Asymptotic behavior of Markov semigroups on preduals of von Neumann algebras

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Received 5 May 2004
Available online 13 May 2005
Submitted by Z.-J. Ruan

Abstract

We develop a new approach for investigation of asymptotic behavior of Markov semigroup on preduals of von Neumann algebras. With using of our technique we establish several results about mean ergodicity, statistical stability, and constrictiveness of Markov semigroups.
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Keywords: von Neumann algebra; Predual; Markov operator; Mean ergodicity; Statistical stability; Domination

1. Introduction

The object of investigation in the paper is Markov operators on the predual of a von Neumann algebra. Let \( \mathcal{M} \) be a von Neumann algebra. It is well known that \( \mathcal{M} \) has a unique predual \( \mathcal{M}_\ast \). A Markov operator on \( \mathcal{M}_\ast \) is a linear operator that maps the set \( \mathcal{S}(\mathcal{M}) \subseteq \mathcal{M}_\ast \) of all normal states on \( \mathcal{M} \) into itself. Let us recall that a Markov operator on a von Neumann algebra \( \mathcal{M} \) is a linear normal positive unital mapping on \( \mathcal{M} \). Hence, a Markov
operator on $\mathcal{M}_*$ is the predual operator for a Markov operator on $\mathcal{M}$. We will not use this duality and deal with Markov operators on the predual of a von Neumann algebra.

During the last two decades several important results were established about asymptotic behavior of semigroups of Markov operators on $L^1(\Omega, \Sigma, \mu)$ with $\sigma$-finite measure $\mu$ (see, for instance, the book of Lasota and Mackey [23] and the paper of Komornik [18] for a survey and applications). However, the technique which was used for obtaining of these results does not work in noncommutative setting. Our main goal in this paper is to find noncommutative variants of these results as well as satisfactory methods for proving them. We point out that several related results were obtained in [12,15,16,32] for Markov semigroups on $\mathcal{M}$.

In the next section we establish a theorem, which states that under certain conditions a semigroup of Markov operators is mean ergodic. This theorem will be used for obtaining of lower-bounds criteria of statistical stability and mean ergodicity of semigroups of Markov operators. Then we prove a theorem about an important geometric property of the predual of a von Neumann algebra, which plays the key role in Sections 4 and 5. In Section 4 we introduce a notion of $\alpha$-constrictor for semigroup of Markov operators and establish several results concerning it. Section 5 is devoted to investigation of inheritance of mean ergodicity of Markov operators under taking a power and under the asymptotic domination.

To avoid any duplication we will formulate those results which hold in both discrete and strongly continuous cases in the same form. With the purpose in the following we will consider discrete or strongly continuous semigroups $T = (T_t)_{t \in J}$ of Markov operators on $\mathcal{M}_*$, briefly called Markov semigroups. Here $J = \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ in the discrete case and $J = \mathbb{R}_+$ in the strongly continuous case.

Given a faithful semifinite trace $\tau$ on a von Neumann algebra $\mathcal{M}$, one may be interested in asymptotic behavior of Markov semigroups on a noncommutative $L^1$-space $L^1(\mathcal{M}, \tau)$. For any $f \in L^1(\mathcal{M}, \tau)$ we define by

$$\mathcal{M} \ni a \mapsto \varphi_f(a) := \tau(fa)$$

a normal linear form $\varphi_f$ on $\mathcal{M}$. The mapping $f \mapsto \varphi_f$ is a linear bipositive isometric surjection of $L^1(\mathcal{M}, \tau)$ onto the predual $\mathcal{M}_*$ of $\mathcal{M}$ (see [28, Theorem 14]). Therefore the results of the paper can be easily applied to Markov semigroups on $L^1(\mathcal{M}, \tau)$.

We point out that some results of Sections 4 and 5 hold also for semigroups of positive operators on more general class of ordered Banach spaces which includes preduals of von Neumann algebras as well as Banach lattices with order continuous norm. We refer the reader who is interested in such types generalization to [10] (see also [7,8] for results closely related to Theorem 8).

2. Lower bounds for Markov semigroups

Fix some necessary notions, which are not fixed in Section 1. Let $\mathcal{M}$ be a von Neumann algebra with the predual $\mathcal{M}_*$ and the dual $\mathcal{M}^*$. It is well known [26] that the selfadjoint part $\mathcal{M}_{sa}^*$ of $\mathcal{M}^*$ and the selfadjoint part $\mathcal{M}_{sa}$ of $\mathcal{M}_*$ are ordered Banach spaces over $\mathbb{R}$ with normal positive cones. For $x \leq y$ in $\mathcal{M}_{sa}$ we will denote by $[x, y]$ the order interval
The following theorem gives important conditions which ensures a Markov semigroup to be mean ergodic. This result is new even in the commutative setting for which it was obtained recently and published in [11].

**Theorem 1.** Let $M$ be a von Neumann algebra, $T$ a Markov semigroup on $M_*$, $g \in M_{++}$, and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$, such that

$$\lim_{t \to \infty} \sup_{f \in M_*} \text{dist}(A(t)f, [-g, g]) \leq \eta$$

for any normal state $f \in M_*$. Then $T$ is mean ergodic.

If, moreover, $M$ is atomic and $T$ consists of completely positive operators, then the space $\text{Fix}(T)$ of all fixed vectors of $T$ is finite-dimensional.

**Proof.** We show first of all that $T$ is mean ergodic. By Sine’s ergodic theorem [29] (see also [20, p. 74]) and simple linearity arguments it is enough to check that for every $T'$-fixed point $0 \neq \psi \in M_{sa}$ there exists a $T'$-fixed point $w \in M_{sa}$ which satisfies $\langle \psi, w \rangle \neq 0$.

Let $M_{sa} \ni \psi \neq 0$ be a fixed point of $T'$. We may assume that $\|\psi_+\| = \|\psi\| = 1$. Set $\varepsilon := (1 - \eta)/3$ and take some $x \in M_{sa}$ which satisfies $\|x\| = 1$ and $\langle \psi_+, x \rangle \geq 1 - \varepsilon$. We have $\|x\| = 1$ and

$$1 \geq \langle |\psi|, |x| \rangle \geq \langle \psi_+, |x| \rangle \geq \langle \psi_+, x \rangle \geq 1 - \varepsilon.$$

Thus $\langle \psi, |x| \rangle = (2\psi_+, |x|) - \langle |\psi|, |x| \rangle \geq 2(1 - \varepsilon) - 1 = 1 - 2\varepsilon$.

Let $x'' \in M^*$ be a $w^*$-cluster point of $\{A(t)|x|\}_{t \in J}$. Then $T'_{t''}x'' = x''$. Since

$$\lim_{t \to \infty} \sup_{f \in M_*} \text{dist}(A_t(T)|x|, [-g, g]) \leq \eta$$

and $[-g, g]$ is weakly compact in $M_*$ by the theorem of Akemann [1, Theorem II.2(2)] (see also [33, Theorem 5.4, p. 149]), we obtain

$$x'' \in [-g, g] + \eta B_{M^*} \subseteq M_* + \eta B_{M^*}.$$
Take the positive projection $R : \mathcal{M}^* \to \mathcal{M}_+$ according to [26, Proposition 1.17.7]. Then
\[ (\text{Id}_{\mathcal{M}^*} - R)x'' \in \eta B_{\mathcal{M}^*}, \]
and
\[ \langle \psi, Rx'' \rangle = \langle \psi_+ + Rx'' \rangle - \langle \psi_-, Rx'' \rangle = \langle x'', \psi \rangle - \eta \geq 1 - 2\varepsilon - \eta = \varepsilon > 0. \]

Moreover,
\[ T_s Rx'' = T_s RT_t'' x'' \geq T_s RT_t'' Rx'' = T_s RT_t Rx'' = T_{s+t} Rx'' \geq 0. \]
Thus $(T_t Rx'')_t$ is decreasing in $\mathcal{M}_{*+}$ and hence, $w := \lim_{t \to \infty} T_t Rx''$ exists. Clearly $T_t w = w$ for all $t \in J$, and $\langle \psi, w \rangle = \langle \psi, Rx'' \rangle > 0$. Thus $T$ is mean ergodic.

Now let $\mathcal{M}$ be atomic and $T$ be completely positive. The ergodic projection $P_T = \lim_{t \to \infty} A_t (T)$ (taken with respect to the strong operator topology) is a completely positive Markov operator. By the theorem of Choi and Effros [4] the range of its adjoint $P_T'$ is a von Neumann algebra, say $\mathcal{N}$. Hence the range $P_T (\mathcal{M}_+)$ is itself the predual of $\mathcal{N}$. Then the real part of the unit ball $B_{\mathcal{N}_{sa}}$ of $\mathcal{N}_*$ satisfies
\[ B_{\mathcal{N}_{sa}} \subseteq [-g, g] + \eta B_{\mathcal{N}_{sa}}, \]
and hence
\[ B_{\mathcal{N}_{sa}} \subseteq \frac{1}{1 - \eta} [-g, g]. \]
Since order intervals of $\mathcal{M}_{sa}$ are compact, the last inclusion shows that the unit ball of $\mathcal{N}_*$ is compact, and hence $\dim \text{Fix} (T) < \infty$. \qed

A Markov semigroup $T$ on $\mathcal{M}_*$ is called *statistically stable* if there exists a normal state $u \in S(\mathcal{M})$ such that
\[ \lim_{t \to \infty} \| T_t f - u \| = 0 \quad (\forall f \in S(\mathcal{M})). \]
Such a normal state $u$ is obviously unique and $T$-invariant.

An element $h \in \mathcal{M}_{*+}$ is called a *lower-bound element* for $T$ if
\[ \lim_{t \to \infty} \| (h - T_t f)_+ \| = 0 \quad (\forall f \in S(\mathcal{M})). \]

The following result is due to Lasota [21] for Markov semigroups on a commutative $L^1$-space. Its generalization for Markov semigroups on the predual of a noncommutative von Neumann algebra is due to Sarymsakov and Grabarnik. It was announced without proof by Ayupov and Sarymsakov in [3] and by Sarymsakov and Grabarnik in [27]. Indeed, it is a simple corollary of Theorem 1 above (cf. also [9]).

**Theorem 2.** Let $\mathcal{M}$ be a von Neumann algebra. Then for any Markov semigroup $T$ on $\mathcal{M}_*$ the following assertions are equivalent:

(i) $T$ is statistically stable;
(ii) there exists a nontrivial lower bound element for $T$. 
Proof. (i) $\Rightarrow$ (ii). Let a normal state $u \in S(M)$ satisfy
\[ \lim_{t \to \infty} \| T_t f - u \| = 0 \quad (\forall f \in S(M)). \]
Then $u$ is a nontrivial lower bound element for $T$.

(ii) $\Rightarrow$ (i). Let $0 \not= h \in M^*_+$ be a nontrivial lower bound element for $T$. Denote
\[ M^* := \{ f \in M^*_+ : \| f + \| = \| f - \| \}. \]
Since
\[ \lim_{t \to \infty} \| (h - T_t f)_+ \| = 0 \quad (\forall f \in S(M)), \]
then
\[ \lim_{t \to \infty} \sup \| (A_t(T)f - h)_+ \| \leq 1 - \| h \| < 1 \quad (\forall f \in S(M)). \]
Theorem 1 applied to the interval $[-g, g] = [-h, h]$ and $\eta = 1 - \| h \|$ implies that $T$ is mean ergodic, and hence there exists a $T$-invariant normal state, say $u$. Obviously
\[ M^*_{sa} = M^*_0 \oplus \mathbb{R} \cdot u \]
since $M^*_{sa}$ has codimension one in $M^*_{sa}$.

Show the following:
\[ \lim_{t \to \infty} \| T_t f - u \| = 0 \quad (\forall f \in S(M)). \]

It is enough to prove that
\[ \lim_{t \to \infty} \| T_t f \| = 0 \quad (\forall f \in M^*_0). \quad (1) \]

Note that $\| T_t f \| \geq \| T_{s+t} f \|$ since $T$ is contractive. Hence
\[ \| f \| \geq \lim_{t \to \infty} \| T_t f \| = \inf_t \| T_t f \| \]
holds for every $f$. If (1) is not true, then there exists an $f \in M^*_0$ with
\[ 2\alpha := \lim_{t \to \infty} \| T_t f \| > 0. \]

Then
\[ 2\alpha = \lim_{t \to \infty} \| T_t f \| = \lim_{t \to \infty} \| T_t (f_+ - f_-) \| = \lim_{t \to \infty} \| (T_t f_+ - \alpha h)_+ - (T_t f_- - \alpha h)_+ \| \]
\[ \leq \lim_{t \to \infty} \left( \| (T_t f_+ - \alpha h)_+ \| + \| (T_t f_- - \alpha h)_+ \| \right) = 2\alpha (1 - \| h \|) \]
that is impossible. Here we use that
\[ \lim_{t \to \infty} \| T_t f_+ - \alpha h - (T_t f_+ - \alpha h)_+ \| = 0, \]
and
\[ \lim_{t \to \infty} \| T_t f_- - \alpha h - (T_t f_- - \alpha h)_+ \| = 0, \]
since $h$ is a lower bound element for $T$. The contradiction shows that assertion (1) holds. \Box
We will call an $h \in \mathcal{M}_{*+}$ a mean lower bound element for a Markov semigroup $T$ if
\[
\lim_{t \to \infty} \|(h - A_t(T)f)_+\| = 0 \quad (\forall f \in \mathcal{S}(\mathcal{M})).
\]
Obviously any lower bound element is a mean lower bound element.

Our final result in this section is another corollary of Theorem 1. In the special case of Markov semigroups on $L^1(\Omega, \Sigma, \mu)$ with $\sigma$-finite measure $\mu$, it was obtained recently in [11].

**Theorem 3.** Let $\mathcal{M}$ be a von Neumann algebra and $T$ be a semigroup of completely positive Markov operators on $\mathcal{M}_*$. Then the following assertions are equivalent:

(i) there exists a $T$-invariant normal state $u$ such that
\[
\lim_{n \to \infty} \|A_n(T)f - u\| = 0
\]
for any normal state $f \in \mathcal{M}_*$;

(ii) there exists a nontrivial mean lower bound element for $T$.

**Proof.** (i) $\Rightarrow$ (ii). Let $u \in \mathcal{M}_{*+}$ satisfy $\lim_{t \to \infty} \|A_t(T)f - u\| = 0$ for every normal state $f$, then $u$ is a nontrivial mean lower bound element for $T$.

(ii) $\Rightarrow$ (i). Let $0 \neq h \in \mathcal{M}_{*+}$ be a nontrivial mean lower bound element for $T$. Then
\[
\lim_{t \to \infty} \sup_{f \in \mathcal{S}(\mathcal{M})} \|(A_t(T)f - h)_+\| \leq \eta
\]
with $\eta := 1 - \|h\|$. By Theorem 1, $T$ is mean ergodic. Thus, we have only to prove that the space $\text{Fix}(T)$ of fix points of $T$ is one-dimensional. Let $P$ be the projection onto $\text{Fix}(T)$ given by $Pf = \lim_{t \to \infty} A_t(T)f$. Since $h$ is a mean lower bound we obtain $f = Pf \geq h$ for all normal states $f \in \text{Fix}(T) \cap \mathcal{S}(\mathcal{M})$. But this in turn implies $f \geq Ph =: h_0$ for all these normal states, and $h_0 \neq 0$ since $\|h_0\| = \|Ph\| = \|\lim_{t \to \infty} A_t(T)h\| = \|h\| > 0$. Now $P$ is a completely positive Markov operator. Hence by the theorem of Choi and Effros, used already in the proof of Theorem 1, $P(\mathcal{M}_*)$ is isometrically and order isomorphic to the predual $\mathcal{N}_*$ of a von Neumann algebra $\mathcal{N}$. Then every positive element $f \in \mathcal{N}_*$ of norm 1 majorizes the nonzero element $h_0$, which obviously implies $\dim(\mathcal{N}_*) = 1$. □

3. A geometric property of the predual of a von Neumann algebra

Let $\mathcal{M}$ be a von Neumann algebra. Recall that for any nonempty subsets $A, B$ of $\mathcal{M}_*$ the nonsymmetric distance is given by $\text{dist}(A, B) := \sup_{x \in A} \text{dist}(x, B)$ and then the Hausdorff distance between $A$ and $B$ is
\[
\text{dist}_H(A, B) = \max(\text{dist}(A, B), \text{dist}(B, A)).
\]
It is well known that $\mathcal{M}_{*\text{sa}}$ and $\mathcal{M}_{*\text{sa}}$ are both ordered by normal and generating positive cones. Normality of a positive cone $C$ is equivalent to the continuity of the mapping
\[
C \times C \ni (x, y) \mapsto \text{dist}_H([0, x], [0, y])
\]
at the point \((0, 0) \in C \times C\). This motivates the following definition. We call a positive cone \(C\) **strongly normal** if the mapping above is continuous on \(C \times C\). Clearly, that \(C\) is strongly normal if and only if
\[
\text{dist}_H([0, x], [0, y]) \to 0
\]
whenever \(\|x - y\| \to 0\).

In this section we prove that the selfadjoint part of the predual of a von Neumann algebra is ordered by a strongly normal cone. This result will play the important role in Sections 4 and 5. We begin with the following technical lemma.

**Lemma 4.** Let \(H\) be a Hilbert space and \(M\) be a von Neumann algebra in \(\mathcal{L}(H)\). Let \(S, T, U \in M_+\) satisfy
\[
0 \leq S \leq T + U = I.
\]
Then for every \(\eta \in H\) the following inequality
\[
|(S\eta | \eta) - (T^{1/2}ST^{1/2}\eta | \eta)| \leq 2\|\eta\|(U\eta | \eta)^{1/2}
\]
holds.

**Proof.** Consider the equality
\[
|(S\eta | \eta) - (T^{1/2}ST^{1/2}\eta | \eta)|
\]
\[
= \left| \|S^{1/2}\eta\|^2 - \|S^{1/2}T^{1/2}\eta\|^2 \right|
\]
\[
= \left( \|S^{1/2}\eta\| + \|S^{1/2}T^{1/2}\eta\| \right) \cdot \left| \|S^{1/2}\eta\| - \|S^{1/2}T^{1/2}\eta\| \right|
\]
and the inequality
\[
\|S^{1/2}\eta\| - \|S^{1/2}T^{1/2}\eta\|^2 \leq \|S^{1/2}(I - T^{1/2})\eta\|^2
\]
\[
= \left( (I - T^{1/2})S(I - T^{1/2})\eta | \eta \right)(I - T^{1/2})^2 \eta | \eta).
\]
Since \(0 \leq T \leq I\) implies \(0 \leq I - T^{1/2} \leq I + T^{1/2}\), it follows \((I - T^{1/2})^2 \leq I - T = U\). So, we obtain
\[
|(S\eta | \eta) - (T^{1/2}ST^{1/2}\eta | \eta)| \leq 2\|\eta\|(U\eta | \eta)^{1/2}.
\]

Now we are in position to show that the selfadjoint part of the predual of a von Neumann algebra is ordered by a strongly normal cone (cf. also [10, Theorem 4]).

**Theorem 5.** Let \(M\) be a von Neumann algebra and let \(M_{ssa}\) be the selfadjoint part of the predual \(M_*\) of \(M\). Then the cone \(M_{sa}^+\) of positive normal linear forms on \(M\) is strongly normal in \(M_{ssa}\).

**Proof.** Let \(0 \leq \mu, v \in M_*\) be arbitrary and set \(\rho = |\mu - v|\). Let \(0 \leq \chi \leq \mu\) be given. Then \(0 \leq \chi \leq v + \rho\). We set
\[
\gamma = \|v + \rho\|^{-1} \quad \text{and} \quad \chi_1 = \gamma \chi, \quad \nu_1 = \gamma v, \quad \rho_1 = \gamma \rho.
\]
Then \( \|v_1 + \rho_1\| = 1 \). Apply the GNS-representation of \( \mathcal{M} \)
\[
\pi_\lambda : \mathcal{M} \to B(\mathcal{H})
\]
induced by \( \lambda \), and let \( \xi \in B(\mathcal{H}) \) be a normalized cyclic vector for \( \pi_\lambda \). Then there exist \( S, T, U \) in the positive cone \( \pi_\lambda(\mathcal{M})'_+ \) of the commutant \( \pi_\lambda(\mathcal{M})' \) of \( \pi_\lambda(\mathcal{M}) \) satisfying
\[
0 \leq S \leq T + U = I
\]
as well as
\[
\chi_1(a) = (\xi | S \circ \pi_\lambda(a)\xi), \quad v_1(a) = (\xi | T \circ \pi_\lambda(a)\xi),
\]
\[
\rho_1(a) = (\xi | U \circ \pi_\lambda(a)\xi)
\]
for all \( a \in \mathcal{M} \).
We set \( \psi(a) = (\xi | T^{1/2} \circ S \circ T^{1/2} \circ \pi_\lambda(a)\xi) \). Then \( 0 \leq \psi \leq v_1 \). Denote
\[
V = S - T^{1/2} \circ S \circ T^{1/2}.
\]
By Lemma 4, we obtain for \( a \in \mathcal{M}_+ \):
\[
|\psi(a) - \chi_1(a)| = |(V \circ \pi_\lambda(a)\xi | \xi)| = |(V \circ \pi_\lambda(a)\xi | \pi_\lambda(a)^{1/2}\xi)|
\leq 2\|\pi_\lambda(a)^{1/2}\xi\| (U \circ \pi_\lambda(a)^{1/2}\xi | \pi_\lambda(a)^{1/2}\xi)^{1/2}
\leq 2\|\pi_\lambda(a)^{1/2}\xi\| \rho_1(a)^{1/2}.
\]
Since \( \psi - \chi_1 \) is selfadjoint, and since the norm is determined on the unit ball of the self-adjoint part \( \mathcal{M}_{*sa} \) of \( \mathcal{M} \), we obtain
\[
|\psi(a) - \chi_1(a)| \leq 2((\rho_1(a_+))^{1/2} + (\rho_1(a_-))^{1/2}) \leq 4\|\rho_1\|^{1/2} \quad (a \in \mathcal{M}).
\]
Dividing this inequality by \( \gamma' \), we get
\[
|\gamma^{-1}\psi(a) - \chi(a)| \leq \frac{4}{\sqrt{\gamma}} \cdot \|\rho\|^{1/2} \leq 4\sqrt{2}\|v + \mu\|\|\rho\| \quad (a \in \mathcal{M}).
\]
Now \( 0 \leq \gamma^{-1}\psi \leq v \) by construction. So we obtain
\[
\text{dist}(\chi,[0,v]) \leq 4\sqrt{2}\|v + \mu\|\|v - \mu\|,
\]
and by straightforward calculation it follows
\[
\text{dist}_H([0,\mu],[0,v]) \leq 4\sqrt{2}\|v + \mu\|\|v - \mu\|,
\]
which yields the strong normality of \( \mathcal{M}_{*+} \). □

4. \( \alpha \)-Constrictors for Markov semigroups

Let \( \mathcal{M} \) be a von Neumann algebra with the predual \( \mathcal{M}_* \), and let \( T = (T_t)_{t \in I} \) be a semigroup on \( \mathcal{M}_* \). Given a nonempty subset \( A \subseteq \mathcal{M}_{*sa} \) and a real \( \alpha \geq 0 \), we call \( A \) an \( \alpha \)-constrictor for the semigroup \( T \) if
\[
\limsup_{t \to \infty} \text{dist}(T_tx, A) \leq \alpha \quad (\forall x \in \mathcal{M}_{*sa}, \|x\| \leq 1).
\]
We denote the set of all $\alpha$-constrictors for $T$ by $\text{Con}_\alpha(T)$. Following to the paper [22] of Lasota, Li, and Yorke, we call $T$ constrictive whenever $T$ possesses a compact 0-constrictor. The following result is due to Phong [24] and Sine [31] in the more general setting of bounded semigroups on Banach spaces. This result shows the importance of constrictive semigroups.

**Theorem 6.** A Markov semigroup $T$ on $\mathcal{M}_*$ is constrictive if and only if there exists a decomposition

$$\mathcal{M}_* := \mathcal{M}_*^0 \oplus \mathcal{M}_*^r$$

into $T$-invariant subspaces $\mathcal{M}_*^0$, $\mathcal{M}_*^r$ such that

$$\mathcal{M}_*^0 = \left\{ x \in \mathcal{M}_*: \lim_{t \to \infty} \| T_t x \| = 0 \right\}$$

and $\dim(\mathcal{M}_*^r) < \infty$.

The theorem implies that one can investigate the asymptotic behavior of constrictive semigroups by means of pure linear algebra. Unfortunately, with the exception of some rather special cases, it is difficult to prove directly that $T$ is constrictive, and the problem arises to find weaker conditions (compared to existence of a compact 0-constrictor) under which $T$ is constrictive. In the commutative setting this problem was exhaustively investigated in the series of papers [7,8,17–19,25]. As we know, this investigation is not made in the noncommutative setting before, so we try to fill this gap in this paper. We begin with the following corollary of Theorem 1.

**Corollary 7.** Let $T$ be a Markov semigroup on $\mathcal{M}_*$ which possesses an $\eta$-constrictor $[-g,g]$ for some $0 \leq \eta < 1$. Then $T$ is mean ergodic.

**Proof.** It can be easy and directly checked that the semigroup $T = (T_t)_{t \in J}$ is mean ergodic. □

**Theorem 8.** Let $\mathcal{M}$ be a von Neumann algebra and $T$ be a Markov semigroup on $\mathcal{M}_*$. Assume that $T$ has an $\eta$-constrictor $[-y,y]$ for some $0 \leq \eta < 1$. Then there exists a limit $w := \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} T_k y$ (respectively $w := \lim_{t \to \infty} \frac{1}{t} \int_0^t T_t y \, dt$), and the set

$$\{ -w, w \}$$

is a 0-constrictor for $T$. In particular, $T$ is weakly almost periodic.

**Proof.** By Corollary 7, the semigroup $T = (T_t)_{t \in J}$ is mean ergodic. We will prove that

$$\frac{1}{1-\eta}[-w, w]$$

is a 0-constrictor for $T$.

Let $\eta < \sigma < 1$ be fixed. We claim that $\frac{1}{1-\sigma}[-w, w] \in \text{Con}_0(T)$. To prove this, it is sufficient to show that for any $x \in \mathcal{M}_{*\sigma}$, $\| x \| \leq 1$ and $\varepsilon > 0$, there exists $t(x, \varepsilon) \in J$ satisfying $T_t x \in \frac{1}{1-\sigma}[-w, w] + \varepsilon L_{\mathcal{M}_{*\sigma}}$ for all $t \geq t(x, \varepsilon)$, where $L_{\mathcal{M}_{*\sigma}}$ is the unit ball of $\mathcal{M}_{*\sigma}$.

Fix $x \in L_{\mathcal{M}_{*\sigma}}$ and $\varepsilon > 0$. By Theorem 5, there exists $\delta > 0$ for which

$$\left\| z - \frac{1}{1-\sigma} w \right\| \leq \delta$$

$$\Rightarrow [-z, z] \subseteq \frac{1}{1-\sigma}[-w, w] + \frac{\varepsilon}{2} L_{\mathcal{M}_{*\sigma}} \quad (0 \leq z \in \mathcal{M}_*).$$ (2)
Since \( w \in \overline{\mathbb{co}}(T_t y: t \in J) \), we can take \( \alpha_q \in \mathbb{R}_+ \) and \( s_q \in J \) where \( q \in \mathbb{I}_m \) such that

\[
\sum_{q=1}^{m} \alpha_q = 1, \quad a_m := \sum_{q=1}^{m} \alpha_q T_{s_q} y, \quad \|a_m - w\| \leq \frac{\delta(1 - \sigma)}{2}.
\]  

(3)

By induction, we construct an increasing sequence \((t_i)_{i=1}^{\infty} \subseteq J\) satisfying

\[
T_{t_i - s_q} x \in \left[ -\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y, \sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} y \right] + \sigma^i B_{\mathcal{M}_{sa}}
\]  

(4)

for all \( i, q \in \mathbb{N} \), \( q \leq m \). The first step of (4) follows directly from \([-y, y] \in \text{Con}_\eta(T)\).

Assume that for some \( i \), we find a \( t_i \) which satisfies (4) for all \( \mathbb{N} \ni q \leq m \). Then, we choose \( u_q^i \) and \( v_q^i \) such that

\[
u_q^i \leq \sigma^i, \quad u_q^i + v_q^i = T_{t_i - s_q} x
\]

for each \( q \) satisfying \( \mathbb{N} \ni q \leq m \). Then for a large enough \( t_{i+1} \), we have

\[T_{t_{i+1} - s_q} x = T_{t_{i+1} - t_i} (u_q^i + v_q^i)
\]

\[
\leq \left[ -\sum_{j=1}^{i+1} \sigma^{j-1} T_{t_{i+1} - t_j} y, \sum_{j=1}^{i+1} \sigma^{j-1} T_{t_{i+1} - t_j} y \right] + \sigma^{i+1} B_{\mathcal{M}_{sa}}
\]

for all \( q \in \mathbb{I}_m \). Thus, we obtain that (4) is true with replacing \( i \) by \( i + 1 \).

Using the fact that \( w \) is a fixed point of \( T \) and the condition (3), we obtain

\[
\left\|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} a_m - \frac{1}{1 - \sigma} w\right\| \leq \left\|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} a_m - \frac{1 - \sigma^i}{1 - \sigma} w\right\| + \frac{\sigma^i}{1 - \sigma} \|w\|
\]

\[
= \left\|\sum_{j=1}^{i} \sigma^{j-1} T_{t_i - t_j} (a_m - w)\right\| + \frac{\sigma^i}{1 - \sigma} \|w\|
\]

\[
\leq \sum_{j=1}^{i} \sigma^{j-1} \|a_m - w\| + \frac{\sigma^i}{1 - \sigma} \|w\|
\]

\[
\leq \frac{1}{1 - \sigma} \|a_m - w\| + \frac{\sigma^i}{1 - \sigma} \|w\|
\]

\[
< \frac{\delta}{2} + \frac{\sigma^i}{1 - \sigma} \|w\|
\]

< \frac{\delta}{2} + \frac{\sigma^i}{1 - \sigma} \|w\|
for all \( i \). Fix a large enough \( i \) such that
\[
\sigma^i \leq \min \left( \frac{\delta(1-\sigma)}{2\|w\|}, \frac{\epsilon}{2} \right),
\]
then the condition (2) implies that
\[
\left[ -\sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} a_m, \sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} a_m \right] \subseteq \frac{1}{1-\sigma} [-w, w] + \frac{\epsilon}{2} B_{M^{*+}}.
\]
(6)

Now, by using (4)–(6), we obtain
\[
T_{i_j} x = \sum_{q=1}^{m} \alpha_q T_{s_q+t_i-s_q} x
\]
\[\in \left[ -\sum_{q=1}^{m} \alpha_q T_{s_q} \sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} y, \sum_{q=1}^{m} \alpha_q T_{s_q} \sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} y \right] + \sum_{q=1}^{m} \alpha_q T_{s_q} (\sigma^i B_{M^{*+}})\]
\[\subseteq \left[ -\sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} a_m, \sum_{j=1}^{i} \sigma^{j-1} T_{i_j-t_j} a_m \right] + \sigma^i B_{M^{*+}}\]
\[\subseteq \frac{1}{1-\sigma} [-w, w] + \frac{\epsilon}{2} B_{M^{*+}} + \sigma^i B_{M^{*+}} \subseteq \frac{1}{1-\sigma} [-w, w] + \epsilon B_{M^{*+}}.
\]
Then
\[
T_i x \in \frac{1}{1-\sigma} [-T_{i-t_i} w, T_{i-t_i} w] + \epsilon T_{i-t_i} (B_{M^{*+}}) \subseteq \frac{1}{1-\sigma} [-w, w] + \epsilon B_{M^{*+}}
\]
for all \( J \ni t \ni t(x, \epsilon) := t_i \). Thus we have shown that \( \frac{1}{1-\sigma} [-w, w] \in \text{Con}(T) \). By arbitrariness of \( \sigma, \eta < \sigma < 1 \), we obtain that \( \frac{1}{1-\eta} [-w, w] \in \text{Con}(T) \). \( \Box \)

Now we establish two corollaries of Theorem 8. They might be of interest in the context of noncommutative probability theory.

**Corollary 9.** Let \( M \) be an atomic von Neumann algebra and \( T \) be a Markov semigroup on \( M_+ \). Then \( T \) is constrictive if and only if \( T \) possesses an \( \eta \)-constrictor \([-y, y]\) for some \( 0 \leq \eta < 1 \) and some \( y \geq M^{*+} \).

**Proof.** The sufficiency follows directly from Theorem 8, since every order interval in the predual of an atomic von Neumann algebra is compact [33, Corollary 5.11, p. 156]. The necessity holds obviously for Markov semigroups on the predual of any (not only atomic) von Neumann algebra. \( \Box \)

Recall that a Markov operator \( T \) on \( M_+ \) is called **irreducible** if its adjoint \( T' \) on \( M \) does not possess \( \sigma(M, M_+) \)-closed invariant hereditary subcones other than \( \{0\} \) or \( M_+ \) (see [14, p. 388]).
Corollary 10. Let $T$ be a completely positive Markov operator on the predual $\mathcal{M}_*$ of an atomic von Neumann algebra $\mathcal{M}$. Assume that $T$ is irreducible and there exists a positive $y \in \mathcal{M}_*$ such that $[-y, y] \subseteq \text{Con}_\eta(T)$ for some real $\eta$, $0 \leq \eta < 1$.

Then there exists a Markov operator $Q$ of finite rank such that $Q^{p+1} = Q$ for some $p \in \mathbb{N}$ and the sequence $(T^n - Q^n)_{n=0}^{\infty}$ converges in the strong operator topology to 0.

Proof. Theorem 9 implies that $T$ is constrictive, so $\dim(\mathcal{M}_r^r) < \infty$, where we use the notation of Theorem 6. By [13], the peripheral point spectrum $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is a finite group. Hence $T$ is periodic on $\mathcal{M}_r^r$ and it is enough to set $Q := TP_r$, where $P_r$ is the Jacobs–Deleuw–Glicksberg projection onto $\mathcal{M}_r^r$ (see [20, Section 2.4, Theorem 4.4]).

Remark. In the last two corollaries we assumed $\mathcal{M}$ to be atomic. It is an open problem whether these results hold for arbitrary von Neumann algebras. In this general setting, Theorem 8 shows immediately that $T$ is weakly almost periodic. If moreover $T$ consists of completely positive operators then the space $\mathcal{M}_r^r$ of reversible vectors is finite-dimensional which follows easily from the theorem of Choi and Effros on the range of completely positive projections on a von Neumann algebra [4, pp. 166–168]. The problem is whether the space $\mathcal{M}_r^f$ of all flight vectors coincides with $\mathcal{M}_r^0 = \{x : \lim_{t \to \infty} T_{tx} = 0\}$. Up to now we can show the coincidence of these two spaces only in the case above ($\mathcal{M}$ is atomic) and in the case $\mathcal{M}_r = \mathcal{M}_1^r \otimes \mathcal{M}_2^r$ where $\mathcal{M}_1$ is commutative and $\mathcal{M}_2$ is finite-dimensional.

5. Inheritance of mean ergodicity

Let $T$ be an arbitrary bounded linear operator on a Banach space $X$. If a power $T^n$ of $T$ is mean ergodic then a moment of reflection shows that $T$ is itself mean ergodic. The converse is wrong even for Koopman operators on $C(K)$ [30]. However it is true for many important classes of positive operators. For instance, this holds in the case of Banach lattices with order continuous norm (the result which is due to Derriennic and Krengel [5, Proposition 4.5]). We prove Theorem 12, saying that this is true also for Markov operators on preduals of von Neumann algebras (cf., also, [10, Theorem 11]). We begin with the following lemma.

Lemma 11. Let $\mathcal{M}$ be a von Neumann algebra. If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{M}_r^r$, and $x_n \xrightarrow{\|\cdot\|} x_0$, then the set $\bigcup_{n=1}^{\infty} [0, x_n]$ is conditionally weakly compact and its norm closure contains the order interval $[0, x_0]$. Moreover if the algebra $\mathcal{M}$ is atomic, then the set $\bigcup_{n=0}^{\infty} [0, x_n]$ is conditionally compact.

Proof. By Theorem 5, $\lim_{n \to \infty} \text{dist}_H([0, x_n], [0, x_0]) = 0$ holds. Hence, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{\infty} [0, x_n] \subseteq \bigcup_{n=0}^{n_\varepsilon} [0, x_n] + \varepsilon B_{\mathcal{M}_r^r}.$$ (7)
Since the set $\bigcup_{n=0}^{\infty} [0, x_n]$ is weakly compact for any $n \geq 1$ [1, Theorem II.2(2)], it follows easily that $\bigcup_{n=1}^{\infty} [0, x_n]$ is conditionally weakly compact. Moreover, since

$$\lim_{n \to \infty} \text{dist}_H ([0, x_0], [0, x_n]) = 0,$$

we obtain that $[0, x_0] \subseteq \text{cl}_{\| \cdot \|} \bigcup_{n=1}^{\infty} [0, x_n]$.

Finally assume that the algebra $M$ is atomic, then all intervals $[0, x_n]$ are compact. Hence $[0, x_0]$ is compact by (8). It follows now from (7) that $\bigcup_{n=0}^{\infty} [0, x_n]$ is conditionally compact.

**Theorem 12.** Let $M$ be a von Neumann algebra and $T$ be a mean ergodic Markov operator on $\mathcal{M}_*$. Then $T^m$ is mean ergodic for any $m \in \mathbb{N}$.

**Proof.** It is enough to show that $\lim_{n \to \infty} \mathcal{A}_n(T^m)f$ exists for all $f \in \mathcal{M}_{*+}$. Fix some $f \in \mathcal{M}_{*+}$. Then

$$0 \leq \mathcal{A}_n(T^m)f = \frac{1}{n} \sum_{k=0}^{n-1} T^{mk}f \leq \frac{1}{n} \sum_{i=0}^{mn-1} T^i f = m \mathcal{A}_{mn}(T)f.$$

By assumption, the sequence $(m \mathcal{A}_{mn}(T)f)_{n=1}^{\infty}$ is convergent in $\mathcal{M}_*$. Lemma 11 says that $\{\mathcal{A}_n(T^m)f\}_{n=1}^{\infty}$ is conditionally weakly compact and consequently it possesses a weak cluster point. Applying Eberlein’s ergodic theorem [20, Theorem 2.1.1], we obtain that the norm limit $\lim_{n \to \infty} \mathcal{A}_n(T^m)f$ exists. $\square$

Before we will give our next result let us give a definition from [6]. Let $S, T$ be operators on $\mathcal{M}_*$. We say that $S$ is asymptotically dominated by $T$ if for any $f \in \mathcal{M}_{*+}$ there exists a sequence $(q^n f)_{n=1}^{\infty} \subseteq \mathcal{M}_*$ such that

$$\lim_{n \to \infty} \|q^n f\| = 0 \quad \text{and} \quad T^n f + q^n f \geq S^n f \quad (\forall n \in \mathbb{N}).$$

While the asymptotic domination is a generalization of usual domination is far away from its special case $S \leq T$. For example, any positive $S$, such that $S^n \to P$ strongly, is asymptotically dominated by $P$.

The following theorem is motivated by the similar result of Arendt and Batty [2] for Banach lattices with order continuous norm (see also [10]).

**Theorem 13.** Let $M$ be a von Neumann algebra and $S$ be a Markov operator on $\mathcal{M}_*$ which is asymptotically dominated by a (not necessarily linear) positive mean ergodic operator $T$. Then $S$ is mean ergodic.

**Proof.** We need only to show that $\lim_{n \to \infty} \mathcal{A}_n(S)f$ exists for any $f \in \mathcal{M}_{*+}$. Fix $f \in \mathcal{M}_{*+}$ and take a sequence $(q^n f)_{n=1}^{\infty} \subseteq \mathcal{M}_*$ which satisfies (9). Set $d^n_0 := 0$ and $d^n f := \frac{1}{n} \sum_{k=0}^{n-1} q^n f_k$, then

$$0 \leq \mathcal{A}_n(S)f \leq \mathcal{A}_n(T)f + d^n f \quad (\forall n \in \mathbb{N}).$$
Since $T$ is mean ergodic, $\lim_{n \to \infty} \| A_n(T)f - u \| = 0$ for an appropriate $u \in M_{s+}$. Moreover,

$$\lim_{n \to \infty} \| A_n(T)f + d_n^f - u \| = 0,$$

since $\lim_{n \to \infty} \| d_n^f \| = 0$. Lemma 11 implies that \{${A_n(S)f}_{n=1}^{\infty}$\} is conditionally weakly compact, and henceforth has a weak cluster point. The Eberlein’s ergodic theorem implies that $\lim_{n \to \infty} A_n(S)f$ exists, what is required.

We finish with the following result, which is a simple consequence of Lemma 11.

**Theorem 14.** Let $M$ be a von Neumann algebra and $0 \leq S \leq T \in L(M)$, where $T$ is a compact dual operator. Then $S$ is weakly compact. Moreover, whenever the algebra $M$ is atomic, $S$ is compact.

**Proof.** Since $M_{s+a}$ is an order ideal in $M_{s+}^*$ the operator $S$ is itself a dual operator, say $S = S'$. Denote also the predual for $T$ by $T_1$.

It is enough to show that for every bounded sequence $(x_n)_{n=1}^{\infty} \subseteq M_{s+}^*$ there exists a subsequence $(x_{n_m})_{m=1}^{\infty}$ such that $(S_1 x_{n_m})_{m=1}^{\infty}$ is weakly convergent in $M_s$.

Fix a sequence $(x_n)_{n=1}^{\infty} \subseteq M_{s+}^*$. Since $T_1$ is compact, there exists a convergent subsequence $(T_1 x_{n_k})_{k=1}^{\infty}$. Then

$$\{S_1 x_{n_k}\}_{k=1}^{\infty} \subseteq \bigcup_{k=1}^{\infty} [0, T_1 x_{n_k}],$$

and henceforth by Lemma 11, the set $\{S_1 x_{n_k}\}_{k=1}^{\infty}$ is conditionally weakly compact. Then there exists a weakly convergent subsequence $(S_1 x_{n_{k_i}})_{i=1}^{\infty}$, which implies that $S_1$ is weakly compact. It is known that in this case its dual $S$ is weakly compact as well.

Whenever the von Neumann algebra $M$ is atomic, each order interval in $M_a$ is compact, and the same arguments with using of Lemma 11 show that the sequence $(S_1 x_{n_m})_{m=1}^{\infty}$ has a norm convergent subsequence, and hence $S_1$ and $S$ are both compact.

**Acknowledgments**

We gratefully acknowledge the useful comments of the anonymous referee which enabled us to make the paper more readable. The first author is grateful for generous support of the Alexander von Humboldt Foundation during his stay at the University of Tübingen in 2000–2002, where the important part of this work was done.

**References**