Coquet-type formulas for the rarefied weighted Thue–Morse sequence∗

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A B S T R A C T

Newman proved for the classical Thue–Morse sequence, \((-1)^{(n)}\)_{n≥0}, that \(c_1N^k < \sum_{n=0}^{N-1} (-1)^{3n} < c_2N^k\) for all \(N \in \mathbb{N}\) with real constants \(\lambda, c_1, c_2\) satisfying \(c_2 > c_1 > 0\) and \(\lambda = \log 3/\log 4\). Coquet improved this result and deduced \(\sum_{n=0}^{N-1} (-1)^{3n} = N^\lambda F(\log_4 N) + o(N)\), where \(F(x)\) is a nowhere-differentiable, continuous function with period 1 and \(\eta(N) \in [-1, 0, 1]\). In this paper we obtain for the weighted version of the Thue–Morse sequence that for every \(r \in [0, 1, 2]\) if and only if the sequence of weights is eventually periodic. From the specific Coquet-type formulas we derive parts of the weak Newman-type results that were recently obtained by Larcher and Zellinger.

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1. Introduction

Given a nonnegative integer \(n\), let \(\{n_i : i \geq 0\}\) be the digits in the binary expansion of \(n\). Thus \(n = \sum_{i≥0} n_i 2^i\). Let \(s(n) = \sum_{i≥0} n_i\). We call \(s(n)\) the binary sum of digits of \(n\). Throughout the paper the set of positive integers is denoted by \(\mathbb{N}\) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Among the multiples of 3 there is a preponderance of values \(n\) such that \(s(n)\) is even over those such that \(s(n)\) is odd. Newman [22] obtained the following inequalities

\[
\frac{3^\alpha}{20} < N^{-\alpha} \sum_{n=0}^{N-1} (-1)^{3n} < 5 \cdot 3^\alpha \quad \text{with} \quad \alpha = \log_4(3).
\]

The sequence \((-1)^{(s(n)}\)_{n≥0} is well known as the Thue–Morse sequence. Sequences of the form \((-1)^{(s^j(pn+j)}\)_{n≥0} with integers \(p \geq 2\) and \(j \in \{0, 1, \ldots, p-1\}\) are often called rarefied Thue–Morse sequences. Several asymptotic properties of the sum \(\sum_{n=0}^{N-1} (-1)^{(s^j(pn+j)}\) for different values of \(p\) and \(j\) were investigated for example in [3–5,11,12,23,24].

Coquet [1] generalized the result of Newman by the following:

**Theorem 1** (Coquet).

\[
\sum_{n=0}^{N-1} (-1)^{3n} = N^\alpha F(\log_4 N) + \frac{\eta(N)}{3},
\]

where \(F\) is a continuous, nowhere-differentiable function with period 1 and where \(\eta(N) \in [-1, 0, 1]\).

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Furthermore, he studied the maximum and the minimum of $F$ and improved and sharpened the constants in Newman’s inequalities.

Existence of such Coquet-type formulas and single precise results were already derived for more general rarefied Thue–Morse sequences, see for example [10–12]. An extended overview of the investigation carried out for the rarefied Thue–Morse sequences can be found in [9].

The formula of Theorem 1 based on a continuous, nowhere-differentiable periodic function strongly reminds one of the formula for the average-growth behavior of the sum of digits obtained by Delange [2]:

**Theorem 2** (Delange).

\[
\frac{1}{N} \sum_{n=0}^{N-1} s(n) = \frac{1}{2} \log_2(N) + F(\log_2(N)),
\]

where $F$ is a continuous, nowhere-differentiable function with period 1.

Similar formulas for the average-growth behavior of various generalized sum of digits can be found e.g. in [6–8, 18–20].

Here we mention a result obtained by Larcher and Pillichshammer [20] on the weighted version of the sum of digits that motivated the investigations carried out in this paper. The properties of the weighted version of the sum of digits play an important role in the investigation of certain digital point sequences (see e.g. [14–16, 21]). Let $\gamma$ be a sequence in $\mathbb{R}$. We define $s_{\gamma}(n)$, the weighted binary sum of digits relative to $\gamma$, by

\[
s_{\gamma}(n) = \gamma_0 n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots,
\]

where the $n_i$ are given by the base 2 expansion of $n = n_0 + n_1 2 + n_2 2^2 + \cdots$.

**Theorem 3** (Larcher and Pillichshammer). There exists a continuous function $G_\gamma$ with period 1 such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} s_{\gamma}(n) = \frac{1}{2} \left( \gamma_{[\log_2(N)]} \cdot [\log_2(N)] + \sum_{i=0}^{[\log_2(N)]-1} \gamma_i \right) + G_\gamma(\log_2(N)) + o(1)
\]

if and only if the sequence $\gamma$ converges. (Here and later on $[x]$ denotes the integer part and $\{x\}$ the fractional part of a real $x$.)

In [13] all weight sequences which lead to a formula as in the theorem above but based on a function with period $k \in \mathbb{N}$ are classified. Existence of Delange-type formulas for further weighted generalized versions is investigated in [17]. Larcher and Zellinger [21] recently investigated Newman’s phenomenon for the weighted version of the Thue–Morse sequence and classified for every $r \in \{0, 1, 2\}$ all weight sequences for which $\sum_{n=0}^{N-1} (-1)^{n \gamma_r(3n+r)}$ is either positive or negative for almost all $N \in \mathbb{N}$.

In this paper we aim for a generalization of Theorem 1. More exactly we search for conditions on the weight sequences such that a Coquet-type formula for the weighted binary sum of digits can exist.

**Definition 4.** Let $\gamma$ be a sequence in $[0, 1]$. We set

\[
\Pi_l(N, \gamma) = \sum_{n=0}^{N-1} (-1)^{\gamma(n)}.\]

We say that $\Pi_l(N, \gamma)$ is Coquet-type if there exists a continuous and periodic function $F_r : \mathbb{R} \to \mathbb{R}$, with period $l$, a bounded function $\phi_r : \mathbb{N} \to \mathbb{R}$ and a positive real constant $\delta$ such that

\[
\Pi_l(N, \gamma) = \frac{\phi_r(N)}{3} + N^\delta F_r(\log_4(N)). \tag{1}
\]

Throughout this paper we often call a formula in the style of (1) a Coquet-type formula.

It turns out that the crucial property of the weight sequence is whether it is periodic or not.

**Definition 5.** A sequence $(a_i)_{i \geq 0}$ is periodic if there exists a nonnegative integer $t$ such that $a_i = a_{i+t}$ for all $i \geq 0$. We call the number $t$ the length of the period and the repeated word $(a_0, a_1, \ldots, a_{t-1})$ the period. A sequence $(a_i)_{i \geq 0}$ is eventually periodic if there exists an index $i_0$ such that $(a_{i+i_0})_{i \geq 0}$ is periodic. Especially, if the period has length 1, then we say that the sequence eventually has a constant value $a_0$.

Before we give an overview of the rest of the paper we would like to state our main result.

**Theorem 6.** Let $\gamma$ be a sequence in $[0, 1]$. There exists a Coquet-type formula for $\Pi_l(N, \gamma)$ for every $r \in \{0, 1, 2\}$ if and only if the sequence of weights is eventually periodic. Furthermore, if the period contains at least one nonzero entry, then the functions $F_r$ are nowhere-differentiable.
The rest of the paper is organized as follows. In Section 2.1 we give a general formula for \( \Pi_r(N, \gamma) \). From this formula, in Section 2.2 we derive a Coquet-type formula under the assumption that the weight sequence is eventually periodic. In Section 2.3 we prove that the continuous and periodic functions that appear in Section 2.2 are nowhere-differentiable, except for the trivial case where there are just finitely many nonzero weights. In Section 2.4 we take a closer look at the Coquet-type formulas that are achieved for the weight sequences which eventually have the constant value 1. From these formulas we derive parts of the weak Newman-type results of Larcher and Zellinger [21]. Finally, in Section 2.5 we ask for a necessary condition on the weight sequence in the case of Coquet-type \( \Pi_r(N, \gamma) \) and obtain that the sequence of weights must be eventually periodic which finally concludes the proof of Theorem 6.

Throughout this paper we write \( N = \epsilon_0 + \epsilon_1 4 + \cdots + \epsilon_4 4^k \), with \( \epsilon_i \in \{0, 1, 2, 3\} \) and \( \epsilon_k \neq 0 \), for the base 4 expansion of \( N \). Furthermore we set \( \xi = e^{2\pi i/3} \) and \( g(n) = (-1)^{\gamma(n)} \xi^n \); note that the latter is a 4-multiplicative function, i.e., \( g(0) = 1 \) and \( g(n) = \prod_{p \geq 0} g(p \sigma_p n) \). We write \( \Re(z) \) for the real part of the complex number \( z \) and \( \Im(z) \) for the imaginary part.

## 2. Results

### 2.1. A general formula

**Theorem 7.** Let \( \gamma \) be a sequence in \( \{0, 1\} \) and \( r \) in \( \{0, 1, 2\} \). We have

\[
\Pi_r(N, \gamma) = \frac{2}{3} \Re(\xi^{-r} H(N)) + \frac{\rho(N)}{3},
\]

where \( \rho(N) \) and \( H(N) \) are defined by \( \rho(N) = \sum_{n=0}^{N-1} (-1)^{\gamma(n)} \) and

\[
H(N) = \sum_{0 \leq p \leq k} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \psi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1})
\]

with \( \psi(0) = 0 \), \( \psi(4^p) = 1 \), \( \psi(2 \cdot 4^p) = (1 + (-1)^{2p} \xi) \), \( \psi(3 \cdot 4^p) = 1 + (-1)^{2p+1} \xi^2 \), \( \kappa(0, 0) = 1 \), \( \kappa(1, 0) = -\sqrt[3]{-1} \), \( \kappa(0, 1) = \sqrt[3]{-1} \), and \( \kappa(1, 1) = 3 \).

**Proof.** We observe for any nonnegative integer \( m \) that

\[
\frac{1}{3} \sum_{i=0}^{2} \xi_{m+3i} = \begin{cases}
1 & \text{if } 3|m \\
0 & \text{otherwise},
\end{cases}
\]

\[
\Re(\xi^m) = \Re(\xi^{2m}),
\]

\[
\Im(\xi^m) = -\Im(\xi^{2m}),
\]

and obtain the following chain of equalities:

\[
\Pi_r(N, \gamma) = \sum_{n=0}^{N-1} (-1)^{\gamma(n)}
\]

\[
= \sum_{n=0}^{N-1} (-1)^{\gamma(n)} \sum_{l=0}^{2 \lfloor \log_4 n \rfloor} \xi^{(n-r)l}
\]

\[
= \frac{1}{3} \sum_{n=0}^{N-1} (-1)^{\gamma(n)} + \frac{2}{3} \Re(\xi^{-r} \sum_{n=0}^{N-1} (-1)^{\gamma(n)} \xi^n)
\]

\[
= \frac{1}{3} \sum_{n=0}^{N-1} (-1)^{\gamma(n)} + \frac{2}{3} \Re(\xi^{-r} H(N)),
\]

where

\[
H(N) = \sum_{n=0}^{N-1} (-1)^{\gamma(n)} \xi^n = \sum_{n=0}^{N-1} g(n).
\]

Since \( g(n) \) is a 4-multiplicative function and \( N = \epsilon_0 + \epsilon_1 4 + \cdots + \epsilon_4 4^k \) it is easy to verify the following equality:

\[
\sum_{n=0}^{N-1} g(n) = \sum_{l=0}^{k} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \left( \sum_{i=0}^{p-1} g(id^p) \right) \sum_{m=0}^{d^p-1} g(m).
\]
We set \( \varphi(\epsilon_{p}A^{p}) = \sum_{i=0}^{\epsilon_{p}-1} g(iA^{p}) \) and get
\[
\begin{align*}
\varphi(0) &= 0, \\
\varphi(4^{p}) &= g(0) = 1, \\
\varphi(2 \cdot 4^{p}) &= g(0) + g(1 \cdot 4^{p}) = 1 + (-1)^{2p} \xi, \\
\varphi(3 \cdot 4^{p}) &= g(0) + g(1 \cdot 4^{p}) + g(2 \cdot 4^{p}) = 1 + (-1)^{2p} \xi + (-1)^{2p+1} \xi^{2}.
\end{align*}
\]

We regard the sum \( \sum_{m=0}^{4^{p}-1} g(m) \) and obtain
\[
\begin{align*}
\sum_{m=0}^{4^{p}-1} g(m) &= \prod_{i=0}^{p-1} (g(0 \cdot 4^{i}) + g(1 \cdot 4^{i}) + g(2 \cdot 4^{i}) + g(3 \cdot 4^{i})) \\
&= \prod_{i=0}^{p-1} (1 + (-1)^{2i} \xi + (-1)^{2i+1} \xi^{2} + (-1)^{2i+1} \xi^{2}) \\
&= : \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1}).
\end{align*}
\]

Short verification of the equalities \( \kappa(0, 0) = 1, \kappa(1, 0) = -\sqrt{3}, \kappa(0, 1) = \sqrt{3}, \kappa(1, 1) = 3 \) completes the proof. \( \square \)

### 2.2. Coquet-type results

Let us briefly consider that the weight sequence eventually has the constant value 0. We distinguish between the case where all weights are 0, i.e., \( \gamma = (0)_{i \geq 0} =: \mathbf{0} \), and the case where there exists at least one nonzero weight, say \( \gamma_{l_{0}} = 1 \) but \( \gamma_{l_{0}+1} = 0 \). The former leads to
\[
\Pi_{x}(N, \mathbf{0}) = \frac{N}{3} + \frac{\eta_{x}(N)}{3}
\]

with \( \eta_{x}(N) \in \{0, \pm 1, \pm 2\} \) and the latter case yields
\[
\Pi_{x}(N, \gamma) = \chi_{x}(N)
\]

with \( \chi_{x} : \mathbb{N}_{0} \to \{0, \pm 1, \ldots, \pm 2^{l_{0}}\} \). Since the constant functions \( F_{x}(x) = 1/3 \) and \( F_{x}(x) = 0 \) are both periodic and continuous we know \( \Pi_{x}(N, \gamma) \) are Coquet-type in these cases.

In the following we restrict to weight sequences that contain infinitely many 1s. Additionally we assume that the sequence of weights is eventually periodic. Note that the period contains at least one nonzero entry, since there are infinitely many 1s in the weight sequence.

Based on the specific structure of the period we define the following parameter.

**Definition 8.** We define \( l \) as the minimum of all integers \( k \) that satisfy \( \gamma_{l} = \gamma_{2k+1} \) for all \( j \) large enough and where the product \( \prod_{i=0}^{k-1} \kappa(\gamma_{2(rk+i)} \gamma_{2(rk+i)+1}) \) is a positive real constant, denoted by \( \Lambda \), for all \( r \) large enough with \( \kappa \) defined in the formulation of Theorem 7.

One can easily see that proper \( l \) and \( \Lambda \) can be found for every (eventually) periodic sequence of weights. It will turn out that \( l \) is related to the period of the functions \( F_{x} : \mathbb{R} \to \mathbb{R} \) and \( \Lambda \) is related to the exponent \( \delta \).

**Example 9.**
- For \( \gamma = (1, 1, 1, \ldots) \) we get \( l = 1 \) and \( \Lambda = 3 \).
- For \( \gamma = (1, 1, 0, 0, 1, 1, 0, 0, \ldots) \) we get \( l = 2 \) and \( \Lambda = 3 \).
- For \( \gamma = (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, \ldots) \) we get \( l = 4 \) and \( \Lambda = 3^{2} \).
- For \( \gamma = (0, 1, 0, 1, 0, 1, 0, 1, \ldots) \) we get \( l = 4 \) and \( \Lambda = 3^{2} \).
- For \( \gamma = (1, 0, 0, 1, 1, 0, 0, 1, \ldots) \) we get \( l = 2 \) and \( \Lambda = 3 \).
- For \( \gamma = (0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, \ldots) \) we get \( l = 12 \) and \( \Lambda = 3^{6} \).
- For \( \gamma = (1, 0, 0, 0, 1, 0, 0, 0, \ldots) \) we get \( l = 8 \) and \( \Lambda = 3^{2} \).

In the following we investigate \( H(N) \) given in (2) under our assumption on the weight sequence. In a first step we straighten out a possible previous period. In the case where a previous period occurs in the sequence of weights we define \( \overline{\gamma} \) as the sequence in \( \{0, 1\} \) which is periodic and fulfills \( \overline{\gamma}_{i} = \gamma_{i} \) for all \( i \) large enough (in fact we change the entries in the
previous period such that we get a periodic sequence). We set $p' = \max\{i \in \mathbb{N}_0 : (y_{2i}, y_{2i+1}) \neq (\bar{y}_{2i}, \bar{y}_{2i+1})\}$ and observe for $k > p'$ (i.e., $N \geq 4^{p'+1}$) that

$$H(N) = \sum_{0 \leq p \leq p'} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \varphi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1})$$

$$= \sum_{0 \leq p \leq p'} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \varphi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1})$$

$$+ \prod_{k=0}^{p'} \kappa(\gamma_{2k}, \gamma_{2k+1}) \sum_{p' < p \leq k} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \varphi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1})$$

where $\bar{g}$ and $\bar{\varphi}$ are defined as $g$ and $\varphi$ but for the sequence $\bar{\gamma}$. Finally we arrive at the equality

$$H(N) = \Delta_{\gamma}(N) + c(\gamma)\overline{H}(N),$$

where $c(\gamma)$, given by

$$c(\gamma) = \prod_{k=0}^{p'} \kappa(\gamma_{2k}, \gamma_{2k+1}) \bar{\kappa}(\gamma_{2k}, \gamma_{2k+1}),$$

is a complex valued constant depending on $\gamma$. $\overline{H}(N)$ is defined as $H(N)$ but for the sequence $\bar{\gamma}$ and $\Delta_{\gamma}(N)$ is defined by

$$\Delta_{\gamma}(N) = \sum_{0 \leq p \leq p'} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \varphi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1}) - c(\gamma) \sum_{0 \leq p \leq p'} \left( \prod_{j=p+1}^{k} g(\epsilon_j 4^j) \right) \varphi(\epsilon_p 4^p) \prod_{i=0}^{p-1} \kappa(\gamma_{2i}, \gamma_{2i+1}).$$

We set

$$\phi(N) = \sum_{n=0}^{N-1} (-1)^{\gamma(n)} + 2\Re \left( \xi^{-N} \Delta_{\gamma}(N) \right).$$

(3)

From the fact that the absolute value of $g(n)$ is 1 it is easily deduced that $\Delta_{\gamma}(N)$ is bounded. This together with the fact that $\left| \sum_{n=0}^{N-1} (-1)^{\gamma(n)} \right|$ is bounded by $4^{p'+1}$ yields $\phi(N) = O(1)$.

Hence it remains to obtain a formula in the shape of

$$\overline{H}(N) = N^d G_{\bar{\gamma}}(\log_4(N))$$

with a positive real constant $\delta$ and a continuous and periodic function $G_{\bar{\gamma}} : \mathbb{R} \to \mathbb{C}$ in order to arrive at a Coquet-type formula in the sense of Definition 4.

In a first step we determine a proper $\delta$. We use the value of $\overline{H}(4^lm)$, i.e.,

$$\overline{H}(4^lm) = A^m = A^{\log_4(N)} = N^{\log A/\log 4},$$

and set

$$\delta = \log A/\log 4.$$

Since we have assumed that there are nonzero weights in the period we know that $A$ is greater than 1 and that $\delta$ is positive.

The following example lists the values of $\delta$ for the weight sequences considered in Example 9. One can see that $\delta$ depends on the density of 1s in the period.

**Example 10.**

- For $\bar{\gamma} = (1, 1, 1, 1, \ldots)$ we get $\delta = \log 3/\log 4$.
- For $\bar{\gamma} = (1, 1, 0, 0, 1, 1, 1, \ldots)$, for $\bar{\gamma} = (1, 1, 0, 0, 0, 0, 1, 1, 1, \ldots)$, for $\bar{\gamma} = (0, 1, 0, 1, \ldots)$, for $\bar{\gamma} = (1, 0, 1, 1, 0, 1, 0, 1, 1, 1, \ldots)$ and for $\bar{\gamma} = (0, 0, 1, 1, 0, 0, 1, 1, 1, \ldots)$ we get $\delta = \log 3/\log 16$.
- For $\bar{\gamma} = (1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, \ldots)$ we get $\delta = \log 3/\log 256$.

We use the uniquely determined integer $m$ such that $m \leq \log N/\log 4^l < (m + 1)$ (note that $m = \lfloor \log N/\log 4^l \rfloor$) and observe the chain of equalities $N^\delta = N^{\log A/\log 4^l} = A^{\log N/\log 4^l} = A^m A^{\log N/\log 4^l}$. Furthermore we extend the base 4 expansion on $N$ by setting $\epsilon_i = 0$ for all $i > \log N/\log 4$. From the definition of $\overline{H}(N)$ we conclude

$$\overline{H}(N)/N^\delta = A^{-(\log N/\log 4^l)} \sum_{q=0}^{m} \sum_{i=0}^{l(m+1)-1} \prod_{j=q4^l+i+1}^{q4^l+i+1} g(\epsilon_j 4^j) \varphi(\epsilon_q 4^q) \prod_{i=0}^{s-1} \kappa(\gamma_{2i}, \gamma_{2i+1}) A^{\delta-m}.$$
We relate each natural number \( N \) to the 4-adic rational \( \vartheta(N) \), defined by \( \vartheta(N) = \frac{N}{4^m} \), and regard its base 4 expansion:

\[
\vartheta(N) = (\epsilon_{m+1}, \ldots, \epsilon_m, \epsilon_{m-1}, \ldots, \epsilon_{(m-1)l}, \ldots, \epsilon_{1}, \ldots, \epsilon_0)_4.
\]

We rewrite this expansion,

\[
\vartheta(N) = (e_{-1}, \ldots, e_0, e_{-1}, \ldots, e_{-2}, \ldots, e_{-(m-1)}, \ldots, e_{-lm00}, \ldots)_4,
\]

and define the two functions \( \bar{g} \) and \( \bar{\psi} \) by the following table

<table>
<thead>
<tr>
<th>( e_2 )</th>
<th>( \bar{g}<em>{\vartheta</em>{2r}, \vartheta_{2r+1}}(e_2) )</th>
<th>( \bar{\psi}<em>{\vartheta</em>{2r}, \vartheta_{2r+1}}(e_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>((-1)^{\vartheta_2} \xi)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>((-1)^{\vartheta_{2r+1}} \xi^2) + 1</td>
<td>((-1)^{\vartheta_2} \xi)</td>
</tr>
</tbody>
</table>
| 3 | \((-1)^{\vartheta_{2r+1}} \xi + (-1)^{\vartheta_2} \xi^2\) | \((-1)^{\vartheta_{2r+1}} \xi + (-1)^{\vartheta_2} \xi^2\),

where \( 0 \leq r < l \) is determined by the residue of \( z \) modulo \( l \). Now we rewrite the formula from above,

\[
\frac{H(N)}{N^\delta} = \Lambda \left\lfloor \log N \right\rfloor_{\log 4} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \left( \prod_{j=-q}^{-s} \bar{g}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_2) \right) \bar{\psi}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_{-q+j}) \sum_{i=0}^{s-1} \kappa(\vartheta_{2i}, \vartheta_{2i+1}) \Lambda^{-q},
\]

and define a function \( G_\vartheta : \mathbb{R} \rightarrow \mathbb{C} \) by

\[
G_\vartheta(x) = \frac{1}{\left\lfloor x/\log 4 \right\rfloor} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \left( \prod_{j=-q}^{-s} \bar{g}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_2) \right) \bar{\psi}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_{-q+j}) \sum_{i=0}^{s-1} \kappa(\vartheta_{2i}, \vartheta_{2i+1}) \Lambda^{-q},
\]

where the \( e_2 \) are determined by the digits of the base 4 expansion of \( 4^{|x/\log 4|} \), i.e., \( 4^{|x/\log 4|} = (e_{-1}, \ldots, e_0, e_{-1}, \ldots)_4 \).

Note that \( G_\vartheta(x) \) has period \( l \), since \( \left\lfloor x/\log 4 \right\rfloor \) has period \( l \). Furthermore, \( \left| \bar{g}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_2) \right| = 1 \), \( \left| \bar{\psi}_{\vartheta_{2r}, \vartheta_{2r+1}}(e_2) \right| \leq 2 \), \( |\prod_{i=0}^{s-1} \kappa(\vartheta_{2i}, \vartheta_{2i+1})| \leq \Lambda \) and \( \Lambda > 1 \) and therefore the series in the definition of \( G_\vartheta(x) \) is absolutely convergent. Finally, it is easy to verify the equality

\[
\frac{H(N)}{N^\delta} = G_\vartheta(\log_4(N))
\]

for all \( N \in \mathbb{N} \).

Summing up, it remains to prove the continuity of \( G_\vartheta(x) \) to arrive at the following theorem.

**Theorem 11.** If the sequence of weights is eventually periodic then there exists a Coquet-type formula for \( \Pi_r(N, \gamma) \) for every \( r \in \{0, 1, 2\} \) in the sense of **Definition 4.**

The main difficulty in the proof of continuity is to get information about the local change of a function without using too much information about the period of \( \vartheta \). The following lemma describes the structure of the period that is considered when setting the parameter \( \Lambda \).

**Lemma 12.** The period in the weight sequence that is used to determine the parameter \( \Lambda \) in **Definition 8**, \( \vartheta_0, \vartheta_1, \ldots, \vartheta_{2l-1} \), can be built by combining the following words consisting of pairs and applying finitely many operations which exchange the order of the pairs but not the inner order of the pairs:

\[
(11), (00), (1001), (1010101), (01010101).
\]

For example the period \( 0110 \) can be built by \( (1001) \) but not by \( (11) \) and \( (00) \).

**Proof.** From **Definition 8** we know that \( l \) is the minimal positive value of \( k \) such that \( \prod_{i=0}^{k-1} \kappa(\vartheta_{2i}, \vartheta_{2i+1}) \) and \( \vartheta_0, \vartheta_1, \ldots, \vartheta_{2k-1} \) is a period of the sequence \( \vartheta \). Now if we regard the value of \( \kappa \) for the 4 different pairs, i.e., \( \kappa(1, 1) = 3, \kappa(1, 0) = -i \sqrt{3}, \kappa(0, 1) = i \sqrt{3} \) and \( \kappa(0, 0) = 1 \), it is clear that both \( (00) \) and \( (11) \) can be used to build up the period \( \vartheta_0, \vartheta_1, \ldots, \vartheta_{2l-1} \). But once a \( (10) \) pair occurs we need either one \( (01) \) pair or three further \( (10) \) pairs to get a real and positive value. Analogously, if once a \( (01) \) pair occurs we need either one \( (10) \) pair or three further \( (01) \) pairs. \( \square \)

Now we can take the period \( \vartheta_0, \vartheta_1, \ldots, \vartheta_{2l-1} \) as a finite combination of pairs that can be rearranged to a combination of words out of the set

\[
\{(11), (00), (1001), (1010101), (01010101)\}.
\]

We observe several lemmas concerning the following two magnitudes.
**Definition 13.** Let $\overline{\gamma}_0, \overline{\gamma}_1, \ldots, \overline{\gamma}_{2l-1}$ be the period of the sequence of weights that lead to the parameter $\Lambda$. We define $f, h : \{0, 1, 2, 3\}^l \rightarrow \mathbb{C}$ by

$$f(e_{j-1}, \ldots, e_1, e_0) = \sum_{s=0}^{l-1} \prod_{r=s+1}^{l-1} \tilde{g}_{\overline{\gamma}_r, \overline{\gamma}_{r+1}}(e_r) \tilde{g}_{\overline{\gamma}_{2r}, \overline{\gamma}_{2r+1}}(e_s) \prod_{i=0}^{l-1} k(\overline{\gamma}_i, \overline{\gamma}_{i+1}),$$

$$h(e_{j-1}, \ldots, e_1, e_0) = \prod_{r=0}^{l-1} \tilde{g}_{\overline{\gamma}_r, \overline{\gamma}_{r+1}}(e_r).$$

**Lemma 14.** We have

$$h(3, 3, \ldots, 3) = 1.$$

**Proof.** By Lemma 12 and the definition of $h(e_{j-1}, e_{j-2}, \ldots, e_1, e_0)$ it suffices to prove this relation for each of the five different words of pairs, which can be done by short and easy computations. $\square$

**Lemma 15.** We have

$$f(3, 3, \ldots, 3) = \Lambda - 1.$$

**Proof.** We regard the definition of $f(e_{j-1}, e_{j-2}, \ldots, e_1, e_0)$ and the definition of $\overline{H}(N)$ and get

$$f(e_{j-1}, e_{j-2}, \ldots, e_1, e_0) = \overline{H}(e_{j-1}4^{l-1} + \cdots + e_14 + e_0).$$

This together with Lemma 14 implies

$$f(3, 3, \ldots, 3) = H(4^l - 1) = H(4) - \overline{H}(4^l - 1) = \Lambda - 1. \quad \square$$

**Lemma 16.** We define $d := \min\{d \in \{0, 1, \ldots, l - 1\} : e_d \neq 0\}$. Then we have

$$f(e_{j-1}, \ldots, e_d, 0, \ldots, 0) = f(e_{j-1}, \ldots, (e_d - 1, 3, \ldots, 3) + h(e_{j-1}, \ldots, (e_d - 1, 3, \ldots, 3).$$

**Proof.** We again use the definition of $f(e_{j-1}, e_{j-2}, \ldots, e_1, e_0)$ and the definition of $\overline{H}(N)$ and get

$$f(e_{j-1}, \ldots, e_d, 0, \ldots, 0) = \overline{H}(e_{j-1}4^{l-1} + \cdots + e_d4^d)$$

$$= \overline{H}(e_{j-1}4^{l-1} + \cdots + e_d4^d - 1) + \overline{g}(e_{j-1}4^{l-1} + \cdots + e_d4^d - 1)$$

$$= f(e_{j-1}, \ldots, (e_d - 1, 3, \ldots, 3) + h(e_{j-1}, \ldots, (e_d - 1, 3, \ldots, 3). \quad \square$$

Using Lemmas 14 and 15 it is not so hard to check the equality of the limits $\lim_{x \to 0^+} G_{\overline{\gamma}}(x)$ and $\lim_{x \to l^-} G_{\overline{\gamma}}(x)$:

$$\lim_{x \to 0^+} G_{\overline{\gamma}}(x) = \tilde{g}_{\overline{\gamma}_0, \overline{\gamma}_1}(1) = 1,$$

$$\lim_{x \to l^-} G_{\overline{\gamma}}(x) = \Lambda^{-1}(\Lambda - 1) \sum_{q=0}^{\infty} \Lambda^{-q} = \frac{\Lambda - 1}{\Lambda} \frac{1}{1 - 1/\Lambda} = 1.$$

Therefore $G_{\overline{\gamma}}(x)$ is continuous at $x = kl$ for every integer $k$.

It remains to prove continuity at any point in the interval $(0, l)$. Since $\Lambda^{-[x/l]}$ is continuous in $(0, l)$, it suffices to prove continuity of $\Psi(4^{[x/l]})$, satisfying

$$\Psi(4^{[x/l]}) = G_{\overline{\gamma}}(x) \cdot \Lambda^{[x/l]}$$

$$= \sum_{q=0}^{\infty} \left( \prod_{j=0}^{q-1} h(e_{j+l+1}, \ldots, e_{j+l}) \right) f(e_{q+l+1}, \ldots, e_{q+l}) \Lambda^{-q},$$

where the $e_i$ are given by the base 4 expansion of $4^{[x/l]}$, i.e., $(e_l, \ldots, e_1e_0, e_1e_2, \ldots, 4)$. We observe that

$$\left| \prod_{j=0}^{q-1} h(e_{j+l+1}, \ldots, e_{j+l}) \right| = 1, \ \text{if} \ |f(e_{l-1}, \ldots, e_0)| \leq 2l\Lambda =: M, \ \text{choose} \ y \ \text{such that} \ 0 \leq x < y < l \ \text{and} \ 1 \leq 4^{[x/l]} < 4^{[y/l]} \leq 4^{[l]} + 4^{-u} \leq 4$$

and distinguish between the following two cases:

- Let $4^{[x/l]}$ and $4^{[y/l]}$ belong to the same interval of the form $[a4^{-b_l}(a + 1)4^{-b_l}]$:

  It is clear that $|\Psi(4^{[x/l]}) - \Psi(4^{[y/l]}|) \leq M \sum_{q>u} \Lambda^{-q} = M \Lambda^{-u}$ and $|\Psi(4^{[y/l]}) - \Psi(4^{[x/l]})| \leq M \Lambda^{-u}$. Hence $|\Psi(4^{[x/l]}) - \Psi(4^{[y/l]}|) \leq 2M \Lambda^{-u}$.
Let $4^{[x]}$ and $4^{[y]}$ belong to neighboring intervals, i.e., $(a - 1)4^{[-1]} < 4^{[x]} < a4^{[-1]} < (a + 1)4^{[-1]}$. The inequality $|\Psi (4^{[y]}) - \Psi (4^{[x]})| \leq \frac{M}{a - 1} \Lambda^{-u}$ still holds. It remains to prove that $|\Psi (a4^{[-1]} - \Psi (4^{[x]})| \leq c \Lambda^{-u}$ for a positive constant $c$ not depending on $u$.

We regard the base 4 expansion of $a4^{[-1]}$, $(a_{l-1}, \ldots, a_0, a_{-1}, \ldots, a_{-l})_4$, set

$$\min \{i \in \{-lu, \ldots, l - 1\} : a_i \neq 0\} = z = -cl + d$$

with $c, d \in \mathbb{Z}$ such that $d \in \{0, 1, \ldots, l - 1\}$ and get

$$a4^{[-1]} = (a_{l-1}, \ldots, a_0, a_{-1}, \ldots, a_{-cl+d}0, \ldots, 0)_4,$$

$$(a - 1)4^{[-1]} = (a_{l-1}, \ldots, a_0, a_{-1}, \ldots, (a_{-cl+d} - 1)3, \ldots, 3) \in \{lu \text{ digits}\}$$

$$4^{[-1]} = (a_{l-1}, \ldots, a_0, a_{-1}, \ldots, (a_{-cl+d} - 1)3, \ldots, 3e_{-l(u+1)+l-1}, \ldots)_4.$$  
We define $E = \prod_{k=-l(u+1)+l-1}^{l-1} \tilde{g}_{\tau_2, \tau_{2u+1}}(a_k)$ and compute

$$|\Psi (a4^{[-1]} - \Psi (4^{[x]})| = |\Psi (a4^{[-1]} - \Psi ((a - 1)4^{[-1]} + \Psi ((a - 1)4^{[-1]} - \Psi (4^{[x]})|$$

$$\leq (\Psi (a4^{[-1]} - \Psi ((a - 1)4^{[-1]} | + \frac{M}{A - 1} \Lambda^{-u}.$$ 

Application of Lemmas 15 and 16 yields

$$\Psi (a4^{[-1]} - \Psi (4^{[x]})| \leq c \Lambda^{-u}$$

with a positive constant $c$ not depending on $u$.

2.3. Nowhere-differentiability of the periodic functions

In this section we study the functions $F_r(x) = \Re (\xi^{-c} \psi (\gamma) G_r(x))$ with $r \in \{0, 1, 2\}$, where $c(\gamma), \psi$ and $G_r : \mathbb{R} \to \mathbb{C}$ are defined as in the last section.

**Theorem 17.** Let $\gamma$ be a sequence in $\{0, 1\}$ that is eventually periodic with a period that contains at least one nonzero entry. For every $r \in \{0, 1, 2\}$ the continuous and periodic function $F_r : \mathbb{R} \to \mathbb{C}$ determined by $F_r(x) = \Re (\xi^{-c} \psi (\gamma) G_r(x))$ is nowhere differentiable.

Our method of proof uses the explicit formula of $G_r(x)$. Tenenbaum [25] gave an alternative proof of the nowhere-differentiability of the function that appears in Theorem 1. His proof uses the equation in Theorem 1 to deduce that the differential quotient $\lim_{h \to 0} \frac{|F(x + h) - F(x)|}{h}$ is not even bounded. Of course his method of proof could also be applied to the assertion in the theorem above.

**Proof.** We return to $\psi$, which was introduced for the proof of continuity in the last section and show that it is nowhere-differentiable. First we prove nowhere-differentiability in $y = 1$, which implies $G_r(x)$ is not differentiable in $x = kl$.

We set $y_k = (1, 0, \ldots, 01)_4$ and $y'_k = (1, 0, \ldots, 02)_4$ and get the following two limits, which do not exist:

$$\lim_{k \to \infty} \frac{\psi (y_k) - \psi (1)}{y_k - 1} = \lim_{k \to \infty} \frac{\tilde{g}_{\tau_0, \tau_1} (1) A^{-k}}{4^{kl}}$$

$$\lim_{k \to \infty} \frac{\psi (y'_k) - \psi (1)}{y'_k - 1} = \lim_{k \to \infty} \frac{\tilde{g}_{\tau_0, \tau_1} (1) \tilde{g}_{\tau_0, \tau_1} (2) A^{-k}}{2 \cdot 4^{kl}}$$

$$= \lim_{k \to \infty} \frac{1}{2} \left(-1\right)^{\psi_0 x} \left(1 + (-1)^{\psi_0 x} \left(\frac{4^l}{A}\right)^k\right).$$
These limits together with the fact that $\Lambda \leq 3^l < 4^l$ yield nowhere-differentiability of $F_r(x) = \mathfrak{N}(\xi^{-r}c(y)g_r(x))$ for $x = 0$ for every $r \in \{0, 1, 2\}$ and every specific value of $c(y)$.

For $y$ in $(1, 4^l)$ with base 4 expansion $(e_{l-1}, \ldots, e_0, e_{-l-2}, \ldots)$ we set $y_k = (e_{l-1}, \ldots, e_0, e_{-l}, \ldots)$ and in the case where $e_{-kl} \in \{0, 1\}$ we set $z_k = (e_{l-1}, \ldots, e_0, e_{-l}, \ldots)$ and as above we can get a similar limit.

In the case where $e_{-kl} = 3$ we need another approach. We regard $\sup \{j \in \{-kl, \ldots, l+1\} : e_j \neq 3\}$ which is denoted by $d$. We use the integers $b, c$, satisfying $d = bl + c$ and $0 \leq c < l$, and set $y_k = (e_{l-1}, \ldots, e_0, e_{-l}, \ldots)$ and $z_k = y_k + 4^{-kl} = (e_{l-1}, \ldots, e_0, e_{-l}, \ldots)$, $(e_{-bl+c+1}, 0, \ldots, 0)$. Note that the supremum exists for $k$ large enough since $y \in (1, 4)$. From the proof of continuity in the last section we already know the value of $\Psi(z_k) - \Psi(y_k)$ and get:

$$\lim_{k \to \infty} \frac{\Psi(z_k) - \Psi(y_k)}{z_k - y_k} = \lim_{k \to \infty} \prod_{j=-kl}^{-l-1} \frac{\tilde{g}_{r_{2j}}r_{2j+1}(e_j)(4^l/A)^k}{\tilde{g}_{r_{2j}}r_{2j+1}(e_j)}.$$

Note that we can modify the limit if we use $y_k$, given by $(e_{l-1}, \ldots, e_0, e_{-l}, \ldots)$, instead of $y_k$.

Altogether for every $r \in \{0, 1, 2\}$ we can take an appropriate limit from above to derive nowhere-differentiability for $F_r(x) = \mathfrak{N}(\xi^{-r}c(y)g_r(x))$ and the proof is complete. □

2.4. Roots of $F_r(x)$ and weak Newman-type results

Larcher and Zellinger [21] classified for every $r \in \{0, 1, 2\}$ all weight sequences such that $\Pi_r(N, \gamma) > 0$ ($< 0$) for all $N$ large enough. From the Coquet-type formulas, obtained in Section 2.2, we can derive some of their results. More precisely, we answer the following question for the different settings of $r \in \{0, 1, 2\}$. Which weight sequences that eventually have the constant value 1 lead to $\Pi_r(N, \gamma) > 0$ ($< 0$) for all $N$ large enough?

**Lemma 18.** If the periodic function $F_r(x)$ in the Coquet-type formula satisfies $F_r(x) > 0$ ($< 0$) for all $x \in \mathbb{R}$, then $\Pi_r(N, \gamma) > 0$ ($< 0$) for all $N$ large enough.

**Proof.** Bearing in mind that $\phi_r(N)/3$ is bounded it is easily seen that $F_r(x) > 0$ for all $x \in \mathbb{R}$ implies

$$\Pi_r(N, \gamma) = \frac{\phi_0(N)}{3} + N^t F_r(\log_4(N)) > 0$$

for all $N$ large enough. An analogous argument yields the result in the case where $F_r(x) < 0$. □

**Lemma 19.** If there exists an $N_0 \in \mathbb{N}$ such that $F_r(\log_4 N_0) = 0$, then $\Pi_r(N, \gamma) = 0$ for infinitely many $N \in \mathbb{N}$.

**Proof.** We regard the exact formula of $\phi_r(N)$ and observe that $\phi_r(N_0 4^k) = 0$ for all $k$ large enough. Hence for those $k$ we get

$$\Pi_r(N_0 4^k, \gamma) = \frac{\phi_0(N_0 4^k)}{3} + (N_0 4^k)^t F_r(\log_4(N_0 4^k)) = 0.$$  □

**Lemma 20.** We have $\Pi_0(N, (1)_{l\geq 0}) > c_0 N^{3/3 \log 4}$ for all $N \in \mathbb{N}$ with a positive real constant $c_0$, $\Pi_1(N, (1)_{l\geq 0}) < -c_1 N^{3/3 \log 4}$ for all $N \geq 2$ with a positive real constant $c_1$ and $\Pi_2(N, (1)_{l\geq 0}) \leq 0$ for all $N \geq 3$ with equality for example for $N = 2 \cdot 4^m$, where $m \in \mathbb{N}$.

**Proof.** Compare [22] and [4, Theorem 1.1] and the proof given therein. □

The lemmas above and the definition of the functions $F_r(x)$ in the last section yield the following theorem.

**Theorem 21.** Let $y$ be a sequence that eventually has the constant value 1 and $Z(y)$, defined by $Z(y) = \# \{i \in N_0 : y_{2i} = 0\} - \# \{i \in N_0 : y_{2i+1} = 1\}$, describe the finer structure of the previous period.

- We get $\Pi_0(N, \gamma) > 0$ for all $N$ large enough if and only if $Z(y) \equiv 0$(mod 4) and $\Pi_1(N, \gamma) < 0$ for all $N$ large enough if and only if $Z(y) \equiv 2$(mod 4).
- We get $\Pi_1(N, \gamma) > 0$ for all $N$ large enough if and only if $Z(y) \equiv 1$ or 2(mod 4). Otherwise we get $\Pi_1(N, \gamma) < 0$ for all $N$ large enough.
- We get $\Pi_2(N, \gamma) > 0$ for all $N$ large enough if and only if $Z(y) \equiv 3$(mod 4) and $\Pi_1(N, \gamma) < 0$ for all $N$ large enough if and only if $Z(y) \equiv 1$(mod 4).
Lemma 20

Let us briefly consider the classical case where \( \gamma = \gamma' = (1)_{i>0} = 1 \). From Lemma 20 we know that for every positive \( \epsilon \) we have \( \Re(\Pi(x)) > c_0 - \epsilon \) and \( \Re(\xi^{-1}\Pi(x)) < -c_1 + \epsilon \) for all \( x \in \mathbb{R} \). But \( \Re(\xi^{-2}\Pi(x)) \leq 0 \) for all \( x \in \mathbb{R} \) and \( \Re(\xi^{-3}\Pi(x)) = 0 \) for all \( x = 1/2 + m \) with \( m \in \mathbb{Z} \). Altogether we know that \( \Pi(x) \) is a the closed curve and it is contained in a specific subset of \( C \), that is sketched in Fig. 1. Note that from the fact that \( \Re(\xi^{-2}\Pi(x)) = 0 \) for every \( x = 1/2 + m \) with \( m \in \mathbb{Z} \) we know that the full border line has to contain infinitely many points of the sequence \( (\Pi(\log_4(N)))_{N \geq 0} \) and each of the dashed border lines has to contain at least one accumulation point of the sequence \( (\Pi(\log_4(N)))_{N \geq 0} \).

Now the previous period, encoded in \( c(\gamma) \), leads to a rotation of the closed curve \( \Pi(x) \) about the origin with a rotation angle in \([0, \pi/2, \pi, 3\pi/2]\) and a minor important dilation with \( 3^{-\#\{\xi\in\mathbb{N}^*:\gamma(\xi)=0\}}/2 \). More exactly the different values of \( Z(\gamma) \) modulo 4 determine different values of \( \arg(c(\gamma)) \) we have

<table>
<thead>
<tr>
<th>( Z(\gamma) \mod 4 )</th>
<th>( \arg(c(\gamma)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( 3\pi/2 )</td>
</tr>
<tr>
<td>2</td>
<td>( \pi )</td>
</tr>
<tr>
<td>3</td>
<td>( \pi/2 )</td>
</tr>
</tbody>
</table>

A little exhausting case differentiation of the angles \([0, \pi/2, \pi, 3\pi/2]\) and the projections determined by \( \Re(\xi^{-r}c(\gamma)\Pi(x)) \), \( r \in \{0, 1, 2\} \) and the application of Lemmas 18 and 19 lead to the desired results. Here we elaborate the case where \( r = 0 \). If \( \arg(c(\gamma)) \) is \( \pi/2 \) or \( 3\pi/2 \), then the graph (the subset) will be rotated about the angle \( \pi/2 \) or \( 3\pi/2 \). Now we regard the real part and see that \( F_0(x) \) has positive and negative values. Hence \( \Pi_0(N, \gamma) \) cannot be \( >0 \) (or \( <0 \)) for all \( N \) large enough. But if \( \arg(c(\gamma)) \) is \( 0 \), then we still have \( F_0(x) > c \) with a positive constant \( c \). Hence \( \Pi_0(N, \gamma) > 0 \) for almost all \( N \). Finally, if \( \arg(c(\gamma)) \) is \( \pi \), then we obtain \( F_0(x) < c \) with a positive constant \( c \) and \( \Pi_0(N, \gamma) < 0 \) for almost all \( N \). Bearing in mind the factors \( \xi^{-1} \) and \( \xi^{-2} \), which are rotations about the angles \( -2\pi/3 \) and \( -4\pi/3 \) and analogous arguments yield the assertions on \( \Pi_1(N, \gamma) \) and \( \Pi_2(N, \gamma) \). \( \square \)

Remark 22. We know that if the weight sequence is eventually periodic and if the function \( F_r(x) \), appearing in the Coquet-type formula, is either positive or negative, then \( \Pi_r(N, \gamma) > 0 \) (or \( <0 \)) for all \( N \) large enough (compare Lemma 18). But if \( \Pi_r(N, \gamma) > 0 \) (or \( <0 \)) for all \( N \) large enough for an eventually periodic weight sequence implies that \( F_r(x) > 0 \) (or \( <0 \)) in the Coquet-type formula in general is not clear. For example we do not know if there exist an eventually periodic \( \gamma \) and an \( r \in \{0, 1, 2\} \) such that \( F_r(\log_4(N)) > 0 \) (or \( <0 \)) for all \( N \in \mathbb{N} \) and \( \Pi_r(N, \gamma) > 0 \) (or \( <0 \)) for all \( N \) large enough but we can find a point \( x \) with \( F_r(x) = 0 \).

2.5. Converse results

In the following we ask if the converse of Theorem 11 holds as well, i.e.: Does a Coquet-type formula for \( \Pi_r(N, \gamma) \) for every \( r \in \{0, 1, 2\} \) imply that the sequence of weights converges?

In this case we have two formulas for \( \Pi_r(N, \gamma) \) for every \( r \in \{0, 1, 2\} \). One is the Coquet-type formula and one is the general formula given in Theorem 7. We equate both and get

\[
\Pi_r(N, \gamma) = \phi_r(N) \frac{3}{3} + N^\delta F_r(\log_4(N)) = \frac{2}{3} \Re(\xi^{-r} H(N)) + \frac{\rho(N)}{3}.
\]

We set \( N = 4^{lm+s} \), where \( l \) is the common period of the functions \( F_0(x) \), \( F_1(x) \) and \( F_2(x) \) and \( m, s \) are nonnegative integers, where \( 0 \leq s < l \). It is easy to compute the value of \( H(4^{lm+s}) \) and we obtain

\[
\Re(\xi^{-r} \prod_{i=0}^{lm+s-1} \kappa(\gamma_{2i}, \gamma_{2i+1})) = 4^{lm+s}\delta F_r(s) + \phi_r(4^{lm+s}) \frac{\rho(4^{lm+s})}{3}.
\]
From the discussion at the beginning of Section 2.2 we know that the case of finitely many 1s in the weight sequence is a trivial one. Note that we have both an eventually convergent weight sequence and a Coquet-type formula. It remains to investigate the case where there are infinitely many nonzero weights.

From the last equality and from the fact that \( \rho(N) \) and \( \phi_i(N) \) are both bounded we know that

\[
H \left( \sum_{i=0}^{l_{m+s-1}} \frac{k(\gamma_{2i}, \gamma_{2i+1})}{4^s} \right) \xrightarrow{m \rightarrow \infty} F_r(s)
\]

for every \( r \in \{0, 1, 2\} \) and every \( s \in \{0, 1, \ldots, l-1\} \). This yields convergence of the sequence \( (P_m)_{m \geq 1} \) with

\[
P_m = \prod_{i=0}^{l_{m+s-1}} \frac{k(\gamma_{2i}, \gamma_{2i+1})}{4^s},
\]

which implies convergence of the sequence \( (z_m(s))_{m \geq 0} \) with

\[
z_m(s) = \prod_{i=lm+s}^{(l(m+1)+s-1)} \frac{k(\gamma_{2i}, \gamma_{2i+1})}{4^s} \xrightarrow{m \rightarrow \infty} 1.
\]

Since there are just four different pairs of weights, \( \gamma_{2i}, \gamma_{2i+1} \), we know that \( z_m(s) \) can only attain finitely many different values. Hence there exists \( m_0 \in \mathbb{N}_0 \) such that for every \( s \) we have \( z_m(s) = 1 \) for all \( m \geq m_0 \). It is easy to check that the equality \( z_m(s) = 1 \) can only be fulfilled for every \( s \) and all \( m \) large enough in the case where \( \gamma_i = \gamma_{i+2} \) for all \( i \) large enough, i.e., the sequence of weights is eventually periodic. Altogether we arrive at Theorem 6.

References