Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 1000–1002
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

Variational theory for physiological flow

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Received 28 September 2006; accepted 20 December 2006

Abstract

Using He's semi-inverse method, a variational principle for physiological flow is established.

Keywords: Variational theory; Blood flow; Semi-inverse method

1. Introduction

Consider an infinite, horizontal elastic tube, uniform in the absence of waves, containing an incompressible, inviscid fluid of density ρ. Assume that the wavelength is large compared with the tube diameter, so that one-dimensional theory can be applied and the internal pressure p, longitudinal fluid velocity u and cross-sectional area A are functions only of longitudinal coordinate x and time t. Assume further that the external pressure is a constant, which we may set equal to zero.

In addition we assume that the pressure in arteries can be expressed in the form [1]

$$p = p(A). \tag{1}$$

Proceeding in this way, the conservation of mass leads to the following equation:

$$A_t + (uA)_x = 0, \tag{2}$$

and the momentum equation reads

$$u_t + uu_x = -\frac{1}{\rho} p_x = -\frac{c^2}{A} A_x, \tag{3}$$

where a suffix denotes partial differentiation, the quantity c has the dimensions of a velocity and is given by

$$c^2 = \frac{A}{\rho} p'(A). \tag{4}$$

Wave phenomena were studied in Ref. [1]. A biologically complete and yet mathematically simple summary of the physics behind the pulse wave in arteries was that given by Lighthill [2].

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.12.043

In this paper we will apply the semi-inverse method [3,4] proposed by Ji-Huan He to establish a variational formulation for the discussed problem. The semi-inverse method has been applied to search for various variational principles directly from field equations [5–10].

2. Variational principle addressed with He's semi-inverse method

In order to use He's semi-inverse method, we introduce a special function Φ defined as

$$\Phi_x = A, \tag{5}$$

$$\Phi_t = -uA, \tag{6}$$

so that Eq. (2) is automatically satisfied.

According to He's semi-inverse method [3,4], we can construct a trial functional in the form

$$J(u, A, \Phi) = \iint \left\{ u\Phi_t + \frac{1}{2}u^2\Phi_x + c^2\Phi_x \ln A + F \right\} \mathrm{d}x\mathrm{d}t, \tag{7}$$

where F is an unknown function of u, A and their derivatives.

There exist alternative approaches to the construction of the trial functionals; see Refs. [11–17].

Making the trial functional, Eq. (7), stationary with respect to Φ results in the following Euler–Lagrange equation:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Phi_x}\right) = 0, \tag{8}$$

where $L = u\Phi_t + \frac{1}{2}u^2\Phi_x + c^2\Phi_x \ln A + F$, so Eq. (8) can be rewritten as

$$-u_t - \frac{1}{2}(u^2)_x - c^2(\ln A)_x = 0, \tag{9}$$

which is equivalent to Eq. (3).

Now the stationary conditions with respect to u and A read

$$\Phi_t + u\Phi_x + \frac{\delta F}{\delta u} = 0, \tag{10}$$

$$\frac{c^2\Phi_x}{A} + \frac{\delta F}{\delta A} = 0, \tag{11}$$

where $\delta F/\delta u$ is called the variational differential with respect to u, defined as $\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial F}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x})$.

Eqs. (10) and (11) should be equivalent to the field equations, Eqs. (5) and (6). We substitute Eqs. (5) and (6) into Eqs. (10) and (11) to identify F:

$$\frac{\delta F}{\delta u} = -\Phi_t - u\Phi_x = uA - uA = 0, \tag{12}$$

$$\frac{\delta F}{\delta A} = -\frac{c^2\Phi_x}{A} = -c^2, \tag{13}$$

and from Eqs. (12) and (13) the unknown F can be identified as

$$F = -c^2 A. \tag{14}$$

We obtain, therefore, the final variational principle for the discussed problem, which reads

$$J(u, A, \Phi) = \iint \left\{ u\Phi_t + \frac{1}{2}u^2\Phi_x + c^2\Phi_x \ln A - c^2 A \right\} \mathrm{d}x\mathrm{d}t. \tag{15}$$

3. Conclusion

The variational approach to solitary solutions has caught much attention recently [18–20]. We obtain a variational principle for the discussed problem using the semi-inverse method which is proven to be a promising method for the search for variational principles directly from field equations without the use of Lagrange multipliers. Applying the Ritz method, we can easily obtain soliton solutions and periodic solutions; the solution procedure is illustrated in detail in Refs. [21,22].

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