the Jordan algebra over the \( \Phi \)-module \( V^- \) with operations \( U_{x,y}^{(a)} y^{-\sigma} = Q_{x^{-\sigma}} Q_{a} y^{-\sigma} \) and \( (x^{-\sigma})^2 = Q_{x^{-\sigma}} a \).

The set Ker a of all \( x^{-\sigma} \in V^- \) such that \( Q_{x} x^{-\sigma} = Q_{a} Q_{x^{-\sigma}} a = 0 \) is an ideal of a, so that the quotient \( V_a^{-\sigma} = (V^-)^{(a)}/\text{Ker } a \) is again a Jordan algebra. This is called the local algebra of V at a.
For triple systems and Jordan algebras, homotopes and local algebras are defined in the same way—just delete the superscripts $\sigma$ from the previous definitions. We refer to [1] for a thorough study of local algebras.

Local algebras can be considered as particular cases of subquotients [27, 0.4] as defined by Loos and Neher [20], so the general results of that theory apply to them. In particular,

0.6. Lemma. Let $V$ be a nondegenerate Jordan pair, $\sigma = \pm$, let $a$ be an element from $V^\sigma$, and let $p: V^{-\sigma} \to V_a^{-\sigma}$ be the canonical map. Then $\text{Soc}(V_a^{\pm \sigma}) = p(\text{Soc}(V^{-\sigma}))$, and if $\text{Soc}(V)$ is simple, then either $\text{Soc}(V_a^{-\sigma})$ is simple or it is zero.

Proof. The first assertion is [20, 2.7(a)]. For the second, suppose that $\text{Soc}(V_a^{\pm \sigma})$ is nonzero; then the same argument is in [20, 2.7(b)] applies, taking into account that the socle of a Jordan pair $V$ is simple if and only if for each pair of orthogonal division idempotents $e_1, e_2$, the Peirce space $V_{12}$ is nonzero [17, Lemma 6 and Theorem 2] and that the Peirce spaces $V_{12}$ and $\text{Soc}(V)_{12}$ coincide.

As a consequence, $\text{Soc}(J_a) \neq 0$ for a Jordan system $J$ implies that $\text{Soc}(J) \neq 0$ (see [1]). Indeed, if we take $a = a^+ \in V(J)^+ = J$, then $V(J)_a = J_a^+$; hence $\text{Soc}(J_a) \neq 0$ implies that $\text{Soc}(V(J)) \neq 0$, and the obvious equality $\text{Soc}(V(J)) = V(\text{Soc}(J))$ gives $\text{Soc}(J) \neq 0$.

0.7

We refer to [1, 14, 33] for the basic notions on primitivity of Jordan systems. Recall that primitive Jordan systems are the building blocks of a structure theory based on the Jacobson radical (Theorem 5.4 of [14], Lemma 6 of [33], and Theorem 8 of [34]):

0.8. Theorem. For a Jordan system $J$, $J/\text{Jac}(J)$ is a subdirect product of primitive Jordan systems.

Unlike for Jordan algebras, primitivity of Jordan triple systems and pairs is defined at a particular element (although it has been proved in [4] that primitivity can be moved to any other element). It is then natural to study primitivity of Jordan systems through their local algebras (see [1, 3, 4]). To do that, the key fact is that primitivity flows from pairs to their local algebras and back [3]. We will only make use of one direction (since the other relies on the structure theory [2, 33]), which we record in the following global-to-local inheritance theorem (Lemma 9 of [33] and Theorem 6.1 of [1]):

0.9. Theorem. If a Jordan pair $V$ is primitive at $b \in V^\sigma$, then the local algebra $V_b^{\pm \sigma}$ is primitive.
For algebras, the situation is neater since primitivity does not happen at a particular element (and since we do make use of the structure theory for algebras). In particular we have (Theorem 4.1 of [8]):

0.10. Theorem. Let J be a Jordan algebra, and \( 0 \neq a \in J \). If J is primitive, then \( J_a \) is primitive.

As mentioned before, our primitive systems will mainly be strongly prime with nonzero socle. For these one has (4.4 of [27]):

0.11. Proposition. Let J be a strongly prime Jordan system. If J has nonzero socle, then

(i) J is primitive (at each element of the socle if J is a pair or a triple system);
(ii) if J is an algebra or pair, then each local algebra is primitive with nonzero socle;
(iii) if J is a triple system, then each local algebra is either primitive with nonzero socle, or a subdirect sum of two primitive algebras with nonzero socle.

0.12

We finally mention some facts from Jordan PI theory. Recall that a polynomial \( f(x_1, \ldots, x_n) \in FJ[X] \), the free Jordan algebra on the set \( X \), is called essential if its image in the free special Jordan algebra \( SFJ[X] \) under the natural homomorphism has a monic leading monomial as an associative polynomial. (Note that there is no notion of a leading monomial as a Jordan polynomial, since Jordan monomials, i.e., monomials obtained from the generators from successive applications of the operations \( U \) and square, do not form a basis of \( FJ[X] \); hence there is no uniqueness in the representation of polynomials as linear combinations of monomials.) A PI–Jordan algebra is a Jordan algebra which satisfies some essential \( f(x_1, \ldots, x_n) \). We recall that the usual linearization process produces a linear essential Jordan polynomial of degree at most \( n \) out of any essential Jordan polynomial of degree \( n \), and therefore a Jordan algebra satisfying an identity of degree \( n \) always satisfies a multilinear identity of degree at most \( n \) (see [35, Sections 1.5 and 5.4]). From Theorems 1.1 and 5.2 of [7] together with the corollary to Theorem 3 of [18], analogues of Kaplansky’s theorem and Posner’s theorem follow:

0.13. Theorem. Let J be a PI–Jordan algebra. If J is primitive, then it is simple with capacity. If it is strongly prime, then the central closure \( \Gamma^{-1}J \) is simple with capacity.
The operant notion of a PI–Jordan triple system or pair is that of homotope-PI triple system or pair. We will use the notations of [1] and [3]. In particular, if \( f(x_1, \ldots, x_n) \) is a polynomial in the free Jordan algebra \( FJ[X] \) on a countable set of generators \( X \), and \( z \) is an element of the free Jordan triple system \( FT[X] \), the polynomial

\[
f(z; x_1, \ldots, x_n) = f^{(z)}(x_1, \ldots, x_n)
\]

is the image of \( f \) under the homomorphism \( FJ[X] \to FT[X]^{\pm} \) which is the identity on \( X \).

A Jordan triple system \( T \) satisfies a homotope polynomial identity (homotope-PI, for short) if there is an essential polynomial \( f(x_1, \ldots, x_n) \) in \( FJ[X] \) such that the polynomial \( f(y; x_1, \ldots, x_n) \) with \( y \in X \) different from the \( x_i \) vanishes under all substitutions of elements \( y, x_i \in T \). This definition extends to Jordan pairs \( V \) by considering their associated triple system \( TV \).

The fact that a Jordan system \( J \) satisfies a homotope–PI means that all homotopes, hence all local algebras, satisfy a given identity. We are interested in a weaker assertion, the existence of some \( a \in J \) for which the local algebra \( J \) is PI. We call such an element a PI-element and write \( \text{PI}(J) \) for the set of PI-elements of \( J \) (\( \text{PI}(V^+) = (\text{PI}(V^+), \text{PI}(V^-)) \) if \( J = V = (V^+, V^-) \) is a Jordan pair). Thus, the fact that \( J \) has a nonzero PI-element can be abbreviated to \( \text{PI}(J) \neq 0 \). Since the present work continues [27], we recall here the main results of that paper.

**Theorem.** Let \( J \) be a nondegenerate Jordan system. Then \( \text{PI}(J) \) is an ideal of \( J \).

We say that a Jordan system \( J \) is rationally primitive if it is primitive and has a nonzero PI-element. This is the Jordan analogue of strongly primitive associative algebras. Rational primitivity is characterized in the following analogue of Amitsur’s theorem on generalized identities.

**Theorem.** Let \( J \) be a Jordan system. The following are equivalent:

(a) \( J \) is rationally primitive.

(b) \( J \) is strongly prime and \( \text{Soc}(T) = \text{PI}(T) \neq 0 \).

(c) \( J \) is strongly prime and the local algebra at some nonzero element is a simple unital PI algebra.
As a consequence one has an analogue for Jordan systems of Kaplansky’s theorem, with homotope polynomial identities on Jordan systems playing the role of polynomial identities on algebras.

0.18. THEOREM. Let J be a primitive Jordan pair or triple system.

(i) If the local algebra at each element of J is PI, then J is simple, equal to its socle.

(ii) If J satisfies a homotope-PI, then J is simple with capacity.

1. THE EXTENDED CENTROID

In this section we define the extended centroid of a nondegenerate Jordan system. The construction is patterned after the corresponding construction in associative theory and its non-associative generalizations [9, 12]. The main difference is that the scalarity conditions in Jordan systems are not linear (since the operations are not linear), so in that respect, the situation is much the same as in the definition of the centroid of a Jordan system (cf. [16]).

1.1

Let T be a Jordan triple system and let I be an ideal of T. A linear mapping \( f: I \to T \) will be called a \( T \)-homomorphism if for all \( y \in I, x, z \in T \) it satisfies:

(i) \( f(P_y y) = P_x f(y) \),

(ii) \( f(P_y T) \subseteq I, \) and \( f^2(P_y x) = P_{f(y)} x \),

(iii) \( f((y, x, z)) = (f(y), x, z) \).

The set of \( T \)-homomorphisms with domain I will be denoted \( \text{Hom}_T(I) \).

For a Jordan pair \( V = (V^+, V^-) \) and an ideal \( I = (I^+, I^-) \), we define a \( V \)-homomorphism as a pair \( f = (f^+, f^-) \) of linear mappings \( f^\sigma: I^\sigma \to V^\sigma, \sigma = \pm \), satisfying, for all \( y^\sigma \in I^\sigma, x^\sigma, z^\sigma \in V^\sigma \),

(i') \( f^\sigma(Q_x y^{-\sigma}) = Q_{f^\sigma(x)} y^{-\sigma} \),

(ii') \( f^\sigma(Q_y V^{-\sigma}) \subseteq I^\sigma, \) and \( (f^\sigma)^2(Q_y x^{-\sigma}) = Q_{f^\sigma(x)} x^{-\sigma} \),

(iii') \( f^\sigma((y^\sigma, z^{-\sigma}, x^\sigma)) = (f^\sigma(y^\sigma), z^{-\sigma}, x^\sigma) \).

We denote by \( \text{Hom}_V(I, V) \) the set of \( V \)-homomorphisms from I into V.

For a Jordan algebra \( J \) and an ideal \( I \), an algebra \( J \)-homomorphism is a \( J \)-homomorphism for the underlying triple system which, in addition to (i),
(ii), and (iii), satisfies

(iv) \( f(y^2) \in I \), and \( f^2(y^2) = f(y)^2 \),
(v) \( f(x \circ y) = x \circ f(y) \).

That is, \( f \) is a \( \hat{J} \)-homomorphism for the underlying triple system of a unital hull \( \hat{J} \) of \( J \). Again we denote the set of \( J \)-homomorphisms defined on \( I \) by \( \text{Hom}_J(I, J) \).

Following [9, 12], a pair \((f, I)\), where \( f \in \text{Hom}_J(I, J) \), for a Jordan system (pair, triple, or algebra) will be called a permissible map if the ideal \( I \) is essential. We will be interested in nondegenerate Jordan systems, and for those, an ideal \( I \) is essential if and only if it has zero annihilator. We recall next some facts about essentiality of ideals. Recall that if \( I, L \) are ideals of the Jordan triple system \( J \) then the set \( I \circ L = P_j L + P_j P_j L \) is again an ideal of \( T \) (see [22, p. 221]), and if \( J \) is a Jordan algebra, \( I \circ L = U_j L \) (since this is already an ideal, hence \( U_j U_j L \subseteq U_j L \)). Also, recall that the annihilator of an ideal \( I \) in a nondegenerate Jordan system can be described as \( \text{Ann}_J(I) = \{ z \in J \mid P_j I = 0 \} \) (see [22, 1.7i]).

1.2. Lemma. Let \( J \) be a nondegenerate Jordan system, then:

(a) If \( I, L \) are essential ideals of \( J \), then \( I \circ L \) is essential.
(b) If \( \Phi \) is a ring of scalars for \( J \) and \( I \) is a \( \Phi \)-ideal, then \( I \) is essential as a \( \Phi \)-ideal if and only if it is essential as a \( \mathbb{Z} \)-ideal.
(c) If \( J \) is an algebra and \( I \) is an algebra ideal, \( I \) is essential as an ideal of the underlying triple if and only if it is essential as an algebra ideal.
(d) If \( J = (V^+, V^-) \) is a Jordan pair and \( I \) is an ideal of \( J \), then \( I \) is essential if and only if \( T(I) \) is essential in \( T(J) \).
(e) If \( J \) is a Jordan triple system and \( I \) is an ideal of \( J \), then \( I \) is essential if and only if \( V(I) \) is essential in \( V(J) \).

Proof. The assertions (a), (b), and (c) follow directly from nondegeneracy of \( J \) and the description \( \text{Ann}_J(I) = \{ z \in J \mid P_j I = 0 \} \) of the annihilator of an ideal \( I \). For (d) and (e), it suffices to note, in addition, the equalities \( \text{Ann}_{T(J)}(T(I)) = T(\text{Ann}_J(I)) \) and \( \text{Ann}_{V(J)}(V(I)) = V(\text{Ann}_J(I)) \), which are also straightforward.

Later, we will need another characterization of the annihilator of an ideal in a nondegenerate Jordan system:

1.3. Lemma. Let \( J \) be a nondegenerate Jordan system, and let \( I \) be an ideal of \( J \). If \( z \in J \) has \( P_j z = 0 \), then \( z \in \text{Ann}_J(I) \).

Proof. Consider the polynomial system \( J[t] \) over \( J \). As in [8, 1.1], it is easy to see that \( J[t] \) is again nondegenerate and that \( I[t] \) is an ideal of
$J[t]$ with $P_{[t]}z = 0$. Take an element $a = P_h P_h h$ with $h \in I[t]$. Then $P_a = P_{P_h h} = P_{P_h h} P_{h} = 0$; hence $a = P_h P_h h = 0$ by nondegeneracy. Thus, setting $h = x + ty$ with $x, y \in I$, the term of degree one in $a$ is $(P_x P_y) z = 0$. Therefore $P_x P_y I = 0$, hence $P_{y, I} I \subseteq I \cap \text{Ann}_t(I) = 0$, so $z \in \text{Ann}_t(I)$. 

1.4

We will define the extended centroid of a nondegenerate Jordan system as a set of equivalence classes of permissible maps modulo a suitable equivalence relation. For linear algebras [9, 12] this set has a ring structure coming from the addition of homomorphisms and from the composition of restrictions of homomorphisms. Defining the corresponding operations in our case requires some preparatory work. In fact we do not even know yet if the sets of homomorphisms introduced above are submodules of the usual $\text{Hom}_a(I, J)$, and, moreover, $J$-homomorphisms do not always restrict to $J$-homomorphisms on smaller ideals (cf. below).

We will deal with these problems for Jordan triple systems, and later on we will reduce the pair and algebra cases to that one. For permissible maps $(f, I)$ and $(g, L)$ of the Jordan triple system $T$, define a relation “$\sim$” by

$(f, I) \sim (g, L)$ if there is an essential ideal $K$ of $T$, contained in $I \cap L$, such that $f(x) = g(x)$ for all $x \in K$.

It is easy to see that this is an equivalence relation. The quotient set $\mathfrak{Q}(T)$ will be called the extended centroid of $T$. We will write $[f, I]$ for the equivalence class of the permissible map $(f, I)$.

1.5. LEMMA. Let $T$ be a nondegenerate Jordan system, let $I$ be an ideal of $T$, and let $f, g \in \text{Hom}_T(I, T)$. Then, for all $x \in I$ and $y \in T$, $fgP_x y = gfP_x y$.

Proof. Notice first that $g(P_2 y)$ and $f(P_2 y)$ belong to $I$; hence $fgP_x y$ and $gfP_x y$ make sense. We will show that $[f, g]P_x y = fgP_x y - gfP_x y$ is an absolute zero divisor, and the result will follow since $T$ is nondegenerate.

We have, for all $z \in T$,

$$P_{[f, g]P_x y z} = P_{f P_x y z} + P_{g P_x y z} - \{fgP_x y, z, gfP_x y\}.$$

Now, $P g P_x y z = g^{2} f^{2} P_x y z$ (by (ii)) = $g^{2} f^{2} P_x P_x z = g^{2} P_{f(x)} P_y z$ (by (ii)) = $g^{2} P_{f(x)} P_y z$ = $P_{f(x)} g^{2} P_x y z$ (by (i) three times, since $g(P_x P_y z) = P_x g(P_y z) \in I$ by (ii)) = $g^{2} f^{2} P_x y z$ (by (ii)) = $f^{2} g^{2} P_x y z$ (by (i) three times, as before) = $f^{2} g^{2} P_x y z = P_{f g P_x y} z$ (by (ii)).

On the other hand, applying (ii) and (iii), $(fgP_x y, z, gfP_x y) = fg^{2}(P_x y, z, fp_x y) = f^{2}(g^{2} P_x y, z, P_x y)$ = $f^{2}(g^{2} P_x y, z, P_x y) = f^{2} g^{2}(P_x y, z, P_x y) = 2 f^{2} g^{2} P_x y z = 2 P_{f g P_x y} z$.

Therefore, $P_{[f, g]P_x y z} = P_{f g P_x y} z + P_{g P_x y z} - 2 P_{f g P_x y} z = 0$. □
1.6. **Lemma.** Let $T$ be a nondegenerate Jordan triple system, and let $I$ be an ideal of $T$. If $f, g \in \text{Hom}_T(I, T)$, then $f + g \in \text{Hom}_T(I, T)$.

**Proof.** Clearly $f + g$ satisfies properties (i) and (iii), since these are linear conditions on the mapping. Note also that $(f + g)P_T \subseteq P_T + gP_T \subseteq I$. Thus it is enough to show that for all $x \in I$ and $y \in T$, $(f + g)^2(P_{f+g} y) = P_{f+g} (f_x + g_{f+g} y)$.

We have $(f + g)^2P_{f+g} y = f^2P_{f+g} y + g^2P_{f+g} y + gfP_{f+g} y + fgP_{f+g} y$. Since $T$ is nondegenerate, we can apply Lemma 1.5 to get $gfP_{f+g} y + fgP_{f+g} y = gf(2P_{f+g} y) = gf(x, y, x) = (f(x), y, g(x))$. Thus $(f + g)^2P_{f+g} y = f^2P_{f+g} y + g^2P_{f+g} y + (f(x), y, g(x)) = P_{f+g} (f_x y + P_{f+g} (g_{f+g} y) + (f(x), y, g(x)) = P_{f+g} (f_{f+g} x + g_{f+g} y)$.

1.7

We now turn to the problem of inducing a $\Phi$-module structure on $\mathcal{H}(T)$ based on the $\Phi$-module structures of the sets $\text{Hom}_T(I, T)$, which are submodules of the corresponding $\Phi$-modules $\text{Hom}_T(I, T)$ by 1.6. The difficulty that we face is that if $\lambda, \mu$ belong to $\mathcal{H}(T)$, and $(f, I) \in \lambda$, $(g, I) \in \mu$ are representatives, it is not clear whether the restrictions of $f$ and $g$ to their common domain $I_f \cap I_g$ are again $T$-homomorphisms, which would be needed to define $\lambda + \mu$ as the class $[f_{f \cap I} + g_{f \cap I}, I_f \cap I_g]$ (here and in the following, $h_L$ denotes the restriction of $h$ to $L$).

**Example.** Consider the associative commutative algebra $\mathbb{Z}[t]$ of polynomials over $\mathbb{Z}$. We can form the triple system $T = \mathbb{Z}[t]^2$ as usual: $P_{t, q} = pqt$. Take the ideal $I = \mathbb{Z}[t]t$ of $T$, and the map $f: I \to T$ given by $f(p(t)) = p(t)$. Then $(f, I)$ is permissible. Now let $L = \mathbb{Z}[t]^2 + \mathbb{Z}[t]^2t$. This is easily seen to be an ideal of $T$ (which is essential since it is nonzero and $T$ is strongly prime), and it is contained in $I$. We have $f(P_{t, 1}) = f(t^4) = t^3 \notin L$; hence $f(P_{t, 1}) \subseteq L$, and the restriction of $f$ to $L$ is not a $T$-homomorphism.

Let $(f, I)$ be a permissible map of the triple system $T$ and let $L$ be a nonzero ideal of $T$ contained in $I$. We will say that $f$ restricts to $L$ if $f_L \in \text{Hom}_T(L, T)$. It is straightforward that a necessary and sufficient condition for $f$ to restrict to $L$ is that $f(P_{t, 1}) \subseteq L$. Fortunately this holds in a number of cases, as we show in the following lemmas.

1.8. **Lemma.** Let $(f, I_f)$, $(g, I_g)$ be permissible maps of a nondegenerate Jordan triple system $T$. Then the set $I_{f, g} = \{x \in I_f \cap I_g \mid f(x) = g(x)\}$ is an ideal of $T$, and both $f$ and $g$ restrict to $I_{f, g}$.

**Proof.** It is clear that if $x, y \in I_{f, g}$, then $x + y \in I_{f, g}$. Also, if $z \in T$, then $f(P_{f, z}) = P_{f, f(x)} = P_{f, f(x)} = g(P_{g, z}) = g(P_{g, x})$; hence $P_T I_{f, g} \subseteq I_{f, g}$. Similarly $(I_{f, g}, T, T) \subseteq I_{f, g}$. 
Now set \( v = f(P_z, z) - g(P_z, z) \) with \( x \) and \( z \) as before. We have

\[
P_v = P_{f(P_z, z)} + P_{g(P_z, z)} - P_{f(g(P_z, z), g(P_z, z))} = f^2 P_{P_z, x} + g^2 P_{P_z, x} - 2 fg P_{P_z, x} \\
= (f^2 + g^2 - 2fg) P_{P_z, x} + P_{g(x) P_z, x} - P_{f(x), g(x) P_z, x} \\
= P_{f(x) P_z, x} + P_{g(x) P_z, x} - P_{f(x), g(x) P_z, x} \\
\text{(since } fg P_{x, z} = P_{f(x), g(x)} \text{)} \\
= P_{f(x) - g(x) P_z, x} = 0 \text{ (since } f(x) = g(x) \text{)}.
\]

Therefore \( v = 0 \) by nondegeneracy of \( T \), and \( P_z, z \in I_{f, g} \); hence \( P_{I_{f, g}} \subset I_{f, g} \), and \( I_{f, g} \) is an ideal of \( T \).

Finally, for all \( x \in I_{f, g} \) and \( z \in T \), \( f(P_z, z) = g(P_z, z) \) gives \( f(g(P_z, z)) = g^2(P_z, z) = P_{g(x) z} = P_{f(x) z} = f^2(P_z, z) \); hence \( f(P_z, z) \in I_{f, g} \), and \( f_{P_{I_{f, g}}} \subset I_{f, g} \). Similarly \( g_{P_{I_{f, g}}} \subset I_{f, g} \); hence both \( f \) and \( g \) restrict to \( I_{f, g} \).

Our next result will allow us to find a common restriction for two permissible maps.

1.9. LEMMA. Let \( T \) be a Jordan triple system and let \((f, I)\) be a permissible map. If \( K, L \) are ideals of \( T \) with \( K \subset I \), then \( f \) restricts to \( K \subset L \). In particular, \( f \) restricts to \( K \subset L \).

Proof. Set \( N = P_K P_T, K + (K \subset L, T, K \subset L) \). Then we have

\[
P_{K, L} \subset P_{P_K P_T, T} + (P_K, L, T, P_T P_K L) + P_{P_{K, L}, T},
\]

and the containments

\[
P_{P_K P_T, T} \subset P_{K, L} P_T, T + \{P_K, L, T, P_K L\} \subset N,
\]

and

\[
P_{P_{K, L}, T} \subset P_{P_K P_T, T} + \{P_T P_K L, T, P_T P_K L\} \subset P_{P_K P_T, T} + \{K \subset L, T, K \subset L\} \subset N + P_T N.
\]

Thus, \( P_{K, L} \subset N + P_T N \) and \( f(P_{K, L}) \subset f(N) + P_T f(N) \) (recall that \( N \subset I \), the domain of \( f \)).

Now, \( f(N) = f(P_K P_T, K + (K \subset L, T, K \subset L)) = P_K P_T, f(K) + (f(K \subset L), T, K \subset L) \subset P_K P_T, T + (T, K \subset L) \subset K \subset L \). Thus, \( f(P_{K, L}) \subset K \subset L \) and \( f \) restricts to \( K \subset L \).

Lemma 1.8 gives a better insight into the equivalence relation which defines the extended centroid.
1.10. Lemma. Let $T$ be a nondegenerate Jordan triple system and let $(f, I_f), (g, I_g)$ be permissible maps. Then the following are equivalent,

1. $(f, I_f) \sim (g, I_g)$,

2. There is an essential ideal $I \subseteq I_f \cap I_g$ such that $f_I = g_I \in \text{Hom}_T(I, T)$,

3. $f_{I_f \cap I_g} = g_{I_f \cap I_g} \in \text{Hom}_T(I_f \cap I_g, T)$.

If in addition $T$ is prime, then these conditions are equivalent to

4. $f(x) = g(x)$ for some nonzero $x \in I_f \cap I_g$.

Proof. (1) $\Rightarrow$ (2). By Lemma 1.8, $I_{f,g} = \{x \in I_f \cap I_g \mid f(x) = g(x)\}$ is an ideal of $T$, and since $f \sim g$, it contains an essential ideal. Also $f_{I_{f,g}} = g_{I_{f,g}} \in \text{Hom}_T(I_{f,g}, T)$, so we can take $I = I_{f,g}$ in (2).

(2) $\Rightarrow$ (3). Consider the mapping $h = f - g : I_f \cap I_g \to T$. Take $x \in I_f \cap I_g$ and $y \in I$. Then $P_{h(x)}P_y = P_{h(x)}P_y + P_{g(x)}P_y - P_{f(x),g(x)}P_y = f^2P_xP_y + g^2P_xP_y - 2fgP_xP_y = P_{f(y)}, g(y) = P_{f(y)}, f(y) = 0$, since $f(y) = g(y)$. Thus the element $v = P_{h(x)}v$ has $P_v = P_{h(x)}v, P_{h(x)}v = 0$. Therefore $v = 0$, since $T$ is nondegenerate, and $P_{h(x)}I = 0$. This means that $h(x)$ belongs to the annihilator of $I$, but this is an essential ideal. Hence $h(x) = 0$; i.e., $f(x) = g(x)$.

(3) $\Rightarrow$ (1). This is clear, since if $I_f$ and $I_g$ are essential, so is $I_f \cap I_g$.

Finally, it is obvious that (4) follows from the above conditions, and, reciprocally, if there is a nonzero $x \in I_f \cap I_g$ with $f(x) = g(x)$, then, with the notation of 1.8, $I_{f,g}$ is nonzero and hence essential in the prime system $T$. 

1.11

We can now define addition in the extended centroid $\mathcal{E}(T)$ of a nondegenerate Jordan triple system $T$ in the following way: if $\lambda, \mu$ belong to $\mathcal{E}(T)$, and $(f, I_f), (g, I_g) \in \mu$ are representatives, then $f$ and $g$ restrict to $I = (I_f \cap I_g) \ast T$, and we define $\lambda + \mu = [f_I + g_I, I]$. Now, $I$ is essential by 1.2 since both $I_f$ and $I_g$ are; hence the map $(f_I + g_I, I)$ is permissible by 1.6 and 1.9. Moreover, the sum is well defined since, if $(h, I_h)$ belongs to $\lambda$, then $f$ and $h$ agree on $I_f \cap I_h$ by 1.10, and, setting $I' = (I_h \cap I_g) \ast T$ and $L = (I_h \cap I_f \cap I_g) \ast T \subseteq I \cap I'$, we have that $f_I, g_I, h_I$ and $h_I' \ast T$ restrict to the essential ideal $L$ by Lemma 1.9, and $(f_I + g_I) = f_L + g_L = h_L + g_L = (h_I + g_I)$. Note also that the ring of scalars $\Phi$ acts on $\mathcal{E}(T)$ by the rule $\alpha[f, I] = [\alpha_f, I]$, and therefore $\mathcal{E}(T)$ is a $\Phi$-module.

To define a multiplication in $\mathcal{E}(T)$ we need the following fact,
1.12. Lemma. Let $T$ be a nondegenerate Jordan triple system and let $I$ be an ideal of $T$. There is a bilinear map

$$\text{Hom}_T(I, T) \times \text{Hom}_T(I, T) \to \text{Hom}_T(I \ast T, T)$$

given by $(f, g) \mapsto fg: x \mapsto f(g(x))$, which satisfies $fg = gf$ for all $f, g \in \text{Hom}_T(I, T)$.

Proof. Since $g(P_I T) \subseteq I$, we have $g(I \ast T) \subseteq I$, and $fg$ is defined on $I \ast T$.

It is obvious that the composition $fg$ defined on $I \ast T$ satisfies conditions (i) and (iii) of the definition of $T$-homomorphism. We next check condition (ii).

Notice first that

$$P_{I \ast T} T \subseteq P_I P_I P_I T + \{I \ast T, T, I \ast T\} + P_I P_I P_I T$$

$$\subseteq P_I(I \ast T) + P_I P_I(I \ast T) + \{I \ast T, T, I \ast T\}.$$  

Thus

$$fg(P_{I \ast T} T) \subseteq P_I fg(I \ast T) + P_I P_I fg(I \ast T) + \{fg(I \ast T), T, I \ast T\}$$

$$\subseteq P_I T + P_I P_I T + \{T, I \ast T\} \subseteq I \ast T.$$  

Now we have to show that $(fg)^2 P_z z = P_{fg(x)} z$ for all $x \in I \ast T$ and $z \in T$. Note that if $x_1, x_2 \in I \ast T$, $(fg)^2 P_{x_1, x_2} z = P_{fg(x_1, x_2)} z$ follows by repeatedly applying (iii). Since $P_{x_1, x_2} z = P_{x_1} + P_{x_2} + P_{x_1, x_2}$, it is clear that it suffices to consider elements $x$ of the forms $x = P_I T$ or $P_I P_I t$ with $y \in I$ and $s, t \in T$. For the first ones, we have $(fg)^2(P_{P_I T} z) = (fg)^2(P_{P_I P_I t} z) = (fg)^2 f^2 P_{P_I P_I t} z$ (by Lemma 1.5) $= f^2 P_{P_I P_I t} z = f^2 P_{P_{P_I P_I t} z}$ $= f^2 f^2 P_{P_I P_I t} z = P_{fg(P_{P_I t})} z$. For the second ones, $(fg)^2(P_{P_I t} z) = (fg)^2(P_{P_I t} P_{P_I t} z) = P_I f^2 P_{P_I t} z$ (by three times, since $h(P_{P_I t} z) \in h(P_{I \ast T}) \subseteq I$ for $h = g$, $fg$ or $gfg$) $= P_{P_{fg(P_{P_I t})}} z$ (by the previous case) $= P_{P_{fg(P_{P_I t})}} z = P_{fg(P_{P_I t})} z$.

Finally, $fg = gf$ follows from Lemma 1.5. 

1.13

We can now define a multiplication in $\mathcal{O}(T)$: if $\lambda, \mu \in \mathcal{O}(T)$, take $(f, I_f) \in \lambda$ and $(g, I_g) \in \mu$. Then $(f, I_f)$ and $(g, I_g)$, where $I = (I_f \cap I_g) \ast T$, are permissible maps by 1.9. We define the product $\lambda \mu$ as the class of $(fg_{I \ast T}, I \ast T)$, which is permissible by 1.12. This is easily seen to be well defined and gives $\mathcal{O}(T)$ a structure of a commutative $\Phi$-algebra.
1.14

In our construction of the extended centroid \( \mathcal{C}(T) \) of a nondegenerate triple system \( T \) we have not paid attention to the ring of scalars \( \Phi \). However, it was implicitly present in the construction since our ideals are \( \Phi \)-ideals and our linear maps are \( \Phi \)-linear. Nevertheless, the resulting ring \( \mathcal{C}(T) \) is independent of the ring of scalars \( \Phi \) with which we started. In fact, if we denote by \( \mathcal{C}(T) \) the extended centroid over the ring \( \mathbb{Z} \) of integers and by \( \mathcal{C}_\Phi(T) \) the extended centroid which arises when we take \( \Phi \) as the ring of scalars for \( T \) (so that, in the second case, we consider \( \Phi \)-ideals and \( \Phi \)-module homomorphisms), there is an obvious mapping \( \mathcal{C}_\Phi(T) \rightarrow \mathcal{C}(T) \) derived from the fact that every essential \( \Phi \)-ideal is an essential \( \mathbb{Z} \)-ideal by 1.2. Moreover, that mapping is trivially a homomorphism of rings. Now, if \( \lambda \in \mathcal{C}(T) \) has a representative \( (f, I) \in \lambda \), then \( I \star T \) is a \( \Phi \)-ideal (since \( \Phi(I \star T) = I \star \Phi T \subseteq I \star T \)) and we can restrict \( f \) to \( I \star T \), so that \( \lambda \) always has a representative \( (f, I) \), for a \( \Phi \)-ideal \( I \). Now, if \( \alpha \in \Phi \), the multiplication by \( \alpha \) defines an element \( \tilde{\alpha} \in \text{Hom}_\mathcal{T}(T, T) \), and we have, by Lemma 1.5, \( \tilde{\alpha} f = f \tilde{\alpha} \) on \( P_T T \) (hence on \( I \star T \) by property (i) of \( T \)-homomorphisms). We can also define in the obvious way permissible mappings \( \alpha f \) and \( f \alpha \) on \( I \) and 1.10 and the equality \( \tilde{\alpha} f = f \tilde{\alpha} \) on \( I \star T \) implies that \( f \alpha = \alpha f \) on \( I \). Thus, \( f \) is in fact \( \Phi \)-linear, and the element \( \lambda \) is the image of the class \( [f, I] \) in \( \mathcal{C}_\Phi(T) \). Therefore the mapping \( \mathcal{C}_\Phi(T) \rightarrow \mathcal{C}(T) \) is surjective, and since it is clearly injective, it is a ring isomorphism.

The basic structural facts about the extended centroid are summarized in the following result.

1.15. Theorem. Let \( T \) be a nondegenerate Jordan triple system; then the extended centroid \( \mathcal{C}(T) \) is a commutative, associative, unital (von Neumann), regular ring. If, in addition, \( T \) is prime, then \( \mathcal{C}(T) \) is a field.

Proof. It is straightforward that \( \mathcal{C}(T) \) is a commutative associative ring with the operations defined above. Its unit element is the class of the identity mapping \( \text{Id} \in \text{Hom}_\mathcal{T}(T, T) \).

Now, let \( (f, I) \) be permissible. Then \( f(I) \) and \( \text{Ker} f \) are ideals of \( T \). Indeed, \( P_T f(I) = f(P_T I) \subseteq f(I) \) by (i), \( f(I, T, T) = f(I, T, T) \subseteq f(I) \) by (ii), and \( P_T f(I) = f^2(P_T I) \subseteq f(I) \) by (iii), so \( f(I) \) is an ideal. On the other hand we have \( f(P_T \text{Ker} f) = P_T f(\text{Ker} f) = 0 \) by (i); hence \( P_T \text{Ker} f \subseteq \text{Ker} f \). Next, \( f((\text{Ker} f, T, T)) = f(\text{Ker} f) = 0 \) by (iii); hence \( \text{Ker} f, T, T) \subseteq \text{Ker} f \). Now, for \( x \in \text{Ker} f \) and \( y \in T \), \( P_{f} T = f^2 P_{P_f} T \) (by (iii)) = \( f^2 P_{P_f} P_{P_f} T = f^2 P_{P_f} P_{P_f} T \) (by (ii)) = \( 0 \). Thus \( f(P_x y) = 0 \) by nondegeneracy of \( T \), and \( P_{\text{Ker} f} \subseteq \text{Ker} f \).

A standard argument applying Zorn’s lemma to the ideals \( K \subseteq I \) of \( T \) with \( K \cap \text{Ker} f = 0 \) produces an essential complement in \( I \), an ideal \( K \) of \( T \) such that \( K \cap \text{Ker} f = 0 \) and \( K' = K \oplus \text{Ker} f \subseteq I \) is essential in \( I \) and
hence in \( T \) (since \( 0 = \text{Ann}_f(K') \supseteq I \cap \text{Ann}_f(K') \) and \( I \) is essential). Moreover \( K' \ast T \subseteq (K \ast T) \oplus \text{Ker} f; \) hence \((K \ast T) \oplus \text{Ker} f \) is essential in \( T \) by 1.2, so taking \( K \ast T \) instead of \( K \) if necessary and applying 1.9, we can assume that \( f \) restricts to \( K \).

Next, the same argument as before gives an ideal \( L \) such that \( L \oplus f(K) \) is essential. Define \( g: L \oplus f(K) \to T \) by \( g(a \oplus f(b)) = b, \) where \( a \in L \) and \( b \in K. \) This is well defined since, if \( f(b) = 0, \) then \( b \in K \cap \text{Ker} f = 0. \) Moreover, \( g \) clearly satisfies properties (i) and (iii) of the definition of \( T \)-homomorphism. Now, if \( x \in L, \) \( k \in K, \) and \( y \in T, \) we have \( P_{x+f(k)y} = P_k^2 y + P_{f(k)}^2 y = P_k y + f^2 P_k y. \) Therefore, using the definition of \( g \) and taking into account that \( f(P_k y) \in K, \) \( g(P_{x+f(k)y}) = f(P_k y) \in f(P_k T) \subseteq f(K), \) and \( g^2(P_{x+f(k)y}) = P_k y = P_{f(k)y} = P_{g^2(x+f(k)y)}; \) hence \( g \) satisfies (ii), and \((g, L \oplus f(K))\) is permissible.

Now, if \( c \in K, \) \( d \in \text{Ker} f, \) and \( d + c \) belongs to the domain of \( fgf, \) we have
\[
fgf(d + c) = fg(f(c)) = f(c) = f(d + c).
\]
This shows that \( fgf \sim f; \) hence for any \( \lambda = [f, I] \) we have found that \( \mu = [g, L \oplus f(K)] \) with \( \lambda \mu \lambda = \lambda. \) This proves that \( \mathcal{O}(T) \) is von Neumann regular.

Finally, if \( T \) is prime, keeping the previous notation, \( \lambda \neq 0 \) implies \( \text{Ker} f = 0 \) by 1.10. Then \( f(I) \) is essential, since if \( N \) is an ideal of \( T, \) then \( N \cap f(I) = 0 \) implies that \( f(N \ast I) = N \ast f(I) = 0; \) hence \( N \ast I \subseteq \text{Ker} f = 0, \) and \( N \subseteq \text{Ann}_f(I) = 0. \) Thus \( \mu = [g, f(I)] \) satisfies \( \lambda \mu = \mu \lambda = 1, \) and therefore \( \mathcal{O}(T) \) is a field.

1.16. Remark. An important property of the elements of the extended centroid is the following “common multiple” property: if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) is a finite set of elements of \( \mathcal{O}(T), \) there are representatives \((f_i, I) \in \mathcal{O}_{\lambda_i} \) which share a common domain \( I. \) This follows by taking representatives \((g_i, I) \in \lambda_i \) and setting \( I = (\cap_{i=1}^n I_i) \ast T. \) Then each \( g_i \) restricts to \( I, \) and this is essential since the \( I_i \)’s are. Thus it suffices to put \( f_i = (g_i)_I. \)

2. EXTENDED CENTROID OF ALGEBRAS AND PAIRS

In this section we carry out the construction of the extended centroid for nondegenerate Jordan algebras and pairs and relate it to the extended centroid of the Jordan triple systems naturally attached to them.

We begin with Jordan algebras. Our first aim is to compare \( J \)-homomorphisms of a nondegenerate Jordan algebra \( J \) with those of its underlying \( T \)-system. It is clear that if \( I \) is an ideal of \( J \) and \( f: I \to J \) is a \( J \)-homomorphism as a Jordan algebra, then it is a \( J \)-homomorphism as a Jordan triple system. We now examine what is needed for the reciprocal.
In the proof of the following result we make use of the identities

\[(*) \quad U_a(b \circ c) = \{a, b, c\} \circ a - \{a^2, b, c\},\]

\[(**) \quad U_{a+b} = U_a U_b + U_b U_a + U_{a,b} U_{a,b} - U_{a^2+b^2},\]

and the partial linearization of (**),

\[U_{a+b,c} = U_{a,c} U_b + U_{b,c} U_a + U_{a,b} U_{c,b} + U_{c,b} U_{a,b} - U_{a+c,b^2},\]

which follow easily from Macdonald's theorem [15, 3.4.16].

2.1. Lemma. If \(J\) is a nondegenerate Jordan algebra, \(I\) is an ideal of \(J\), and \(f : I \to J\) is a \(J\)-homomorphism of the underlying Jordan triple, then it is a \(J\)-homomorphism of Jordan algebras if \(f(y^2) \in I\) for all \(y \in I\).

Proof. Take \(x, y \in I\). We show first that \(f(x) \circ y = x \circ f(y)\). Indeed, if \(z \in J\), we have \(U_z(f(x) \circ y) = \{z, f(x), y\} \circ z - \{z^2, f(x), y\}\) (by \((*)\)) = \(\{z, x, f(y)\} \circ z - \{z^2, x, f(y)\}\) (by (i) and (iii)) = \(U_z(x \circ f(y))\) (by (**)).

Therefore \(U_z(f(x) \circ y - x \circ f(y)) = 0\), and \(f(x) \circ y - x \circ f(y)\) belongs to \(\text{Ann}_J(I) = 0\) by 1.3.

Next, take \(x \in J\) and \(y \in I\). We will prove that \(f(x \circ y) = x \circ f(y)\). To see that, take \(z \in U_I J = I \ast J\). By identity (**) and properties (i), (ii), and (iii), we have

\[U_{f(y) \circ z} - f^2(z) = (U_{f(y)} U_x + U_x U_{f(y)} + U_{f(y), x} U_{f(y), x} - U_{f(y)^2, x}) z .\]

Now \(U_{f(y) \circ z} - f^2(z) = f(y) \circ (f(y), z, x^2) - U_{f(y)^2} (z \circ x^2)\) (by identity (**) = \(y \circ f(f(y), z, x^2) - U_{f^2(z) \circ x^2}\) (since \((f(y), z, x^2) \in I \ast J\) and \(z \circ x^2 \in I \ast J\)) = \(y \circ (f^2(z), x^2) - U_{f^2(z)} (z \circ x^2)\) (by (i), (ii), and (iii), taking into account that \(f(z) \in I\) and \(f^2(z \circ x^2) = f^2(z, x, x) = f^2(z, x, x) = f^2(z) \circ x^2\) by (iii)) = \(U_{f^2(z)} f^2(z)\), again by (**). Therefore,

\[U_{f(y) \circ z} = (U_{y} U_x + U_x U_{y} + U_{x,y} - U_{x^2, y}) f^2(z) = U_{y \circ f^2(z)},\]

by (**).

Next, setting \(a = y, c = f(y)\), and \(b = x\) in the identity (**) and taking \(z \in U_J I\), we obtain

\[U_{y \circ f(y) \circ z} = (U_{y, f(y)} U_x + U_x U_{y, f(y)} + U_{y, x} U_{f(y), x} + U_{f(y), x} U_{y, x} - U_{y, f(y), x}) z = (U_{y, U_x} + U_x U_{y, y} + 2U_{x, y} f(z) = U_{y \circ f(y), x^2} z.\]
Now, \( U_{y \cdot f(y) \cdot x^2, z} = f(y) \circ (y, z, x^2) + y \circ (f(y), z, x^2) - (f(y), z \circ x^2, y) \) (by linearized \((*)\)) = \( y \circ f((y, z, x^2) + y \circ (f(y), z, x^2) - (f(y), z \circ x^2, y) \)(by \(I = 2y \circ (y, f(z), x^2) - (y, f(z) \circ x^2, y) \)(by \(i\)) and \((iii)\), taking into account that \( z \circ x^2 = (z, x, x) \)) = 2U_{y \cdot x^2, f(z)}.

Therefore,

\[
U_{y \cdot x \cdot f(y) \cdot x \cdot z} = \left( 2(U_{I \cdot I} + U_{I \cdot U_y} + U_{I \cdot U_y}^2) \right) f(z) = 2U_{y \cdot x} f(z),
\]

by \((*)\), and

\[
U_{I \cdot y \cdot x \cdot f(y) \cdot x \cdot z} = fU_{y \cdot x \cdot f(y) \cdot x \cdot z} = f2U_{y \cdot x} f(z) = 2U_{y \cdot x} f^2(z),
\]

by \((i)\).

Therefore, if \( z \in U_{I \cdot J}, U_{I \cdot y \cdot x \cdot f(y) \cdot x \cdot z} = U_{I \cdot y \cdot x} z + U_{I \cdot y} z - U_{I \cdot y \cdot x \cdot f(y) \cdot x \cdot z} = 2U_{y \cdot x} f^2(z) - 2U_{y \cdot x} f^2(z) = 0, \) and, by nondegeneracy of \( J, f(y \circ x) - f(y) \circ x \in \text{Ann}_{J} (U_{I \cdot J}), \) the annihilator of \( U_{I \cdot J} \). But since \( I \) is essential, \( I \cdot J = U_{I \cdot J} \) is essential; hence \( \text{Ann}_{J} (U_{I \cdot J}) = 0, \) and \( f(y \circ x) = f(y) \circ x \) for all \( x \in J, y \in I. \)

Also, for all \( z \in J, \)

\[
\{f(y)^2, z, f^2(y^2)\}
\]

\[
= f^2 \{f(y)^2, z, y^2\}
\]

\[
= f^2 \{f(y) \circ (f(y), z, y^2)\} - f^2U_{I \cdot y} (z \circ y^2) \quad (\text{by \((*)\)})
\]

\[
= f^2f^2 \{y \circ (y, z, y^2)\} - f^2f^2U_{I \cdot y^2} (z \circ y^2) = f^4U_{I \cdot y^2} z
\]

\[
= U_{I \cdot y^2, f(y) \circ x \cdot z} \quad (\text{since \( f(y^2) \in I \)}).
\]

Therefore, \( U_{I \cdot y^2, f(y) \circ x \cdot z} = U_{I \cdot y^2} + U_{I \cdot y} - 2U_{I \cdot y^2} = U_{I \cdot y^2} - 2U_{I \cdot y} \)

\[
= U_{I \cdot y^2} + f^2U_{y^2} - 2U_{I \cdot y^2} \quad (\text{by \((ii)\)}) = U_{I \cdot y^2} + f^2U_{y^2} - 2U_{I \cdot y^2} \quad (\text{since \( f^2U_{y^2} = f^2U_{y} fU_{y} = f^2fU_{y} \) by \((i)\) two times, noticing that \( U_{I \cdot J} \subseteq I \text{ by \((ii)\)}\)) = U_{I \cdot y^2} + U_{I \cdot y^2} - 2U_{I \cdot y^2} \quad (\text{by \((ii)\) two times, notic- ing that \( f(y^2) \in I \)} = 0, \) and \( f^2(y^2) = f(y^2) \) since \( J \) is nondegenerate.

Thus, \( f \) satisfies \((iv)\) and \((v)\) of the definition of \( J\)-homomorphism for algebras. \(\square\)

2.2. **Remark.** Since the condition \( f(I^2) \subseteq I \) is part of condition \((iv)\) in the definition of \( J\)-homomorphism of algebras, this lemma implies that for a nondegenerate \( J, \) condition \((v)\) follows from \((i)-(iv).\) Moreover, since we did not make use of the condition \( f(I^2) \subseteq I \) in the proof of \((v),\) it follows that \((v)\) is a consequence of \((i)-(iii)\) for a nondegenerate \( J.\)
2.3

This lemma shows that permissible maps \((f, I)\) of a nondegenerate Jordan algebra \(J\) are just permissible maps of the corresponding Jordan triple system that satisfy \(f(I^2) \subseteq I\). The equivalence relation that defines the extended centroid of a triple system can be restricted to algebra ideals and algebra \(J\)-homomorphisms to obtain the extended centroid of the algebra \(J\), which we temporarily denote by \(\mathcal{E}_\text{alg}(J)\). This set can be given a ring structure as for triple systems. Indeed, if \((f, I)\) and \((g, I)\) are permissible maps of the algebra \(J\), then \((f + g)(I^2) \subseteq f(I^2) + g(I^2) \subseteq I\), and \(fg((U_I x))^2 = (U_I f g)^2 = f(U_I g^2) + g^2 f(U_I J) \subseteq f(U_I U_J I) + f(I) \subseteq U_I U_J(I) \subseteq U_I U_J \subseteq U_I U_J\). Therefore we can define addition and multiplication as for triple systems, and the natural mapping \(\mathcal{E}_\text{alg}(J) \to \mathcal{E}(J)\) becomes a homomorphism.

2.4. PROPOSITION. For a nondegenerate Jordan algebra \(J\), the previous mapping \(\mathcal{E}_\text{alg}(J) \to \mathcal{E}(J)\) is an isomorphism.

Proof. Let \(I\) be an ideal of \(J\) as a triple system, and set \(I_{\text{alg}} = \{ y \in I \mid y^2 + y \circ y \subseteq I \}\). We claim that \(I_{\text{alg}}\) is an ideal of \(J\) as an algebra and \(I \ast I \subseteq I_{\text{alg}}\).

First, if \(y, y' \in I_{\text{alg}}\), then \((y + y') \circ I \subseteq I\), and \((y + y')^2 = y^2 + (y')^2 + y \circ y' \subseteq I\); hence \(y + y' \in I_{\text{alg}}\).

Next, if \(z \in J\) and \(y \in I_{\text{alg}}\), then \(U_z y \in I\), and \((U_z y)^2 = U_z U_y z^2 \subseteq U_z U_y I \subseteq U_z I \subseteq I\). Now, if \(x \in J, U_z y \circ x = \{z, y, z \circ x\} - U_z(y \circ x) \subseteq \langle J, I, J \rangle + U_z(I_{\text{alg}} \circ J) \subseteq I + U_z I \subseteq I\). Therefore \(U_z I_{\text{alg}} \subseteq I_{\text{alg}}\). Also, if \(y \in I_{\text{alg}}\), then \(y^2 \in I, y^4 = U_y y^2 \subseteq I,\) and for all \(x \in J, y^2 \circ x = (y, y, x) \subseteq I;\) hence \(I_{\text{alg}} \subseteq I_{\text{alg}}\).

Now, if \(z \in J, y \in I_{\text{alg}}\), then \((z \circ y)^2 = U_z y \circ y + U_z y^2 + U_z z^2 \subseteq J \circ I_{\text{alg}} + U_z I + U_z J \subseteq I\), and, if \(x \in J, (z \circ y) \circ x = \{z, y, x\} + \{x, z, y\} \subseteq \langle J, I, J \rangle \subseteq J_{\text{alg}}\). Thus \(I_{\text{alg}} \circ J \subseteq I_{\text{alg}}\).

Finally, if \(y \in I, z \in J\), then \(U_z y \in I, (U_z y)^2 = U_z U_y y^2 \subseteq U_z J \subseteq I,\) and for all \(x \in J, U_z \circ x = \{y, z, y \circ x\} - U_y(x \circ z) \subseteq \langle I, J, J \rangle + U_J J \subseteq I;\) hence \(U_z J \subseteq U_z J \subseteq I_{\text{alg}}\).

Therefore \(I_{\text{alg}}\) is an ideal of the algebra \(J\), and \(I \ast J = U_I J + U_J U_J \subseteq I_{\text{alg}} + U_J I_{\text{alg}} \subseteq I_{\text{alg}}\).

Now, if \((f, I)\) is a permissible map of the triple system \(J\), then \(K = I_{\text{alg}}\) is an essential ideal of \(J\) both as an algebra and as a triple system (since it contains the essential ideal of the triple \(I \ast J\), see 1.2), and \(f\) restricts to \(K \ast J = U_K J\).

Moreover, for all \(k \in K\), \(z \in J\), \(f((U_z x))^2 = f(U_z U_k k^2) = U_z U_k f(k^2)\) (since \(k^2 \in I \subseteq U_k J\), and \(f(U_k J \circ U_k J) \subseteq f(U_k J) \circ U_k J\) by 2.2) \(\subseteq J \circ U_k J \subseteq U_k J\). Therefore \(f((U_K J)^2) \subseteq U_K J\), so setting \(L = U_K J\), the map \((f, L)\) is permissible for the algebra \(J\) by Lemma 2.1.
This shows that every $\lambda \in \mathcal{C}(J)$ has a representative $(f, L) \in \lambda$ where $L$ is an algebra ideal of $J$, and $f$ is a $J$-homomorphism of algebras. Therefore, $\lambda$ is the image of $[f, L] \in \mathcal{C}_{\text{alg}}(J)$, and the mapping $\mathcal{C}_{\text{alg}}(J) \to \mathcal{C}(J)$ is surjective.

Finally, it is easy to see that the mapping is injective, and, therefore, it is an isomorphism. 

In view of this result we can identify the extended centroids of an algebra $J$ and of its underlying triple system, and denote it simply by $\mathcal{C}(J)$.

2.5

We now consider Jordan pairs. These are related to Jordan triple systems through their attached polarized systems. Also, if $V = (V^+, V^-)$ is a Jordan pair and $I = (I^+, I^-)$ is an ideal of $V$, then the set $T(I) = I^+ \oplus I^-$ is, up to a natural isomorphism, an ideal of $T(V) = V^+ \oplus V^-$. Thus for any $f = (f^+, f^-) \in \text{Hom}_\pi(I, V)$, we obtain a mapping $t(f) = f^+ \oplus f^- : T(I) \to T(V)$, given by $t(f)(x^+ \oplus x^-) = f^+(x^+) \oplus f^-(x^-)$, which is easily seen to belong to $\text{Hom}_{T(V)}(T(I), T(V))$.

The lemmas of Section 1 have then a direct translation to $V$, and we can form the extended centroid of a nondegenerate $V$ as the set of equivalence classes of permissible maps of $V$ under the equivalence relation "~" given by $(f, I) \sim (g, L)$ if there is an essential ideal $K$ if $V$ with $K^\sigma \subseteq I^\sigma \cap L^\sigma$ and $f^\sigma(x) = g^\sigma(x)$ if $x \in K^\sigma$, for $\sigma = \pm$. The quotient set, which we denote by $\mathcal{C}(V)$, has a structure of a commutative ring and will be called the extended centroid of $V$.

Notice that since every permissible map $((f^+, f^-), (I^+, I^-))$ of $V$ gives rise to a permissible map $(f^+ \oplus f^-, I^+ \oplus I^-)$ of $T(V)$ (see 1.2(d)), it determines an element of $\mathcal{C}(T(V))$. In fact, this defines a homomorphism of $\mathcal{C}(V)$ into $\mathcal{C}(T(V))$ due to the following:

2.6. Lemma. If $f, g$ are permissible maps of the nondegenerate Jordan pair $V$, then $f \sim g$ if and only if $t(f) \sim t(g)$.

Proof. If $t(f)$ and $t(g)$ agree on the essential ideal $K$ of $T(V)$, then they agree on the polarized ideal $(Q_{\pi_{-}V_{\pi_{-}}}K + Q_{\pi_{+}V_{\pi_{+}}} - K) \oplus (Q_{\pi_{+}V_{\pi_{-}}}K + Q_{\pi_{-}V_{\pi_{+}}} - K)$, which is also essential by 1.2. Then $f$ and $g$ agree on $(T(V)^*K)^+, (T(V)^*K)^-$; hence $f \sim g$. The reciprocal is clear. 

Thus, we have a monomorphism $\mathcal{C}(V) \to \mathcal{C}(T(V))$.

2.7. Proposition. For a nondegenerate Jordan pair $V$, the mapping $\mathcal{C}(V) \to \mathcal{C}(T(V))$ is an isomorphism.

Proof. We only have to show that the mapping is surjective. For that, if $\lambda \in \mathcal{C}(T(V))$ and $(f, I) \in \lambda$, we can restrict $f$ to the polarized ideal $(T(V)^*I)^+T(V)$ by Lemma 1.9, which is essential by 1.2, so we can
assume that \( I \) is polarized, \( I = I^+ \oplus I^- \), with \( I^\sigma \subseteq V^\sigma \). Thus, to show that \( \lambda \) comes from an element of \( \mathcal{O}(V) \), it suffices to show that \( f \) is polarized, \( f(I^\sigma) \subseteq V^\sigma \), \( \sigma = \pm \), since in this case we will have \( f = f^+ \oplus f^- = \iota(f^+, f^-) \), where \( f^\sigma(x^\sigma) = f(x^\sigma) \in V^\sigma \) for any \( x^\sigma \in I^\sigma \).

Now, if \( x^\sigma \in I^\sigma \), put \( f^\sigma(x^\sigma) = y^+ \oplus y^- \). Then \( 0 = P_y V^\sigma \); hence \( 0 = f^2(P_x V^\sigma) = P_{f(x)} V^\sigma = Q_{y^+ \oplus y^-} V^\sigma = Q_{y^+} V^\sigma = Q_{y^-} T(V) \). Thus \( y^- = 0 \) by nondegeneracy, and \( f(x^\sigma) = y^\sigma \in V^\sigma \). \( \blacksquare \)

3. EXTENDED CENTRAL CLOSURE

3.1

In this section we define the scalar extension of a Jordan system which is naturally associated to its extended centroid. We carry out the construction for a nondegenerate Jordan triple system \( T \), but we point out that the results are also valid for algebras and pairs with the suitable modifications in the latter case, which we mention at the end of this section.

First form the free scalar extension \( \mathcal{O}(T) \otimes T \). Since, at the end, the result will be independent of the ring of scalars \( \Phi \), we will not worry about subscripts of the tensor product, so it can be taken over any ring of scalars \( \Phi \) for \( T \).

We consider the set
\[
R = \left\{ \sum_i (\rho_i \lambda_i \otimes x_i - \rho_i \otimes f_i(x_i)) \right\} : \lambda_i, \rho_i \in \mathcal{O}(T), (f_i, I_i) \in \lambda_i, \text{ and } x_i \in I_i \right\}.
\]

Notice that \( R \) is a \( \mathcal{O}(T) \)-submodule of \( \mathcal{O}(T) \otimes T \) (and hence a \( \Phi \)-submodule if tensors are taken over \( \Phi \)).

3.2. LEMMA. \( R \) is an ideal of \( \mathcal{O}(T) \otimes T \).

Proof. For any multiplication operator \( M \in \mathcal{M}(T) \) and for any \( u = \rho \lambda \otimes x - \rho \otimes f(x) \in R \) we have \( 1 \otimes M(u) = \rho \lambda \otimes Mx - \rho \otimes f(Mx) \in R \), using the obvious identification of multiplication algebras \( \mathcal{M}(\mathcal{O}(T) \otimes T) = \mathcal{O}(T) \otimes \mathcal{M}(T) \) and the definition of \( T \)-homomorphism. Thus \( R \) is an outer ideal.

Now, if \( u = \rho \lambda \otimes x - \rho \otimes f(x) \in R \) and \( \mu \otimes z \in \mathcal{O}(T) \otimes T \), we have
\[
P_{t}(\mu \otimes z) = P_{\rho \lambda \otimes x, \rho \otimes f(x)}(\mu \otimes z) = \mu \rho \lambda^2 \otimes P_z + \mu \rho^2 \otimes P_{f(x)} = \mu \rho^2 \lambda^2 \otimes P_z - \mu \rho^2 \otimes f^2 P_z = (\mu \rho^2 \lambda^2 \otimes P_z - \mu \rho^2 \otimes f(P_z)) = (\mu \rho^2 \lambda^2 \otimes f(P_z)) \in R.
\]
Thus \( R \) is an ideal of \( \mathcal{O}(T) \otimes T \). \( \blacksquare \)
3.3

The first step in the construction of the scalar extension associated to the extended centroid is to form the quotient $\tilde{T} = \mathcal{F}(T) \otimes T/R$. This is a triple system over the ring $\mathcal{F}(T)$ (since the ideal $R$ is clearly $\mathcal{F}(T)$-invariant), and the natural injection $T \to \mathcal{F}(T) \otimes T$, $a \mapsto 1 \otimes a$, induces a homomorphism $T \to \tilde{T}$. To show that this is a scalar extension we must show that this is also an injection; that is, $(1 \otimes T) \cap R = 0$. We follow the ideas of [9, 12] with the changes needed to deal with the nonlinearity of the multiplication in $T$.

Let $I$ be an essential ideal of $T$. We denote by $(\mathcal{F}(T) \otimes T)$, the $\Phi$-submodule of elements $a \in \mathcal{F}(T) \otimes T$ that can be written in the form $a = \sum \lambda_i \otimes x_i$ for some elements $\lambda_i \in \mathcal{F}(T)$ which admit representatives $(f_i, I) \in \lambda_i$ whose domain is the ideal $I$. For each $y \in I$ we want to define a map $F_y : (\mathcal{F}(T) \otimes T)_y \to T$ by $F_y(a) = \sum f_i(P_y x_i)$ where $a = \sum \lambda_i \otimes x_i$ and $(f_i, I) \in \lambda_i$ as before. Note that since $y \in I$, $P_y x_i$ belongs to $I$ and hence to the domain of the $f_i$'s.

3.4. Lemma. The mappings $F_y$ are well defined.

Proof. We must show that if $\sum \lambda_i \otimes x_i = 0$, where each $\lambda_i \in \mathcal{F}(T)$ has a representative $(f_i, I) \in \mathcal{F}(T)$, then $\sum f_i(P_y x_i) = 0$. We can assume that the elements $x_i$ are part of a system of generators of the $\Phi$-module $T$. Then, by (I.8.8) of [30], there are elements $\alpha_{ij} \in \Phi$ and a finite number of elements $\mu_j \in \mathcal{F}(T)$ such that $\sum \alpha_{ij} x_i = 0$ for every $j$ and $\lambda_i = \sum \alpha_{ij} \mu_j$ for every $i$.

Take an essential ideal $L$ of $T$ such that the $f_i$’s restrict to $L$ and we can find representatives $(g_j, L) \in \mu_j$. Then, the equality $\lambda_i = \sum \alpha_{ij} \mu_j$ means that for all $z \in L$, $f_i(z) = \sum \alpha_{ij} g_j(z)$ by Lemma 1.10. Thus, if $z \in L$,

$$P_L \left( \sum f_i(P_y x_i) \right) = \sum f_i(P_x P_y x_i)$$

$$= \sum \alpha_{ij} g_j(P_x P_y x_i) = \sum g_j(P_x P_y \sum \alpha_{ij} x_i) = 0.$$

Therefore, $P_L \sum_i f_i(P_y x_i) = 0$ and $\sum_i f_i(P_y x_i) \in \text{Ann}_T(L) = 0$, by 1.3 and essentiality of $L$. □

3.5. Lemma. The mapping $T \to \tilde{T} = \mathcal{F}(T) \otimes T/R$ is injective.

Proof. Suppose that $1 \otimes x \in (1 \otimes T) \cap R$. Then there are $\lambda_i, \rho_i \in \mathcal{F}(T)$ and $x_i \in T$ such that $1 \otimes x = \sum (\rho_i \lambda_i \otimes x_i - \rho_i \otimes f_i(x_i))$, where $(f_i, I_i) \in \lambda_i$ and $x_i \in I_i$. We can find an essential ideal $L$ of $T$ such that all products $\tau_1 \tau_2$ with $\tau_k = 1, \rho_i$, or $\lambda_i$ have representatives whose com-
mon domain is $L$; say $\lambda_i = [f_i, L]$ and $\rho_i = [g_i, L]$. Then, for all $y \in L$ we have $F_y(\rho_i \lambda_i \otimes x_i - \rho_i \otimes f_i(x_i)) = f_i g_i(P_x x_i) - g_i P_x f_i(x_i) = f_i g_i(P_x x_i) - f_i g_i(P_x x_i) = 0$. Thus $F_y(1 \otimes x) = 0$.

On the other hand, $F_y(1 \otimes x) = P_y x$; hence we obtain $P_y x = 0$ and $x \in \Ann_T(L)$ (by 1.3). Therefore $x = 0$.

3.6

According to the previous lemma, we can identify $T$ with its image in $\tilde{T}$.

Now, for any ideal $I$ of $T$, $\mathcal{O}(T) \otimes I$ is an ideal of $\mathcal{O}(T) \otimes T$, and $\tilde{I} = (\mathcal{O}(T) \otimes I + R)/R$ is an ideal of $\tilde{T}$. If $I_1 \subseteq I_2$ are ideals of $T$ then $\Ann_{\tilde{T}}(\tilde{I}_1) \supseteq \Ann_{\tilde{T}}(\tilde{I}_2)$, and therefore the set $\tilde{M} = \cup \{\Ann_{\tilde{T}}(\tilde{I}) : I \text{ is an essential ideal of } T\}$ is an ideal of $\tilde{T}$. Moreover, since $\Ann_{\tilde{T}}(\tilde{I}) \cap T \subseteq \Ann_T(I) = 0$ for any essential ideal $I$ of $T$, we have $\tilde{M} \cap T = 0$. The system $\mathcal{O}(T)/T = \tilde{T}/M$ will be called the extended central closure of $T$.

Since $T \cap \tilde{M} = 0$, we can make the identification $T \subseteq \mathcal{O}(T)T$, and $\mathcal{O}(T)T$ is a scalar extension of $T$.

3.7. Lemma. Any ideal $\tilde{N}$ of $\tilde{T}$ with $\tilde{N} \cap T = 0$ is contained in $\tilde{M}$.

Proof. Take an ideal $N \supseteq R$ of $\mathcal{O}(T) \otimes T$ with $\tilde{N} = N/R$. Any $a \in N$ has the form $a = \sum \lambda_i \otimes x_i$, with $x_i \in T$, $\lambda_i \in \mathcal{O}(T)$, and we can take an ideal $I$ of $T$ such that all $\lambda_i$ and $\lambda_i \lambda_i$ have representatives $(f_i, I)$ and $(P, f_y, I)$, respectively, whose domain is $I$. Now, writing $\equiv$ for the congruence mod $R$, for any $y \in I$ we have $P_{1 \otimes y} a = 1 \otimes (\sum_i f_i(P_i x_i))$. Thus, $1 \otimes \sum_i f_i(P_i x_i) \in N + R = N$, and it also belongs to $T$. Since $T \cap N = 0$, we get $P_{1 \otimes y} a = 0$.

Similarly, one proves the vanishing of $P_{1 \otimes y, 1 \otimes z} a, (a, 1 \otimes y, 1 \otimes z)$, $\{a, 1 \otimes z, 1 \otimes y\}$, and $\{1 \otimes y, a, 1 \otimes z\}$ for all $y, y' \in I$ and all $z \in T$.

Also, since $\lambda_i \lambda_i$ admits a representative $f_i f_i$ whose domain is $I$, for all $y \in I$, we obtain $P_y (1 \otimes y) = \sum_i P_{\lambda_i \otimes x_i} (1 \otimes y) + \sum_{i < j} P_{\lambda_i \otimes x_i, \lambda_j \otimes x_j} (1 \otimes y) = \sum_i \lambda_i^2 \otimes x_i y + \sum_{i < j} \lambda_i \lambda_j \otimes P_{x_i, x_j} y = 1 \otimes (\sum_i f_i^2(P_i x_i) + \sum_{i < j} f_i f_j(P_i x_j y)) \in I \otimes T$. Thus, $P_y (1 \otimes y) \in N \cap (1 \otimes T) = 0$.

Similarly, one proves the vanishing, for all $y \in I$ and all $z \in T$, of $P_{1 \otimes y} P_a (1 \otimes z)$, $P_{1 \otimes y, 1 \otimes z} P_a (1 \otimes z)$, $P_a P_{1 \otimes y} (1 \otimes z)$, and $P_a P_{1 \otimes y, 1 \otimes z} (1 \otimes z)$.

These equalities imply that $P_\tilde{a} \tilde{a} = P_\tilde{a} P_\tilde{a} \tilde{a} = P_\tilde{a} P_\tilde{a} \tilde{a} = P_\tilde{a} \tilde{a} = (\tilde{a}, \tilde{T}, \tilde{I}) = (\tilde{a}, \tilde{T}, \tilde{I}) = 0$, where $\tilde{a} = a + R \in \tilde{T}$. Therefore $\tilde{a} \in \Ann_{\tilde{T}}(\tilde{I}) \subseteq M$, and $N \subseteq M$.

3.8. Theorem. The extended central closure $\mathcal{O}(T)T$ of a nondegenerate Jordan system $T$ is a tight scalar extension of $T$. Therefore it is nondegenerate, and if $T$ is strongly prime, then $\mathcal{O}(T)T$ is strongly prime.

Proof. $T \subseteq \mathcal{O}(T)T$ is tight by Lemma 3.7. The other assertions follow from this and [24, 2.9].
3.9

To close this section we make some remarks on the construction of the extended central closure for algebras and pairs.

If $J$ is a nondegenerate Jordan algebra, the previous construction can be carried out in exactly the same way, and it produces a Jordan algebra. To see that, it is enough to notice that all the ideals involved in the construction ($R$ and $M$) are algebra ideals. This follows by allowing $z = 1$ in the proof of 3.2 and making use of conditions (iv) and (v) of the definition of $J$-homomorphism. The resulting algebra $\mathcal{E}(J)J$ will be called the extended central closure of $J$. Note that, by 2.4, the extended central closure of $J$ is the same whether we consider $J$ as an algebra or a triple system (or to be precise, the underlying triple system of the extended central closure of the algebra $J$ is the extended central closure of $J$ as a triple system).

As for nondegenerate Jordan pairs $V = (V^+, V^-)$, the construction can be carried out with the obvious modifications. We start from the free scalar extension $\mathcal{E}(V) \otimes V = (\mathcal{E}(V) \otimes V^+, \mathcal{E}(V) \otimes V^-)$ and consider the pair-ideal $R = (R^+, R^-)$, where $R^\sigma$ is the linear span of the elements of the form $\rho \lambda \otimes x^\sigma - \rho \otimes f^\sigma(x^\sigma)$ where $\lambda$ has a representative $((f^+, f^-), (I^+, I^-))$ and $x^\sigma \in I^\sigma$. The rest of the construction is the same as for triple systems, using the pair version of the corresponding notions, and it produces a Jordan pair which we denote by $\mathcal{E}(V)V = (\mathcal{E}(V)V^+, \mathcal{E}(V)V^-)$ and call the extended central closure of $V$. Notice that there is an obvious identification $T(CV) = \mathcal{E}(T)V$.

In both cases (algebras and pairs) the extended central closure is a tight scalar extension of the original Jordan system; hence it is nondegenerate and inherits strong primeness from the original system.

4. PROPERTIES OF THE EXTENDED CENTRAL CLOSURE

4.1

As for associative algebras, the extended central closure of a Jordan system should be considered as a localization of the algebra (cf. [31]). To reflect this we introduce the following notions.

Let $V \subseteq \tilde{V}$ be Jordan pairs, let $I$ be an ideal of $V$, and let $a \in \tilde{V}^\sigma$, $\sigma = \pm$. An element $x \in V^\sigma$ will be called a $V$-denominator of $a$ into $I$ if the elements $Q_s a, Q_s x$ and the sets $Q_s Q_s^* V^{-\sigma}, Q_s Q_s^* V^{\sigma}, D_s a V^{-\sigma}, D_s a V^{\sigma}$ are all contained in $I$. We denote by $\mathcal{D}_V(a, I)$ the set of denominators of $a$ into $I$. If $I = V$, we simply write $\mathcal{D}_V(a)$.

For Jordan triple systems $T \subseteq \tilde{T}$, $a \in \tilde{T}$, and an ideal $I$ of $T$ we define the $T$-denominators of $a$ into $I$ similarly, by deleting the superscripts $\pm$ in
the previous definition. The resulting set will be denoted by \( \mathcal{D}_I(a, I) \) or simply \( \mathcal{D}_I(a) \) if \( I = I \).

Finally, for Jordan algebras \( J \subseteq \hat{J}, a \in \hat{J} \), and an ideal \( I \) of \( J \), we add the conditions \( U^{-}U_{a}^{x}J + U_{a}^{x}J + \{a, x, \hat{J}\} \subseteq I \), where \( \hat{J} \) is a unital hull of \( J \). Again, the set of \( J \)-denominators of \( a \) into \( I \) will be denoted by \( \mathcal{D}_I(a, I) \) or \( \mathcal{D}_I(a) \) if \( I = I \). Clearly, \( \mathcal{D}_I(a, I) \) for the algebra \( J \) coincides with \( \mathcal{D}_I(a) \cap J \) for the triple system \( \hat{J} \).

4.2. Lemma. Let \( V \subseteq \hat{V} \) be Jordan pairs, let \( I \) be an ideal of \( V \), and let \( a \in V^\sigma \), \( \sigma = \pm \). Then \( \mathcal{D}_I(a, I) \) is an inner ideal of \( V \).

Proof. If \( x, y \in \mathcal{D}_I(a, I) \subseteq V^{-\sigma} \), \( D_{x+y, a}V^{-\sigma} \) is contained in \( I^{-\sigma} \), and \( Q_a(x + y) \) and \( D_{a, x+y}V^\sigma \) are contained in \( I^\sigma \) by linearity.

Next, \( Q_{x+y}Q_a = Q_{x}Q_Qa + Q_aQ_{x+y} \) by JP9 of [16]. Thus, \( Q_{x+y}Q_aV^{-\sigma} \subseteq I^{-\sigma} \), hence \( Q_{x+y}Q_aV^{-\sigma} = Q_{x}Q_aV^{-\sigma} + Q_aQ_{x+y} \), and \( Q_{x+y}Q_a = Q_aQ_{x+y} \subseteq I^{-\sigma} \), and \( Q_{x+y}Q_a = Q_aQ_{x+y} \subseteq I^{-\sigma} \).

Now, by JP6 of [16],

\[
Q_{x}Q_{a}V_{x, y} = D_{a, y}D_{a, x} + D_{a, Q_{x}, a} - D_{a, Q_{x}, a},
\]

and

\[
D_{a, Q_{x}, a} = D_{a, x} + D_{a, y, x}
\]

by linearized JP2 of [16]. Thus, \( Q_{a}Q_{x, y} = D_{a, y}D_{a, x} - D_{a, y, x} \), and \( Q_{x}Q_{a}V_{x, y} \subseteq I^\sigma \). Therefore \( Q_{a}Q_{x+y}V^\sigma \subseteq I^\sigma \), and we finally get \( x + y \in \mathcal{D}_I(a, I) \).

Now take \( z \in V^\sigma \). Then, for any \( x \in \mathcal{D}_I(a, I) \), we have \( Q_aQ_z \in I^\sigma \) and \( Q_{x+y}Q_a \in Q_zQ_aI^{-\sigma} \subseteq I^{-\sigma} \).

We also have the containments

\[
Q_{x+y}Q_aV_{x, y} \subseteq Q_aQ_{x+y}V_{x, y} \subseteq I^\sigma,
\]

\[
Q_{a}Q_{x}Q_{y}V^{-\sigma} = Q_{x+y}Q_aV^{-\sigma} \subseteq Q_aQ_{x+y}I^{-\sigma} \subseteq I^{-\sigma},
\]

\[
D_{x+a, z}V^{-\sigma} \subseteq D_{x+a, z}V^{-\sigma} + D_{z+a, z}V^{-\sigma} \quad \text{(by linearized JP2 of [16])}
\]

\[
\subseteq D_{x, z}V^{-\sigma} + D_{x+a, z}V^{-\sigma} \subseteq I^{-\sigma},
\]

and

\[
D_{a, Q_{x+a}}V^{-\sigma} \subseteq D_{a, Q_{x+a}}V^{-\sigma} + D_{a, Q_{x+a}}V^{-\sigma} \quad \text{(by linearized JP2 of [16])}
\]

\[
\subseteq D_{a, Q_{x+a}}V^{-\sigma} + D_{a, Q_{x+a}}V^{-\sigma} \subseteq I^\sigma.
\]

Thus \( Q_aQ_z \in \mathcal{D}_I(a, I) \).

Similarly, we obtain that denominator sets are inner ideals in Jordan triple systems and algebras.
The proximity between a Jordan system and its extended central closure can be expressed in terms of denominators.

4.3. Lemma. Let $T$ be a nondegenerate Jordan triple system. Then

1. Every nonzero inner ideal of $C(T)T$ has nonzero intersection with $T$.

2. For every $a \in C(T)T$ and every essential ideal $I$ of $T$, $C_T(a, I)$ contains an essential ideal of $T$.

Proof. We first prove (2). Write $a = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \in C(T)$ and $x_i \in T$. For each $i$ we take a representative $(f_i, I_0) \in \lambda_i$. Set $L = I_0 \cap I$. Then $L \cdot L \subseteq I_0$, and $f_i$ restricts to $L \cdot L$ and to $L \cdot T$ by Lemma 1.9, for all $i$. Now, using that the elements $\lambda_i$ act as the corresponding $T$-homomorphisms $f_i$ on the elements of $L \cdot L$ and $L \cdot T$, we obtain $\lambda_i L \cdot L = f_i(L \cdot L) = L \cdot f_i(L) \subseteq L \cdot T \subseteq L \subseteq I$ and $\lambda_i \lambda_j L \cdot L = f_i f_j(L \cdot L) \subseteq f_i(L \cdot T) \subseteq L \subseteq I$, and a direct computation shows that $L \cdot L \subseteq C_T(a, I)$. Moreover, since $I$ and $I_0$ are essential ideals, $L$ is essential and hence $L \cdot L$ is essential.

Now, to prove (1), let $K$ be a nonzero inner ideal of $C(T)T$, and take $0 \neq a \in K$. By (2) there is an essential ideal $I \subseteq C_T(a)$. By the characterization [22, 1.7(i)] of the annihilator, $P_a I = 0$ implies $P_a C(T)I = 0$; hence $a \in \Ann_{C(T)T}(C(T)I)$ since $C(T)T$ is nondegenerate and $C(T)I$ is an ideal of $C(T)T$. Then $0 \neq \Ann_{C(T)T}(C(T)I) \cap \overline{T} \subseteq \Ann_T(I)$ by tightness, and this is a contradiction since $I$ is essential. Thus $K \cap T \supseteq P_a I$ is nonzero.

The key fact about the extended centroid of strongly prime systems is that it captures scalars from the centroid of any extension $T \subseteq \tilde{T}$ as soon as they do not remove some nonzero ideal from $T$: if $\lambda \in \Gamma(T)$ has $\lambda I \subseteq T$ for some nonzero ideal $I$ of $T$, then $\lambda \in C(T)$. The argument is essentially the same as in Lemmas 5.1 and 5.2 of [2]. In fact, inner ideals suffice due to the following result on generation of ideals (see Lemma 5.1 of [2]). Recall that a structural transformation $(f, g): T \rightarrow T'$ is pair of mappings $f: T \rightarrow T'$ and $g: T' \rightarrow T$ satisfying $P_{f(x)} = fP_x g$ for all $x \in T$ and $P_{g(y)} = gP_y f$ for all $y \in T'$. (We will also say that $f$ is structural if there is $g$ such that $(f, g)$ is structural.)

4.4. Lemma. Let $T$ be a Jordan system and let $\mathcal{M}(T)$ be its (unital) multiplication algebra (spanned by the identity on $T$ and all the operators $P_{x, y}$ for $x, y \in T$ together with $P_{x, 1}$ if $T$ is an algebra). If $K$ is an inner ideal of $T$, then $\mathcal{M}(T)K$ is an ideal of $T$.

Proof. Clearly $\mathcal{M}(T)K$ is outer; hence it suffices to show that this is an inner ideal.
Notice that if \(x, y \in T\), \(L_{x,y} = \text{Id} + P_x P_y - B_{x,y}\) is a sum of structural transformation, and the composition of structural transformations is again structural. Thus \(\mathcal{M}(T)\) is generated as a \(\Phi\)-module by structural transformations. Then, any element from \(\mathcal{M}(T)K\) has the form \(z = \Sigma_i f_i(k_i)\) for structural transformations \(f_i = \text{Id}, P, \text{or } B_{x,y}\), or compositions of these, and elements \(k_i\) from \(K\). (We write \(g_i\) for the second part of the structural transformation \((f_i, g_i)\).

Now, we have \(P_{f(k_i)}T = f_i P_k g_i T\) (by structurality of \(f_i\)) \(\subseteq f_iK\) (by innerness of \(K\)) \(\subseteq \mathcal{M}(T)K\), and \(P_{f(g(k_i))}T \subseteq L_{f(g(k_i)), T} f_i(k_i) \subseteq \mathcal{M}(T)K\), yielding innerness of \(\mathcal{M}(T)K\).

4.5. Proposition. Let \(T \subseteq \tilde{T}\) be Jordan systems. Suppose that \(T\) is strongly prime, and let \(\lambda \in \Gamma(T)\). Then there is a permissible map \((f, I)\) of \(T\) such that \(f(x) = \lambda x\) for all \(x \in I\) if and only if there is a nonzero inner ideal \(K\) of \(T\) such that \(\lambda K \subseteq T\).

Moreover, this always happens for \(\lambda = \mu^2\), \(\mu \in \Gamma(\tilde{T})\), if there is a nonzero \(x \in T\) with \(\mu x \in T\) and in particular if \(T \cap \mu T \neq 0\).

Proof. The “only if” part is clear. Now, if \(\lambda K \subseteq T\) for some nonzero inner ideal \(K\) of \(T\), then \(\lambda \mathcal{M}(T)K = \mathcal{M}(T)\lambda K \subseteq T\). So, denoting by \(L\) the ideal \(\mathcal{M}(T)K\), we have \(\lambda L \subseteq T\). This is a nonzero ideal (since it contains \(K\)); hence it is essential by primeness of \(T\).

It is easy to see that \(P_{L \ast T} T \subseteq P_T P_L L + P_T P_L L + \{L, T, L \ast T\}\) and, from this, that \(\lambda P_{L \ast T} T \subseteq L \ast T\). Thus, setting \(I = L \ast T\), we can consider the mapping \(f : I \rightarrow T\) given by \(f(x) = \lambda x\), which is permissible. This proves the first assertion.

Finally, if for \(\mu \in \Gamma(T)\) there is some nonzero \(\mu x \in T \cap \mu T\), then \(\mu^2 P_T T = \mu \mu T \subseteq T\), and the previous assertion applies to \(\lambda = \mu^2\).

As a first consequence we can relate the extended centroid \(\mathcal{C}(T)\) of a strongly prime Jordan system \(T\) and the extended centroid \(\mathcal{C}(\mathcal{C}(T)T)\) of its central closure. Note that since the centroid \(\Gamma(\mathcal{C}(T)T)\) contains (an isomorphic copy of) \(\mathcal{C}(T)\) (because \(\mathcal{C}(T)T\) is a vector space over \(\mathcal{C}(T)\)) and \(\Gamma(\mathcal{C}(T)T)\) can be seen as a subring of \(\mathcal{C}(\mathcal{C}(T)T)\), we can consider \(\mathcal{C}(\mathcal{C}(T)T)\) as a field extension of \(\mathcal{C}(T)\) in the obvious way.

4.6. Lemma. Let \(T\) be a strongly prime Jordan system; then \(\mathcal{C}(\mathcal{C}(T)T)^2 \subseteq \mathcal{C}(T)\).

Proof. Take \(\lambda \in \mathcal{C}(\mathcal{C}(T)T)\) and a representative \((f, I) \in \lambda\). Since \(T \subseteq \mathcal{C}(T)T\) is tight, we can take a nonzero element \(y \in I \cap T\). Then, by 4.3 there is an essential ideal \(L\) of \(T\) contained in \(\mathcal{S}_T(f(y))\), and the inner ideal \(P_L L\) of \(T\) is nonzero since \(\text{Ann}_T(L) = 0\) and \(y \neq 0\) (see 1.7 of [22]). Now we have \(f^2(P_L L) = P_{f(y)} L \subseteq T\); i.e., \(\lambda^2 P_L L \subseteq T\). Thus it suffices to apply 4.5.
4.7

In what remains of this section we examine to what extent the extended central closure is determined by the properties in 4.3. Extended central closures of linear algebras can be seen in light of the notions of localization theory, through the category of modules over the corresponding multiplication algebras and some of its subcategories (see [31]). In our case, however, multiplication algebras do not seem to be adequate tools for studying extended central closures since they do not reflect the nonlinearity of multiplications in Jordan systems. Nevertheless, our next results have some “injective flavour,” since they deal with liftings of homomorphisms. Consequently, they can be considered as a very preliminary step toward a categorical approach to the extended central closure, in which the use of modules over the multiplication algebra \( M(T) \) of \( T \) is substituted by extensions of the Jordan system. These are, of course, modules for \( M(T) \), but they also take into account the nonlinear action \( y \mapsto p, y \).

Let \( T \) be a nondegenerate Jordan system over \( \Phi \), and let \( \tilde{T} \) be an extension of \( T \) over the \( \Phi \)-algebra \( \Omega \). We denote by \( \Omega \cap \mathcal{P}(T) \) the set \( \{ \omega \in \Omega \mid \text{there is a permissible map } (f, I) \text{ of } T \text{ such that } f(y) = \omega y \text{ for all } y \in I \} \).

If \( \omega \in \Omega \cap \mathcal{P}(T) \) and \( (f, I), (g, L) \) are permissible maps of \( T \) with \( f(x) = \omega x \) for all \( x \in I \) and \( g(y) = \omega y \) for all \( y \in L \), then it is clear that \( (f, I) \) and \( (g, L) \) are equivalent; hence they define a unique element \( \sigma(\omega) \) of \( \mathcal{P}(T) \). In this way we get a mapping \( \sigma: \Omega \cap \mathcal{P}(T) \to \mathcal{P}(T) \).

4.8. Lemma. Let \( T \) be a nondegenerate Jordan system over \( \Phi \), and let \( \tilde{T} \) be an extension of \( T \) over the \( \Phi \)-algebra \( \Omega \). Then, the map \( \sigma \) is a homomorphism of \( \Phi \)-algebras. If, moreover, \( \Omega \) acts faithfully on \( \tilde{T} \) and \( \tilde{T} \) is tight over \( T \), \( \sigma \) is a monomorphism.

Proof. That \( \sigma \) is a homomorphism of \( \Phi \)-algebras is quite straightforward. Now, if \( \sigma(\lambda) = 0 \), then there is an essential ideal \( I \) of \( T \) such that \( \lambda I = 0 \). Then the ideal \( \lambda \tilde{T} \cap T \) is contained in \( \text{Ann}_\Phi(I) = 0 \); hence the ideal \( \lambda \tilde{T} \) is zero by tightness.

With this notation, Proposition 4.5 can then be rephrased as \( \Gamma(\tilde{T}) \cap \mathcal{P}(T) = \{ \lambda \in \Gamma(\tilde{T}) \mid \text{there is a nonzero inner ideal } K \text{ of } T \text{ such that } \lambda K \subseteq T \} \) if \( T \) is strongly prime. Moreover, in this case, if \( \lambda \in \Gamma(\tilde{T}) \) has \( \lambda \tilde{T} \cap T \neq 0 \), then \( \lambda^2 \in \Gamma(\tilde{T}) \cap \mathcal{P}(T) \).

4.9. Lemma. Let \( T \) be a nondegenerate Jordan system over \( \Phi \), and let \( \tilde{T} \subseteq \tilde{T} \) be an extension of Jordan \( \Phi \)-systems. Suppose that \( \tilde{T} \) is a system over the \( \Phi \)-algebra \( \Omega \); then there is a unique homomorphism \( \phi: (\Omega \cap \mathcal{P}(T))T \to \mathcal{P}(T)T \) which is the identity on \( T \). The homomorphism \( \phi \) is given by \( \phi(\sum \omega x_i) = \sum \sigma(\omega_i)x_i \).
Proof. To see that \( \phi \) is well defined, suppose that \( \Sigma \omega_i x_i = 0 \), where \( \omega_i \in \Omega \) and \( x_i \in T \subseteq \tilde{T} \). We must show that \( z = \Sigma \sigma(\omega_i)x_i = 0 \).

Take representatives \( (f_j, I) \in \sigma(\omega_i) \) with common domain \( I \), such that \( f_j(x) = \omega_i x \) for all \( x \in I \), and let \( y \in I \). Then \( P_y z = \Sigma f_i(\omega_i)P_y x_i = \Sigma f_i(\omega_i P_y x_i = P_y(\Sigma f_i(\omega_i x_i) = \Sigma f_i(\omega_i x_i) = \Sigma f_i(\omega_i P_y x_i = P_y(\Sigma f_i(\omega_i x_i) = P_y z = 0 \). Thus \( P_y z = 0 \), and then \( P_{\varphi(T)T} z = 0 \). Therefore \( z \in \text{Ann}_{\varphi(T)T}(\varphi(T)T) \) (by 1.3) = 0.

Now, if \( \psi: (\Omega \cap \varphi(T))T = \varphi(T)T \) is a homomorphism with \( \psi_T = \text{Id} \), take \( \lambda \in \Omega \cap \varphi(T) \) and \( x \in T \). There is a nonzero ideal \( I \) of \( T \) with \( \lambda I \subseteq T \), and for all \( y \in I \) we have \( P_y(\psi(\lambda x) - \psi(\lambda x)) = P_y(\psi(\lambda x) - \psi(\lambda x)) = P_y(\psi(\lambda x) - \psi(\lambda x)) = P_y(\psi(\lambda x) - \psi(\lambda x)) = 0 \). Thus \( P_y(\psi(\lambda x) - \psi(\lambda x)) = 0 \), and \( P_{\varphi(T)T}(\psi(\lambda x) = \psi(\lambda x)) = 0 \); hence \( \phi(\lambda x) - \psi(\lambda x) \in \text{Ann}_{\varphi(T)T}(\varphi(T)T) \) (by 1.3) = 0. Therefore \( \phi = \psi \).

4.10. Proposition. Let \( T \subseteq \tilde{T} \) be an extension of nondegenerate Jordan systems. If any nonzero inner ideal of \( \tilde{T} \) meets \( T \) and for any \( x \in \tilde{T} \), \( \mathcal{D}(x) \) contains an essential ideal of \( T \); then there is a unique homomorphism \( \phi: \Gamma(\tilde{T})^2T(\subseteq \tilde{T} \rightarrow \varphi(T)T \) which is the identity on \( T \). Moreover, \( \phi \) is a monomorphism.

Proof. For any nonzero \( x \in T \), there is a nonzero essential ideal \( I \) of \( T \) contained in \( \mathcal{D}(\lambda x) \). Since \( x \notin \text{Ann}_{\varphi(T)}(I) \), there is a nonzero \( y \in P_I \) by 1.3, and \( xy \in P_I \lambda x \subseteq T \cap \lambda T \); hence \( \lambda x \in \Gamma(T) \cap \varphi(T) \) by 4.5. Then the homomorphism of Lemma 4.9 gives a homomorphism \( \phi: \Gamma(\tilde{T})^2T \rightarrow \varphi(T)T \) which is unique, extending the identity on \( T \). (See the proof of uniqueness in 4.9.)

Now, let us show that \( \phi \) is a monomorphism. Indeed, if \( \tilde{x} = \Sigma \lambda_i x_i \) with \( \lambda_i \in \Gamma(\tilde{T})^2 \) and \( x_i \in T \), has \( \phi(\tilde{x}) = 0 \), then take an essential ideal \( I \) of \( T \) such that all the elements \( \sigma(\lambda_i) \) and \( \sigma(\lambda_i) \sigma(\lambda_i) \), where \( \sigma \) is the homomorphism of 4.7, have representatives defined on \( I \). Then it is easy to see that \( \tilde{x} \in \text{Ann}_{\varphi(T)}(I) \). But this is inner ideal [16, 10.3] and satisfies \( \text{Ann}_{\varphi(T)}(I) \cap T \subseteq \text{Ann}_{\varphi(T)}(I) = 0 \). Therefore \( \text{Ann}_{\varphi(T)}(I) = 0 \), and \( \tilde{x} = 0 \).

4.11

When the characteristic is two, the extended central closure \( \varphi(T)T \) of a Jordan system \( T \), as we have defined it, may not really be a closure in the sense that it is a closure for linear algebras, since we may not have \( \Gamma(\varphi(T)T) = \varphi(T) \) or even \( \varphi(\varphi(T)T) = \varphi(T) \). The problem here is the same as in the construction [26] of a (ordinary) central closure of a Jordan algebra, that is, a central extension \( \tilde{J} \) of a strongly prime Jordan algebra \( J \) whose centroid is the field of fractions of the centroid of \( J \). It is not difficult, however, to obtain a true closure of a Jordan system following the ideas of [26]. To do that one can iterate the construction of the extended...
centroid and the extended central closure of $T$ by transfinite induction: start from a strongly prime system $T$, and set $C_0(T) = C(T)$; then define $C_0\beta(T) = C(C_0\alpha(T)T)$ and $C_\beta(T)T = C(C_0\alpha(T)(C_\alpha(T)T))$ if $\beta = \alpha + 1$ is not a limit ordinal and $C_\beta(T) = \bigcup_{\alpha < \beta} C_\alpha(T)$, $C_\beta(T)T = \bigcup_{\alpha < \beta} C_\alpha(T)T$ if $\beta$ is a limit. By 4.6, at each step we get an extension of exponent two of the previous extended centroid, so we can take all them inside the perfect closure of $C(T)$ and define $C_\infty(T)$ as the limit of these extensions. In this way we can get a extended central closure which is really a closure. Nevertheless, we will only make use of the first step $C(T)$ since it will be enough for our purposes.

5. JORDAN SYSTEMS WITH NONZERO PI-ELEMENTS

In this section we prove our main result, which describes strongly prime Jordan systems having nonzero PI-elements through their extended central closures:

5.1. THEOREM. Let $J$ be a strongly prime Jordan system. If $\text{PI}(J) \neq 0$, then the extended central closure $C(J)J$ of $J$ is rationally primitive; hence it has nonzero socle equal to $\text{PI}(C(J)J)$, and $\text{PI}(J) = J \cap \text{Soc}(C(J)J)$.

As commented before, this theorem should be considered as a Jordan analogue of Martindale’s theorem on prime associative algebras satisfying a generalized identity. Since its proof is somewhat involved, we first sketch the argument, and we will devote the rest of this section to the proof.

5.2

We first change our notation and write $J_0$ for the original system and $J$ for its extended central closure. The theorem asserts that $J$ is rationally primitive, and according to 0.17 this is equivalent to the existence of an element $a \in J$ whose local algebra $J_a$ is simple and PI. To prove that there is such an element in $J$, suppose first that there is $a \in \text{PI}(J)$ such that $J_a$ is strongly prime. By the Jordan analogue 0.13 of Posner’s theorem the central closure of $J_a$ is simple and unital; it suffices to show that the centroid $\Gamma_a = \Gamma(J_a)$ is already a field. Now, since $J_a$ is an algebra over the extended centroid $C(J_0)$ of $J_0$, which is a field, $\Gamma_a$ is an algebra over $C(J_0)$. Moreover, $\Gamma_a$ is a domain ($J_a$ being strongly prime), so it suffices to show that $\Gamma_a$ is algebraic over $C(J_0)$. That will be achieved by using Amitsur’s ultraproduct construction much in the same spirit as it was used by Martindale [21]. Thus, we will get a primitive embedding $J \subseteq \tilde{J}$, where $\tilde{J}$ is a Jordan system over a $C(J_0)$-field $\tilde{\Omega}$, so that we will be able to represent some elements of the local centroid $\Gamma_a$ globally in $\tilde{\Omega}$, and capture a power of hem back in $C(J_0)$ by the key fact 4.5.
5.3

We start with the construction of a primitive embedding for an strongly prime system \(J\) with nonzero PI-elements. Following Amitsur's construction, we first need to get rid of the Jacobson radical.

Let \(J\) be a Jordan system (triple, algebra, or pair). We denote by \(\text{PNil}(J)\) the properly nil radical of \(J\) (see [19]). The following result is Theorem 6.3(a) of [27].

5.4. Theorem. Let \(J\) be a nondegenerate Jordan system, then \(\text{PI}(J) \cap \text{PNil}(J) = 0\). In particular, if \(J\) is prime, then \(\text{PI}(J) \neq 0\) implies \(\text{PNil}(J) = 0\).

Now we can get a semiprimitive system from a strongly prime \(J\) by means of a scalar extension.

5.5. Lemma. Let \(J\) be a strongly prime Jordan system over a large field \(\Omega\): \(|\Omega| > \dim_\Omega J + 2\). If \(\text{PI}(J) \neq 0\), then \(J\) is semiprimitive.

Proof. If \(a \in \text{Jac}(J)\) belongs to the Jacobson radical of \(J\), then for all \(b \in J\), \(\bar{a} \in \text{Jac}(J^{(b)})\) by [16, 4.18]. Now, \(|\Omega| > \dim_\Omega J + 2 = \dim_\Omega J^{(b)} + 2\); hence \(\text{Jac}(J^{(b)})\) is nil by [15, 4.5.8] applied to a unital hull \(J^{(b)}\) of \(J\) (note that \(\dim_\Omega J^{(b)} \leq \dim_\Omega J^{(b)} + 1\), and that \(\text{Jac}(J^{(b)}) \subseteq \text{Jac}(J^{(b)})\) holds trivially since \(\text{Jac}(J^{(b)})\) is a quasi-invertible ideal of \(J^{(b)}\)). Therefore \(a\) is nilpotent in \(J\) for all \(b \in J\); hence \(\text{Jac}(J)\) is properly nilpotent, and thus \(\text{Jac}(J) = \text{PNil}(J)\). Now, since \(\text{PI}(J) \neq 0\), \(\text{PNil}(J) = 0\) by 5.4.

Using this result, we can get a scalar extension of \(J\) which is a subdirect sum of primitive systems. The embedding we are looking for will result as an ultraproduct of that family of primitive images. To deal with it we first study ultraproducts of the kind of Jordan systems we are interested in. We refer to [29] for basic facts about ultraproducts and elementary sentences.

5.6. Lemma. Let \(\{J_\lambda\}_{\lambda \in \Lambda}\) be a family of Jordan systems, let \(\mathcal{F}\) be an ultrafilter on \(\Lambda\), and let \(\bar{J} = \prod J_\lambda / \mathcal{F}\) be the corresponding ultraproduct. If the set \(\{\lambda \in \Lambda \mid J_\lambda\text{ is strongly prime with nonzero socle}\}\) belongs to \(\mathcal{F}\), then \(\bar{J}\) is strongly prime with nonzero socle.

Proof. Put \(\Lambda_0 = \{\lambda \in \Lambda \mid J_\lambda\text{ is strongly prime with nonzero socle}\}\).

Nondegeneracy is expressed by the elementary sentence

\[ \forall x \exists y, \quad x \neq 0 \Rightarrow P_0 y \neq 0, \]

and since it holds in \(J_\lambda\) for all \(\lambda \in \Lambda_0\) and \(\Lambda_0 \in \mathcal{F}\), it holds in \(\bar{J}\). Therefore \(\bar{J}\) is nondegenerate.

Next, each \(J_\lambda\) with \(\lambda \in \Lambda_0\) has nonzero socle; hence it contains a simple element by nondegeneracy (see [17]). For nondegenerate systems this
means that the elementary sentence
\[ \exists x \, \forall y \, \exists z, \quad x \neq 0 \quad \& \quad (P_{x,y} \neq 0 \Rightarrow P_{x,y}z = x), \]
holds in each \( J_\lambda \) for \( \lambda \in \Lambda_0 \). Therefore it holds in \( \tilde{J} \), and \( \tilde{J} \) has nonzero simple elements; hence it has nonzero socle.

Finally, the elementary characterization 0.3 of primeness for example, for triple systems
\[ \exists x, y \exists z_1, z_2, u, v_1, v_2, \]
\[ (x \neq 0 \& y \neq 0) \Rightarrow (P_z P_z P_z P_y P_{y_1} \neq 0 \& P_z P_z P_y P_{y_2} \neq 0) \]
holds in each \( J_\lambda \) with \( \lambda \in \Lambda_0 \) since these are strongly prime systems with nonzero socle. Thus, it holds in \( \tilde{J} \), and it is a strongly prime system. \[ \]

Note that the full strength of the results of [5] implies that the ultra-
product of strongly prime systems is again strongly prime.

5.7

Let us return to the situation we are interested in. After our change of
notation, \( J_0 \) is the original system and \( J = \mathbb{C}(J_0)J_0 \) is its central closure.
Now we construct a primitive embedding for \( J \) as in Theorem 4 of [33].
Take an algebraically closed field \( \Omega \) containing \( \mathbb{C}(J_0) \) with cardinality
\( |\Omega| > \dim_{\mathbb{C}(J_0)} J + 2 \), and form a tight scalar extension \( \Omega J \). By 5.5,
\( \text{Jac}(\Omega J) = 0 \) (since \( \dim_{\mathbb{C}(J_0)} \Omega J \leq \dim_{\mathbb{C}(J_0)} J \)); hence by 0.8, \( \Omega J \) is a subdirect
product of its primitive images \( (J_{\lambda})_{\lambda \in \Lambda} \). Let \( \pi_J : \Omega J \to J_{\lambda} \)
be the corresponding projection, and consider the system of sets \( \Lambda_x = \{ \lambda \in \Lambda \mid \pi_J(x) \neq 0 \} \), for \( 0 \neq x \in \Omega J \), which is directed by strong primeness of \( \Omega J \) (which
is tight over \( J \)). Let \( \mathcal{F} \) be an ultrafilter containing the system \( \{ \Lambda_x \mid 0 \neq x \in \Omega J \} \), and form the ultraproduct \( \tilde{J} = \prod_{J_{\lambda}} / \mathcal{F} \), which is again a Jordan
system of the same type (pair, triple, or algebra) over the field \( \tilde{\Omega} = \Omega^\Lambda / \mathcal{F} \), the
ultrapower of \( \Omega \). Notice that \( J \) embeds in \( \tilde{J} \) (see [2, 33]), so we will
consider \( J \) as a subsystem of \( \tilde{J} \).

We next show that PI-elements of \( J \) are still PI-elements of \( \tilde{J} \). This
follows from the following extension principle

5.8. LEMMA. \textit{Any generalized polynomial}
\[ p(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, b_1, \ldots, b_m) \]
\textit{for fixed} \( b_i \in J \) \textit{and} \( f(x_1, \ldots, x_n, y_1, \ldots, y_m) \) \textit{of the free Jordan system} \( FT[X \cup Y] \) \textit{has} \( f(\Omega J, \ldots, \Omega J, b_1, \ldots, b_m) = 0 \) \textit{implies that} \( f(\tilde{J}, \ldots, \tilde{J}, b_1, \ldots, b_m) = 0 \).
Proof. This follows since the vanishing of \( f(x_1, \ldots, x_n, b_1, \ldots, b_m) \) for all \( x_i \) is an elementary sentence that is inherited by quotients of \( \Omega J \). 

Notice that in this lemma we do not need that \( \mathcal{F} \) be an ultrafilter. Indeed, if \( \tilde{J} = \prod_{\lambda \in \Lambda} J_\lambda / \mathcal{I} \) is a quotient of the product \( \prod_{\lambda \in \Lambda} J_\lambda \), let \( \tilde{\delta}: J \to \tilde{J} \) be the map induced by the map \( x \to (\pi(x)) \), \( J \to \prod_{\lambda \in \Lambda} J_\lambda \), and let \( f(\Omega J, \ldots, \Omega J, b_1, \ldots, b_m) = 0 \) hold. Then, for any \( a_1, \ldots, a_n \) in \( \prod_{\lambda \in \Lambda} J_\lambda \) we can take elements \( a_{\lambda,i} \in \Omega J_\lambda \), \( i = 1, \ldots, n \) and \( \lambda \in \Lambda \), such that \( (\pi(a_{\lambda,i}))_{\lambda \in \Lambda} = a_i \). Now, we have, for all \( \lambda \in \Lambda \), \( f(a_{\lambda,1}, \ldots, a_{\lambda,n}, b_1, \ldots, b_m) = 0 \); hence, \( f(\pi(a_{\lambda,1}), \ldots, \pi(a_{\lambda,n}), \pi(b_1), \ldots, \pi(b_m)) = 0 \), and \( f(a_1 + I, \ldots, a_n + I, \delta(b_1), \ldots, \delta(b_m)) = 0 \).

Notice also that the condition on \( f \) holds if for any linearization \( g \) of \( f \) in \( x_1, \ldots, x_n \) we have \( g(J, \ldots, J, b_1, \ldots, b_m) = 0 \). In particular, if \( f \) is linear in the variables \( x_i \), then \( f(J, \ldots, J, b_1, \ldots, b_m) = 0 \) suffices.

5.9. Lemma. PI(J) = PI(\tilde{J}) \cap J, and \( \tilde{J} \) is primitive at each nonzero \( a \in \text{PI}(J) \).

Proof. Let \( a \) be a nonzero element in \( \text{PI}(J) \); then \( J_a \) satisfies some multilinear identity \( f(x_1, \ldots, x_n) = 0 \) and hence \( J \) satisfies \( f(a; J, \ldots, J) = 0 \) where \( f(z; x_1, \ldots, x_n) \) is the \( z \)-homotope of \( f \). By the previous lemma, \( \tilde{J} \) still satisfies \( f(a; \tilde{J}, \ldots, \tilde{J}) = 0 \), and \( \tilde{J}_a \) has the PI-f. Thus we get \( \text{PI}(J) \subseteq J \cap \text{PI}(\tilde{J}) \). Since the reverse containment is obvious, we obtain \( \text{PI}(J) = J \cap \text{PI}(\tilde{J}) \).

Now, for each \( \lambda \in \Lambda_a \), the projection \( a_\lambda \) of \( a \) into \( J_\lambda \) is a nonzero PI-element because \( (J_\lambda) a_\lambda \) is a homomorphic image of \( J_a \). Therefore, \( J_\lambda \) is strongly prime with nonzero socle by 0.17 since \( J_a \) is primitive. Thus Lemma 5.6 implies that \( \tilde{J} \) is strongly prime with nonzero socle equal to \( \text{PI}(\tilde{J}) \) by 0.17. In particular, \( \tilde{J} \) is primitive at each \( a \in \text{PI}(J) \) by 0.11.

5.10. Lemma. If \( \tilde{b}_1, \tilde{b}_2 \in J_a \) generate orthogonal ideals: \( \text{id}_{J_a}(\tilde{b}_1) \cap \text{id}_{J_a}(\tilde{b}_2) = 0 \), then they generate orthogonal ideals in \( J_a \). In particular, if \( J_a \) is prime, then \( J_a \) is prime.

Proof. We prove this for Jordan triple systems, although the same proof works for pairs and algebras.

Suppose that \( c_i \in \tilde{J} \) has \( \tilde{c}_i = c_i + \text{Ker } a \in \text{id}_{J_a}(\tilde{b}_i) \), for \( i = 1, 2 \), then there are algebra polynomials \( q_i = q_i(x_{1i}, \ldots, x_{ni}, y_i) \) for \( i = 1, 2 \), all of whose monomials contain the variable \( y_i \), such that \( \tilde{c}_i = q_i(a; d_{1i}, \ldots, d_{ni}, b_i) + \text{Ker } a \) for some \( d_{ij} \in J \). Then, setting

\[
g(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, y_1, y_2, y_3) = P_{\tilde{y}_3} P_{\tilde{q}_1} P_{\tilde{q}_2} \tilde{c}_2^{(y_3)},
\]

we get \( g(\Omega J, \ldots, \Omega J, b_1, b_2, a) = 0 \) since \( q_i^{(a)}(\Omega J, \ldots, \Omega J, b_i) + \text{Ker } a \subseteq \text{id}_{J_a}(\tilde{b}_i) \) and \( \text{id}_{J_a}(\tilde{b}_1) \cap \text{id}_{J_a}(\tilde{b}_2) = 0 \). Thus, by 5.8, \( g(\tilde{J}, \ldots, \tilde{J}, b_1, b_2, a) = 0 \) and hence, in particular, \( P_{\tilde{c}_1} P_{\tilde{c}_2} = 0 \); i.e., \( U_{\tilde{c}_1} \tilde{c}_2 = 0 \).
Therefore $U_{id/J}\overline{f}_0(id/J_2) = 0$, and $id/J_0(id/J_2) \cap id/J_2 = 0$ by nondegeneracy of $J_a$.

As stated in 5.2, our aim is to find an element $a \in J$ such that the local $J_a$ is prime. By the fact just proved, for $J_a$ to be prime it suffices that $\tilde{J}_a$ be prime, and this will follow easily from 0.11 if our systems are algebras or pairs. However, for triple systems the situation is not as neat as for pairs or algebras due to the fact that local algebras of primitive triple systems need not be primitive (see [1]). This forces us to split the proof and consider separately Jordan and triple systems. Before treating them, we first focus on pairs and algebras (here $P$ denotes the operators $Q$ and $U$ of pairs and algebras, respectively, and we reserve the $U$-notation for local algebras).

5.11. End of the Proof of 5.1 for Pairs and Algebras. Take a nonzero $a \in P(J)$. Then $a \in P(\tilde{J})$ by 5.9, and $\tilde{J}$ is primitive with nonzero socle, equal to $P(\tilde{J})$ by 0.17. Thus $J_a$ is primitive with nonzero socle by 0.11(ii), so, in particular, it is prime. Thus $J_a$ is prime by 5.10. Moreover, since $a \in P(\tilde{J})$, $\tilde{J}_a$ is PI, and therefore it is simple with capacity by 0.13. The same argument applies to each $J_{\lambda}$ with $\lambda \in \Lambda_a$: $\pi(a) = a_{\lambda} \in P(J_{\lambda})$; hence $P(J_{\lambda})$ is nonzero, and $J_{\lambda}$ is rationally primitive. Thus $a_{\lambda} \in \text{Soc}(J_{\lambda})$, and $(J_{\lambda})_{a_{\lambda}}$ is simple with capacity.

Now, since the local algebra $J_{\lambda}$ is strongly prime and PI, by 0.13 its central closure $\Gamma_{\lambda}^{-1}J_{\lambda}$, where $\Gamma_{\lambda} = \Gamma(J_{\lambda})$ is the centroid of $J_{\lambda}$, is simple and unital, equal to its socle. Our aim is to show that $\Gamma_{\lambda}$ is in fact a field (so that $J_{\lambda}$ is its own central closure, and therefore it is simple, equal to its socle). To see that we show that $\Gamma_{\lambda}$ is algebraic over $\mathfrak{g}(J_{\lambda})$.

We can find $\gamma \in \Gamma_{\lambda}$ and $z \in J_{\lambda}$ such that $1 = \gamma^{-1}z$ is the unit element of $J_{\lambda}$. In particular, $U_z = U_{\gamma(z)}$ belongs to $\Gamma_{\lambda}$. This shows that there are nonzero elements $z \in J_{\lambda}$ such that $U_z$ belongs to the centroid $\Gamma_{\lambda}$.

Claim 1. For any $\bar{z} \in J_{\lambda}$ such that $U_{\bar{z}} \in \Gamma_{\lambda}$, there is a $\mu \in \tilde{\Omega}$ such that $U_{\bar{z}}x = \mu x$ for all $x \in J_{\lambda}$; i.e., $P_{\bar{z}}, x = \mu x$ for all $x \in J_{\lambda}$.

Indeed, for all $\lambda \in \Lambda_a$, the operator $U_{\lambda}$, where $z_{\lambda}$ is the image of $z$ in $J_{\lambda}$, belongs to the centroid $\Gamma_{\lambda}$ of $(J_{\lambda})_{a_{\lambda}}$. Since this is a simple unital algebra over $\Omega$, $\Gamma_{\lambda}$ is a $\Omega$-field contained in $(J_{\lambda})_{a_{\lambda}}$. Also, $|\Omega| > \dim_{\Omega} J + 2 \geq \dim_{\Omega} J_{\lambda} \geq \dim_{\Omega} (J_{\lambda})_{a_{\lambda}} + 2$. Thus, by Amitsur’s trick (see [15, 4.5.9]), $\Gamma_{\lambda} = \Omega$ since $\Omega$ is algebraically closed. Thus there is $\mu_{\lambda} \in \Omega$ such that $U_{\lambda}x = \mu_{\lambda}x$ for all $x \in J_{\lambda}$. Therefore $P_{\bar{z}}, x_{\lambda} = \mu_{\lambda} x_{\lambda}$ for all $x_{\lambda} \in J_{\lambda}$.

Now, the choice of the previous $\mu_{\lambda}$ for each $\lambda \in \Lambda_a$, and $\mu_0 = 0$ otherwise, defines an element $\mu \in \tilde{\Omega}$ which satisfies $P_{\bar{z}}, x = \mu x$ for all $x \in J$.

This proves the claim.

Note that $\mathfrak{g}(J_{\lambda}) \subseteq \Gamma(J) \subseteq \Gamma_{\lambda}$; hence $\Gamma_{\lambda}$ is a $\mathfrak{g}(J_{\lambda})$-algebra.

Claim 2. For all $\bar{z} \in J_{\lambda}$ such that $U_{\bar{z}} \in \Gamma_{\lambda}$, $U_{\bar{z}}^2 \in \mathfrak{g}(J_{\lambda})$. 


If we take $x \in J$ and $\mu \in \tilde{\Omega}$ as before, we get $\mu P_a x = P_a P_a P_a x \in J$; hence $\mu P_a J \subseteq J$. Thus, by Lemma 4.5, $\mu \in \tilde{\Omega} \cap \mathcal{S}(J)$. Now, the mapping $\sigma: \tilde{\Omega} \cap \mathcal{S}(J) \to \mathcal{S}(J)$ defined in 4.7 is a monomorphism of $\Gamma(J)$-algebras by 4.8 and hence of $\mathcal{S}(J_a)$-algebras. Since, by 4.6 we have $\mathcal{S}(J)^2 \subseteq \mathcal{S}(J_a)$, we obtain $\sigma(\mu^2) = \sigma(\mu)^2 \in \mathcal{S}(J_a)$; hence $\mu^2 \in \mathcal{S}(J_a)$. On the other hand, for all $x \in J_a$, $P_a^2 x = \mu^2 x$ by claim 1; hence $P_a^2 = \mu^2 \in \mathcal{S}(J_a)$.

Claim 3. $\Gamma_a^4 \subseteq \mathcal{S}(J_a)$.

Take $y \in \Gamma_a$ and $0 \neq \xi \in J_a$ such that $U_{\xi} \in \Gamma_a$, and by Claim 2, the elements $U_{\xi}^2 = \delta$ and $U_{\xi}^2 = \rho$ belong to $\mathcal{S}(J_a)$. Thus $\eta^4 \delta = \eta^4 U_{\xi}^2 = U_{\xi}^2 = \rho$; hence $\eta^4 = \delta^{-1} \rho \in \mathcal{S}(J_a)$.

Thus $\Gamma_a$ is algebraic over the field $\mathcal{S}(J_a)$; hence it is a field too.

For future reference we state next what we have proved:

5.12. Remark. If $J = \mathcal{S}(J_a)J_0$ is a Jordan pair or Jordan algebra, then for any $a \in \Pi(J)$, $J_a$ is strongly prime and the local centroid $\Gamma_a = \Gamma(J_a)$ has $\Gamma_a^4 \subseteq \mathcal{S}(J_a)$; hence $\Gamma_a$ is algebraic over $\mathcal{S}(J_a)$.

We now turn to Jordan triple systems, for which we first need some preparatory work.

5.13

We say that a strongly prime Jordan triple system $T$ is of polarized type if it has a polarized ideal $I = I^+ \oplus I^-$, with $P_1 T \subseteq I^\sigma$, $P_\sigma T \subseteq I^{-\sigma}$, and $\{I^\sigma, T, T\} \subsetneq I^\sigma$, for $\sigma = \pm$. Equivalently, $J$ is of polarized type if its double $V(J)$ is not tight (see Lemma 2.4 of [5]).

5.14. Lemma. Let $T$ be a primitive Jordan triple system with nonzero socle, and let $0 \neq a \in \text{soc}(T)$. Then $T_a$ is simple with capacity if $T$ is not of polarized type, and $T_a$ is a direct sum of at most two simple algebras with capacity if $T$ is polarized. In particular $T_a$ is prime if and only if it is simple.

Proof. By 0.11(iii) $T_a$ is a subdirect sum of at most two primitive algebras with nonzero socle. Moreover, since $a \in \text{soc}(T)$ and obviously $\text{soc}(V(T)^+) = V(\text{soc}(T))$, we can consider $a$ as an element of $\text{soc}(V(T)^+)$. Then $T_a = V(T)^+ \ominus$ has capacity by 0.7 of [27], and therefore it is a finite direct sum of simple algebras with capacity by [15, 6.4.1], so it is a direct sum of at most two simple algebras with capacity.

Suppose now that $T$ is not of polarized type, then $V(T)$ is a tight double of $J$. Moreover, $V(T)$ is primitive at any element of $\text{soc}(V(T)) = V(\text{soc}(T))$ (see 0.11). Therefore, considering $a$ as an element of $\text{soc}(V(T))^+$ as before, $T_a = V(T)^+$ is primitive by 0.11(ii) and hence strongly prime, and therefore it has only one simple summand; hence it is simple with capacity.


5.15. **Lemma.** Let $T$ be a primitive Jordan triple system with nonzero socle, and let $a \in \text{Soc}(T)$. If $\bar{I}$ and $\bar{L}$ are nonzero ideals of $T_a$, which are orthogonal, $\bar{I} \cap \bar{L} = 0$, then for any nonzero $\bar{b} = b + \text{Ker } a \in \bar{I}$, the local algebra $T_{p,b}$ is primitive.

**Proof.** Since, by hypothesis, $T_a$ is not prime, 5.14 implies that $T_a$ is a direct sum of two simple algebras $J_1, J_2$. Now, if $\pi_i: T_a \to J_i$ is the corresponding projection, $\pi_i(\bar{L}) \neq 0$ or $\pi_i(\bar{L}) \neq 0$. Suppose that $\pi_i(\bar{L}) \neq 0$. Now, we have $U_{\pi_i(\bar{I})} \pi_i(\bar{L}) = 0$, and since $\pi_i(\bar{L}) \neq 0$, we have $\pi_i(\bar{I}) = 0$ by strong primeness of $J_i$. Then $\bar{I} \subseteq \text{Ker } \pi_i$, and since $\text{Ker } \pi_1 \cap \text{Ker } \pi_2 = 0$, $U_i \text{Ker } \pi_2 = 0$. In particular $\text{Ker } \pi_2 \subseteq \bar{b}$ since $b \in \bar{I}$, and $(T_a)_{b} = (T_a/\text{Ker } \pi_2)b/\text{Ker } \pi_2 = (J_2)_{b}/\text{Ker } \pi_2$ is primitive by [8], so it suffices to note that $(T_a)_{b} \cong T_{p,b}$ by [27, 0.5].

Now we return to the notation introduced in 5.7, so that $J$ is a Jordan triple system and $\hat{J}$ is a ultraproduct of the primitive images $J_\lambda$, $\lambda \in \Lambda$ of the scalar extension $\Omega J$.

5.16. **Lemma.** With the notation of 5.7, if $\hat{J}_a$ is primitive for $a \in \text{Pl}(J)$, then the set $\Lambda_a$ of $\lambda \in \Lambda_a$ such that $(J_\lambda)_{a_\lambda}$ is primitive belongs to the ultrafilter $\mathcal{F}$.

**Proof.** Since $a \in \text{Pl}(\hat{J})$ and $\hat{J}$ is primitive, $\hat{J}$ is rationally primitive; hence $\text{Pl}(\hat{J}) = \text{Soc}(\hat{J})$. Thus $a \in \text{Soc}(\hat{J})$, and the algebra $\hat{J}_a$ is either simple with capacity or a direct sum of two simple algebras with capacity by 5.14. Since it is primitive, hence prime, it must be simple. Similarly, if $\lambda \in \Lambda_a$, $0 \neq a_\lambda \in \text{Pl}(J_\lambda)$ and as before, all algebras $(J_\lambda)_{a_\lambda}$ have capacity, so they are either simple or a direct sum of two simple algebras according to whether they are strongly prime or they are not. Since these algebras have nonzero socle (in fact, they equal their socles), both options can be expressed by the elementary sentences 0.3. Denote by $\Lambda_1$ the set of $\lambda \in \Lambda$ such that

$$\forall x, y \exists z, r, u, v \quad \left( P_{a_\lambda} x \neq 0 \& P_{a_\lambda} y \neq 0 \right)$$

$$\Rightarrow \left( P_{a_\lambda} x P_{a_\lambda} y P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} \neq 0 \& P_{a_\lambda} x P_{a_\lambda} y P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} P_{a_\lambda} r \neq 0 \right)$$

holds in $J_a$. And let $\Lambda_2$ be the complementary of $\Lambda_1$. Then $\Lambda_2$ is the set of indexes $\lambda$ such that the negation of that sentence, which is again elementary, holds in $J_a$.

Now, one of the sets $\Lambda_1, \Lambda_2$ belongs to $\mathcal{F}$. If $\Lambda_2 \in \mathcal{F}$, then $\hat{J}_a$ can not be simple since the corresponding elementary sentence would hold in $\hat{J}$. Therefore $\Lambda_1$ belongs to $\mathcal{F}$, and $\Lambda_0 = \Lambda_1 \cap \Lambda_a \in \mathcal{F}$. ■
5.17. LEMMA. With the notation of 5.7, if $J$ is of polarized type, then there is a nonzero element $a \in \mathcal{P}(J)$ such that $J_a$ and $J_a$ are prime. If $J$ is not of polarized type, there is also a nonzero element $a \in \mathcal{P}(J)$ such that $J_a$ is prime.

Proof. If $J$ is of polarized type, take $(I^+, I^-)$ a nonzero ideal of $V(J)$ with $I^+ \cap I^- = 0$. Note that $\mathcal{P}(J)$ is an ideal by 0.16, and for $\sigma = \pm$, $P_{I^\sigma} \mathcal{P}(J) \subseteq I^\sigma \cap \mathcal{P}(J)$ is nonzero, since otherwise $I^\sigma$ would annihilate $\mathcal{P}(J)$ by 1.7 of [22], contradicting primeness of $J$. Thus we can take nonzero elements $a^\sigma \in \mathcal{P}(J) \cap I^\sigma$ and set $a = a^+ + a^-$. Then $J_a$ has nonzero ideals $I^\sigma = I^\sigma + \ker a/\ker a$ for $\sigma = \pm$, and it is easy to see that they are orthogonal $\tilde{I}^+ \cap \tilde{I}^- = 0$. Take nonzero elements $\tilde{a}^\sigma = b^\sigma + \ker a \in \tilde{I}^\sigma$. Then they generate orthogonal ideals in $\tilde{J}$ by 5.10; hence the algebras $\tilde{J}_{p_i, b}$ are prime by 5.15, and so are the algebras $J_{p_i, b}$ by 5.10. Also $P_{I^\sigma} \mathcal{P}(J)$ since $\mathcal{P}(J)$ is an ideal and $a \in \mathcal{P}(J)$.

Finally, if $J$ is not polarized and $J_a$ is not prime, there exist nonzero ideals $\tilde{I}$ and $\tilde{L}$ of $J_a$ with $\tilde{I} \cap \tilde{L} = 0$, and the same argument as before, using 5.10 and 5.15, shows that we can find a nonzero element $P_{I^\sigma} b$ such that $J_{p_i, b}$ is prime and $P_{I^\sigma} b \in \mathcal{P}(J)$ since $\mathcal{P}(J)$ is an ideal of $J$. (As before, we get here that $J_{p_i, b}$ is prime, but note that if $J_a$ was already prime we cannot conclude that $J_a$ is also prime.)

5.18

The previous result will allow us to apply the same argument as in 5.11 to the proof of Theorem 5.1 in the case where $J$ is of polarized type. We will deal with the case where $J$ is not of polarized type by applying the already proved results to the double $V(J)$. To do that we need some information on the extended centroid of $V(J)$ (which is a field since $V(J)$ is strongly prime if it is tight).

For a nondegenerate Jordan triple system $J$ we can define a mapping $\mathcal{C}(J) \rightarrow \mathcal{C}(V(J))$ by simply doubling $J$-homomorphisms: If $\lambda = [f, I] \in \mathcal{C}(J)$, then the pair $((f^+, f^-), (I^+, I^-))$, where $f^+ = f^- = f$ and $I^+ = I^- = I$, defines a permissible mapping of $V(J)$ (see 1.2(e)), and therefore, an element of $\mathcal{C}(J)$. Moreover, if $K = (K^+, K^-)$ is an essential ideal in $V(J)$, then $(K^+, K^-) \cap (K^+, K^-) \neq 0$; hence $K$ contains the essential ideal $V(L)$, where $L = K^+ \cap K^-$ is an essential ideal of $J$. Using that, it is easy to see that this is a well-defined monomorphism of rings, so that we can view $\mathcal{C}(J)$ as contained in $\mathcal{C}(V(J))$.

5.19. LEMMA. Let $J$ be a strongly prime Jordan system, and suppose that it is not of polarized type, so that $V(J)$ is a tight double. Then $\mathcal{C}(J) \subseteq \mathcal{C}(V(J))$ is a degree two extension of fields.

Proof. If $\lambda = [(f^+, f^-), (I^+, I^-)]$ is an element of $\mathcal{C}(V(J))$, we define $\lambda^\sigma$ as the class of $((f^+, f^-), (I^-, I^+))$. This is easily seen to be a well-defined
mapping, and it is straightforward to check that this is an automorphism of \( \mathcal{E}(V(J)) \) of period two.

Now, if \( \lambda = \lambda^* \) is a \( * \)-invariant element of \( \mathcal{E}(J) \) and \( ((f^+, f^-), (I^+, I^-)) \) are a representative of \( \lambda \), then \( ((f^+, f^-), (I^+, I^-)) \) and \( ((f^-, f^+), (I^-, I^+)) \) are equivalent permissible mappings; hence setting \( L = I^+ \cap I^- \) and \( g = f_L^* = f_L^- \) (see 1.10), \( (g, L) \) is a permissible mapping of \( J \), and it is clear that \( \lambda = [g, L] \) under the identification \( \mathcal{E}(J) \cong \mathcal{E}(V(J)) \). Thus, the set of \( * \)-invariant elements of \( \mathcal{E}(V(J)) \) is precisely \( \mathcal{E}(J) \), and this proves the lemma. \( \blacksquare \)

5.20. Lemma. Let \( J \) be a Jordan algebra and let \( \tilde{J} \) be a nondegenerate scalar extension of \( J \). Then there is an embedding \( \tau: \Gamma(J) \to \Gamma(\tilde{J}) \) such that, for any \( \gamma \in \Gamma(J) \) and any \( x \in J \), \( \gamma(x) = \tau(\gamma)(x) \).

Proof. Take \( \gamma \in \Gamma(J) \) and any \( \tilde{x} \in \tilde{J} \). Since \( \tilde{J} \) is a scalar extension of \( J \), \( \tilde{J} = \Gamma(J)J \); hence there exist elements \( \omega_i \in \Gamma(J) \) and \( x_i \in J \) such that \( \tilde{x} = \sum \omega_i x_i \). We want to define \( \tau(\gamma)(\tilde{x}) \) by the expression \( \sum \omega_i \gamma(x_i) \). To see that this is well defined it suffices to show that if \( \tilde{x} = 0 \), then \( z = \sum \omega_i \gamma(x_i) = 0 \). Now, if \( y \in J \), \( P_y = \sum \omega_i \gamma(x_i) y + \sum \omega_i \omega_j \gamma(x_i) \gamma(x_j) y \) is of period two.

5.21. End of the Proof of 5.1 for Triple Systems. By 5.17, if \( J \) is of polarized type, we can find a nonzero element \( a \in \text{PI}(J) \) such that both \( \tilde{J}_a \) and \( J_a \) are prime. Then, by 5.16, the set \( \Lambda_0 \) of indexes \( \lambda \in \Lambda \) such that \( (J_a)_{\lambda a} \) is prime (hence simple and PI by 5.14) belongs to \( \mathcal{F} \). Then the proof given in 5.11 for pairs and algebras applies verbatim if we take \( \Lambda_0 \) instead of \( \Lambda_a \).

Next, if \( J \) is not of polarized type, the double \( V = V(J) \) is a strongly prime Jordan pair. Now, by the last assertion of 5.17, there is a nonzero \( a \in \text{PI}(J) \) such that \( \tilde{J}_a \) is prime (hence strongly prime, since nondegeneracy is obviously inherited by local algebras), and \( \Gamma_g \) is a domain. Now, setting \( a'' = a \in V^\sigma \), the local \( V^+_a = J_a \) is PI, and therefore \( V \) has PI-elements and we can apply 5.12 to conclude that the local centroid \( \Gamma(\mathcal{E}(V)V^+_a) \) is algebraic over \( \mathcal{E}(V) \). Now, since \( \mathcal{E}(V) \) is algebraic over \( \mathcal{E}(J) \) by 5.19 and \( \mathcal{E}(J) \) is algebraic over \( \mathcal{E}(J_0) \) by 4.6, the field \( \Gamma(\mathcal{E}(V)V^+_a) \) is algebraic over \( \mathcal{E}(J_0) \).

Finally note that \( \mathcal{E}(V)V^+_a \) is a central extension of \( V^+_a = J_a \); hence \( \Gamma_g = \Gamma(J_a) \) embeds canonically into \( \Gamma(\mathcal{E}(V)V^+_a) \) by 5.20. Therefore \( \Gamma_g \) is algebraic over \( \mathcal{E}(J_0) \), and since it is a domain, it must be a field. Thus, again we conclude that \( J_a \) is its own central closure, and since it is PI, it is simple. \( \blacksquare \)
6. HOMOTOPE POLYNOMIAL IDENTITIES

As we have already mentioned, D’Amour and McCrimmon’s work [1, 2] suggests that an operant PI theory for general Jordan systems should deal with homotope identities rather than identities (polynomials of the free system which vanish under substitutions but are nonzero in the free special system). In [27] we proved 0.18, an analogue of Kaplansky’s theorem for homotope-PIs on primitive Jordan systems. As a consequence of our main theorem we can prove here an analogue of Posner’s theorem for general Jordan systems:

6.1. THEOREM. Let $J$ be a prime nondegenerate Jordan system.

(i) If the local algebra at each element of $J$ is PI, then the extended central closure $C(J)J$ is simple, equal to its socle.

(ii) If $J$ satisfies a homotope-PI, then $C(J)J$ is simple with capacity.

Proof. In both cases $C(J)J$ is primitive by 5.1 and we can apply 0.18.

Admittedly, this is a weak version of Posner’s theorem or rather of the sharper Posner–Rowen theorem, since here we need the extended central closure instead of the (ordinary) central closure $\Gamma^{-1}J$. If $J$ is a Jordan algebra, we now see from 0.13 that the stronger version holds, but there is a basic reason for finding an obstruction in its extension to Jordan systems; namely, there cannot be central polynomials in Jordan systems other than algebras. A possible substitute would be central polynomials of the multiplication algebra, and indeed this detour works for exceptional systems (see [33, p. 517; 2]). However it relies on the outer simplicity of exceptional systems; hence there is not a straightforward way of proving their general existence. Nevertheless, we state it here as a conjecture:

6.2. Conjecture 1. The extended centroid of a strongly prime homotope-PI Jordan system coincides with the field of fractions of its centroid.

6.3

In spite of the preference for homotope polynomial identities mentioned before, Theorem 2 of [33] suggests that a general PI theory of Jordan systems could be worked out. On the other hand, it was proved in [27] that any PI–Jordan algebra satisfies a homotope-PI. Call a Jordan system $J$ a PI system if there is an essential polynomial which vanishes under all substitutions of its variables, where a polynomial is essential if it is not a special identity and its image in the free special Jordan system viewed as an associative polynomial has a monic leading monomial. We state

6.4. Conjecture 2. Every PI–Jordan system satisfies a homotope-PI.
6.5

Theorem 5.1 will be used in [13] to relate inner ideals of $J$ a special Jordan algebra and one-sided ideals of its $\ast$-tight envelopes. Here, however, we give an application which illustrates the strength of that result. We generalize D’Amour and McCrimmon’s primitive form theorem (see [2]), which states that if a simple Jordan pair $V$ is a scalar form of some classical pair $\tilde{V}$ of Clifford type, then $\tilde{V}$ is primitive.

6.6. Theorem. If a simple Jordan system $J$ with $\text{PI}(J) \neq 0$ is a scalar form of some nondegenerate $\tilde{J}$, then $J$ is primitive. Moreover, if $J$ satisfies a homotope-PI, then it has capacity.

Proof. Nondegeneracy passes down to $J$ since $\tilde{J}$ is a scalar extension of $J$. Thus, $J$ is strongly prime, and $\mathcal{F}(J)J$ is primitive by 5.1. Now $\mathcal{F}(J) = \Gamma(J)$ since $J$ is simple; so $J = \Gamma(J)J$ is primitive. If, in addition $J$ satisfies a homotope-PI, we can apply 0.18 to conclude that $J$ has capacity. 

(A different approach would be as follows: If $\text{Jac}(J) = J$, a suitable tight scalar extension $J'$ is properly nil (see [2, 8.6]), and then it is McCrimmon radical by [27, 6.3(b)], which is impossible since $J \subseteq J'$ is tight, and $J$ is nondegenerate.)

6.7

As a final remark let us point out that 6.1 improves the Zelmanov–D’Amour–McCrimmon classification of strongly prime Jordan systems (see [2] or [6]): the extended central closure of Clifford, bi-Cayley, and Albert forms is already Clifford, bi-Cayley, or Albert (instead of the less definite scalar extensions required in the original proofs).

REFERENCES


