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ScienceDirect
Discrete Mathematics 306 (2006) 2270–2274
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

The carvingwidth of hypercubes ☆

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Received 15 September 2005; received in revised form 29 January 2006; accepted 21 April 2006
Available online 3 July 2006

Abstract

The notion of the carvingwidth of a graph was introduced by Seymour and Thomas [Call routing and the ratcatcher, Combinatorica 14 (1994) 217–241]. In this note, we show that the carvingwidth of a d-dimensional hypercube equals 2^{d-1}.

Keywords: Carvingwidth; Hypercube; Discrete isoperimetric inequalities

1. Introduction

In recent years many "width" parameters of graphs were studied in connection to decomposing a given graph, in order to deal with certain algorithmic problems or to prove some important graph theoretic results. The treewidth, pathwidth, branchwidth, carvingwidth, cliquewidth, etc. are some examples. In spite of their slightly nonintuitive, technical-sounding definitions, these width parameters have proved to be of great use in graph theory as well as in algorithms. Many of these parameters were introduced by Robertson and Seymour in their series of fundamental papers on graph minors. The notion of the carvingwidth of a graph G, denoted as $\mathrm{cw}(G)$ (see Section 2 for definition) was introduced in [13] by Seymour and Thomas, to develop an algorithm to decide whether there exists a routing tree with congestion less than k, when the underlying graph is planar. In that paper, they had introduced another important parameter called branchwidth. This parameter approximates the treewidth by a constant factor. They gave an $O(n^4)$ algorithm to construct a optimal branch decomposition of a planar graph. The key part of this algorithm is to construct an optimal carving decomposition of a corresponding graph, called the medial graph. Recently Gu and Tamaki [5] improved the situation by giving an $O(n^3)$ algorithm to construct the optimal carving decomposition of the medial graph, thus making it possible to construct the optimal branch decomposition of the planar graph in $O(n^3)$ time. An algorithm to decide whether the carvingwidth of a graph is at most k (for a fixed k) is given by Thilikos et al. [14].

A d-dimensional hypercube H_d on 2^d vertices is defined as follows: the vertices of H_d correspond to the 2^d binary strings each of length d, two of the vertices being adjacent *if and only if* they differ in exactly one bit position. Hypercubes are a well-studied class of graphs, which arise in the context of parallel computing, coding theory, algebraic graph theory and many other areas. Hypercubes are popular among graph theorists because of their symmetry, small diameter

☆ The result of this paper was announced in the conference COCOON 2003, along with some other results.
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0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.04.012

and many other interesting graph-theoretic properties. In this paper, we show that:

$$\mathrm{cw}(H_d) = 2^{d-1}.$$

The result is obtained as a consequence of a relation we derive between the carvingwidth of a graph G and a certain type of edge isoperimetric property of G. We explain this relation below:

Let $G=(V, E)$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. Given a subset S of $V(G)$, its edge boundary $\delta_G(S)$ be defined as

$$\delta_G(S) = \{e \in E(G)\colon \text{ exactly one end point of } e \text{ is in } S\}. \tag{1}$$

Then the edge isoperimetric problem is to minimise $|\delta_G(S)|$ over all subsets S of $V(G)$ with $|S| = \ell$ for a given integer ℓ. We define

$$b_e(\ell, G) = \min_{S\subseteq V(G), |S|=\ell} |\delta_G(S)|.$$

Let s be any integer such that $1 \leqslant s \leqslant |V(G)|$. We show that, for any graph G:

$$\mathrm{cw}(G) \geqslant \min_{s/2 \leqslant x \leqslant s} b_e(x, G). \tag{2}$$

Discrete isoperimetric inequalities (both the edge and vertex versions) are well-studied in graph theory. Though Macaulay had published an isoperimetric theorem in 1921, the first appearance of the term “isoperimetric” in the combinatorial literature seems to be in a paper by Harper [8] published in 1966. See [12] for a brief survey on isoperimetric inequalities. The recent monograph by Harper [10] gives a much broader and deeper exposition of the results on combinatorial isoperimetric problems.

Isoperimetric inequalities turn out to be useful in various situations. In fact, Harper [8] studied the vertex isoperimetric problem on hypercubes in order to estimate the *bandwidth* of H_d. Chandran and Kavitha [2] used isoperimetric inequalities to give lower bounds for the treewidth and pathwidth of H_d. Harper applied again the isoperimetric inequalities to approximately estimate the bandwidth of Hamming graphs [9]. Chandran and Subramanian [3] have shown an inequality relating the vertex isoperimetric problem with the treewidth of a graph, which is very much like Inequality 2 in form, $\mathrm{cw}(G)$ being replaced by treewidth(G) and $b_e(x, G)$ being replaced by the corresponding term for the vertex isoperimetric problem. They derived a lower bound for the treewidth of a graph in terms of girth and average degree, making use of their inequality.

2. Definitions and notations

Let G be a simple unweighted finite undirected graph. Let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

Definition 1. Let $|V(G)| \geqslant 2$. Two subsets $A, B \subseteq V(G)$ cross if and only if $A \cap B$, $A - B$, $B - A$, $V(G) - (A \cup B)$ are all non-empty.

For a subset $A \subseteq V(G)$, let $\overline{A}$ denote its complement. That is $\overline{A} = V(G) - A$. The following observation is immediate from the above definition:

Lemma 1. *Two subsets $A, B \subseteq V(G)$ cross if and only if $\overline{A}, \overline{B}$ also cross.*

Definition 2. A carving in $V(G)$ (also called a carving decomposition of G) is a set $\mathscr{C}$ of subsets of $V(G)$ such that

1. $\emptyset, V(G) \notin \mathscr{C}$,
2. no two members of $\mathscr{C}$ cross,
3. $\mathscr{C}$ is maximal subject to the two conditions above.

The width of a carving $\mathscr{C}$ in $V(G)$ is defined to be width$(\mathscr{C}) = \max_{A\in\mathscr{C}} |\delta_G(A)|$. The carvingwidth of G is defined to be the minimum of width$(\mathscr{C})$, over all possible carvings $\mathscr{C}$ in $V(G)$.

Rooted trees and subtrees: Once we specify a certain vertex of a tree T as the root, then T can be seen a rooted tree. (See [4], Section 1.5 for the definition of a rooted tree.) Now, let T be a rooted tree with r as the root. Given any vertex $u \in V(T)$, we denote by T_u, the subtree of T, rooted at u. In particular, $T = T_r$.

Bijections: Let $\tau : A \to B$ be a bijection from the set A to B. Then for any subset $S \subseteq B$, $\tau^{-1}(S) = \{x \in A : \tau(x) \in S\}$.

Some special induced subgraphs of H_d: Recall that the vertices of H_d can be associated in one to one correspondence with the set of binary d-strings. Let s be a fixed binary string of length k, where $k \leqslant d$. Then the set of vertices corresponding to the binary d-strings with prefix s, clearly induces in H_d a graph isomorphic to H_{d-k}. We will denote this induced subgraph by H_d^s. For example, the induced subgraph on the set of vertices corresponding to the binary d-strings starting with 00 will be denoted by H_d^{00}. It is easy to see that H_d^{00} is isomorphic to H_{d-2}. Note that if s is the empty string, then H_d^s represents H_d itself. The length (i.e. the number of characters) of a binary string s will be denoted by $|s|$.

Edges between two disjoint sets: Let A, B be disjoint subsets of $V(G)$. Then $E(A, B) = \{(u, v) \in E(G) : u \in A \text{ and } v \in B\}$. (That is the set of edges with one end point in A and the other end point in B.)

The set of leaves of a tree: Let T be a tree. Then we denote by $L(T)$ the set of leaves (i.e. vertices of degree 1 or 0) of T. Also, let $\ell(T) = |L(T)|$.

3. Lower bound for cw(G)

The following lemma about carvings in $V(G)$ is proved in [13].

Lemma 2 (*Seymour and Thomas [13]*). *Let V be a finite set with $|V| > 2$, let T be a tree in which every vertex has degree* 1 *or* 3. *Let $\tau : V \to L(T)$ be a bijection. For each edge e of T, let $T_1(e)$ and $T_2(e)$ be the two components of $T \backslash e$; and let*

$$\mathscr{C} = \{\tau^{-1}(L(T_i(e))):\ e \in E(T), i = 1, 2\}.$$

Then $\mathscr{C}$ is a carving in V. Conversely, every carving in V arises from some tree T and bijection τ in this way.

Lemma 3. *Let T be a tree with at least three vertices and such that each of its vertices has degree either* 3 *or* 1. *For any edge e, let $T_1(e)$ and $T_2(e)$ be the components of $T \backslash e$. Then, for any integer x, such that $1 \leqslant x \leqslant n$ (where $n = \ell(T)$), there exists an edge $e \in E(T)$ such that either $x/2 \leqslant \ell(T_1(e)) \leqslant x$ or $x/2 \leqslant \ell(T_2(e)) \leqslant x$.*

Proof. Let v be an internal vertex (i.e., a vertex of degree 3) of T. For the purpose of this proof, consider T as a rooted tree, rooted at v. If $x = n$, then the lemma is trivial: since v has three children there exists a child z of v such that $\ell(T_z) < n/2$. So $\ell(T) - \ell(T_z) \geqslant n/2$ and we can infer that (v, z) is an edge of the required type. Now, assume that $1 \leqslant x < n$. Let $s_1, s_2, \ldots, s_k$ be a maximal sequence of vertices of T, satisfying the following three conditions: (1) $s_1 = v$; (2) s_{i+1} is a child of s_i in the rooted tree T; (3) $\ell(T_{s_i}) > x$ for all $i \leqslant k$.

First observe that $\ell(T_{s_1}) = \ell(T) = n > x$ and thus a sequence of the required type exists. Note that the last vertex of this sequence, s_k cannot be a leaf vertex, since in that case $\ell(T_{s_k}) = 1 \leqslant x$, and condition (3) is violated. Therefore, s_k is an internal vertex of T. Let u, w be the two children of s_k. Without loss of generality, let $\ell(T_u) \geqslant \ell(T_w)$. Then, clearly $\ell(T_u) > x/2$, since $\ell(T_{s_k}) > x$. Also, $\ell(T_u) \leqslant x$ since otherwise $s_1, s_2, \ldots, s_k$ cannot be maximal. We infer that (s_k, u) is an edge of the required type. $\square$

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices and let $1 \leqslant x \leqslant n$. Then $\mathrm{cw}(G) \geqslant \min_{x/2 \leqslant i \leqslant x} b_e(i, G)$.*

Proof. Let $\mathscr{C}$ be a carving in $V(G)$ that has the minimum width over all possible carvings in $V(G)$. By Lemma 2, there exists a tree T whose vertices are of degree 1 or 3, and a bijection $\tau : V(G) \to L(T)$ such that for each set $A \in \mathscr{C}$ there exists an edge $e \in E(T)$, where $A = \tau^{-1}(L(T_1(e)))$ or $A = \tau^{-1}(L(T_2(e)))$, where $T_1(e)$ and $T_2(e)$ are the components of $T \backslash e$. By Lemma 3, for any x, $1 \leqslant x \leqslant n$, there exists an integer s, $x/2 \leqslant s \leqslant x$ and an edge $e \in E(T)$ such that either $s = \ell(T_1(e))$ or $s = \ell(T_2(e))$. Without loss of generality, let $s = \ell(T_1(e))$ and let $X = \tau^{-1}(L(T_1(e)))$. Since $x/2 \leqslant |X| = s \leqslant x$, $|\delta_G(X)| \geqslant \min_{x/2 \leqslant i \leqslant x} b_e(i, G)$. It follows that $\mathrm{cw}(G) \geqslant \min_{x/2 \leqslant i \leqslant x} b_e(i, G)$. $\square$

Now, we use Theorem 1 to give a lower bound for the carvingwidth of H_d. The edge isoperimetric problem for the hypercube was studied and solved by Harper [7] and later independently by Hart [11]. Let m be an integer, where $1 \leqslant m \leqslant 2^d$. Let M be the subset of vertices of H_d, which correspond to the binary representations (assuming d bits per representation) of $\{0, 1, \ldots, m-1\}$. The following result was proved in [11,7]. (Also see the presentation by Bollobás [1], Chapter 16, proof of Theorem 2 therein.)

Lemma 4 (*Hart [11], Harper [7]*). $|\delta_{H_d}(M)| = b_e(m, H_d)$.

Lemma 5. $\mathrm{cw}(H_d) \geqslant 2^{d-1}$.

Proof. By Theorem 1, it is sufficient to show that there exists an integer x such that for every m, where $x/2 \leqslant m \leqslant x$, $b_e(m, H_d) \geqslant 2^{d-1}$. We take $x = 2^{d-1}$. Now let $m = 2^{d-2} + t$ where $0 \leqslant t \leqslant 2^{d-2}$. Let M be the subset of vertices of H_d which correspond to the binary representation of $\{0, 1, \ldots, m-1\}$. Clearly, $M = V(H_d^{00}) \cup X$,where $X \subseteq V(H_d^{01})$ and $|X| = t$. Since for any vertex $u \in V(H_d^0)$, there is a unique vertex $v \in V(H_d^1)$ such that $(u, v) \in E(H_d)$, we have $|E(M, V(H_d^1))| = |M| = 2^{d-2} + t$. Moreover, since for every vertex $u \in V(H_d^{01})$ there is a unique vertex $v \in V(H_d^{00})$ such that $(u, v) \in E(H_d)$, we have $|E(M, V(H_d^0) - M)| \geqslant |V(H_d^0) - M| = 2^{d-2} - t$. It follows that $|\delta(M)| \geqslant 2^{d-2} + t + 2^{d-2} - t = 2^{d-1}$, and therefore $\mathrm{cw}(H_d) \geqslant 2^{d-1}$ as required. $\square$

Remark. The proof of Lemma 5 can be shortened if we use Guu's characterisation [6] of the function $b_e(i, H_d)$.

4. Upper bound for cw(H_d)

To complete the proof, we will demonstrate a carving in $V(H_d)$ with width 2^{d-1}. For a binary string p with $|p| \leqslant d$, let $V_p = V(H_d^p)$. Let $\mathscr{S}$ be the set of all non-empty binary strings of length at most d. We define a family $\mathscr{F}$ of subsets of $V(H_d)$ as follows:

$$\mathscr{F} = \{V_p, \overline{V_p} : p \in \mathscr{S}\}.$$

Lemma 6. *$\mathscr{F}$ is a carving in $V(H_d)$.*

Proof. We need to verify that the three properties of Definition 2 are satisfied. Property 1 is trivial to verify. We verify Properties 2 and 3 below:

Property 2. Let $A, B \in \mathscr{F}$. We need to show that they do not cross.

Case 1: $A = V_p$ and $B = V_{p'}$. Without loss of generality let $|p| \leqslant |p'|$. If p is a prefix of p' then $V_p \subseteq V_{p'}$ and thus $A - B = \emptyset$. Else if p is not a prefix of p', then $A \cap B = \emptyset$.

Case 2: $A = \overline{V_p}$ and $B = \overline{V_{p'}}$. By Lemma 1, A and B cross if and only if $\overline{A}$ and $\overline{B}$ cross. But $\overline{A}$ and $\overline{B}$ do not cross by case 1.

Case 3: $A = V_p$ and $B = \overline{V_{p'}}$. If p is a prefix of p', then $A \subseteq \overline{B}$ and thus $A \cap B = \emptyset$. If p' is a prefix of p then $V(H_d) - (A \cup B) = \emptyset$. On the other hand, if neither p a prefix of p' nor p' a prefix of p, then a $A - B = \emptyset$.

Thus in all cases, A and B do not cross.

Property 3. Consider any subset $T \notin \mathscr{F}$ where $\emptyset \neq T \subset V(H_d)$. We will show that there exists a subset $S \in \mathscr{F}$ such that S crosses with T, thus proving that $\mathscr{F}$ is a maximal set with respect to Properties 1 and 2, as required. Let p be the longest string such that p is a common prefix for all the binary strings corresponding to the vertices in T. Clearly $|p| < d$, since $T \notin \mathscr{F}$. Let T_0 (respectively, T_1) be the set of vertices in T such that the binary digit at the $|p| + 1$th position of its corresponding string is 0 (respectively, 1). Without loss of generality let $|T_0| \leqslant |T_1|$. Also let "$p0$" (respectively, "$p1$") be the string obtained by concatenating the bit 0 (respectively, 1) at the end of the string p. The reader may not that $T_0 \neq \emptyset$: otherwise "$p1$" will be a prefix longer than p that is common to all the strings corresponding the vertices in T, contradicting the assumption. Moreover $T_0 \neq V_{p0}$: otherwise $T = T_0 \cup T_1 = V_{p0} \cup V_{p1} = V_p \in \mathscr{F}$ contradicting the assumption that $T \notin \mathscr{F}$. We consider the following cases:

Case 1: $T_1 \neq V_{p1}$. Then clearly $T_i \subset V_{pi}$ for $i = 0, 1$. In this case it is easy to verify that V_{p0} crosses with T: that is, all the four sets $V_{p0} \cap T$, $V_{p0} - T$, $T - V_{p0}$ and $V(H_d) - (V_{p0} \cup T)$ are non-empty.

Case 2: $T_1 = V_{p1}$ and $|p| \geqslant 1$. In this case also it can be easily checked that V_{p0} crosses with T.

Case 3: $T_1 = V_{p1}$ and $|p| = 0$. Unlike the previous two cases, in this case V_{p0} does not cross with T since $V(H_d) - (V_{p0} \cup T) = \emptyset$. So we need another candidate. Note that since $T_1 = V_{p1}$, $T' = \overline{T} \subset V_{p0}$. Let p' be the longest prefix which is common to all the strings in T'. Clearly $|p'| \geqslant 1$ since $T' \subset V_{p0}$. Thus by Cases 1 and 2, there exists a set $S' \in \mathscr{F}$ which crosses with T'. Now, by the definition of $\mathscr{F}$, $\overline{S'} \in \mathscr{F}$. Take $S = \overline{S'}$. Since S' and T' cross, by Lemma 1, $T = \overline{T'}$ and $S = \overline{S'}$ also should cross, as required. $\square$

Lemma 7. *The width of the carving* $\mathscr{F}$ *equals* 2^{d-1}.

Proof. Noting that $\delta_{H_d}(\overline{V_p}) = \delta_{H_d}(V_p)$, it is clear that the width of the carving $\mathscr{F}$ equals $\max_{p \in \mathscr{S}}(|\delta_{H_d}(V_p)|)$. Since H_d^p, the induced subgraph on V_p is isomorphic to a $(d - |p|)$-dimensional hypercube, $|\delta_{H_d}(V_p)| = |p|2^{d-|p|}$. This expression maximises when $|p| = 1$, and thus the width of the carving $\mathscr{F}$ equals 2^{d-1}. $\square$

Combining Lemmas 5 and 7, we infer the following:

Theorem 2. $\mathrm{cw}(H_d) = 2^{d-1}$. *The carving* $\mathscr{F}$ *constructed above is an optimum carving in* $V(H_d)$.

Acknowledgment

We thank both the anonymous referees for helpful comments.

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