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Averages by the Sieve Method

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In this note we remark on the ready adaptability of the machinery of the principle of inclusion–exclusion to averaging problems, and give examples. The main message here is that to compute averages, one does *not* need first to find the number of objects with exactly j properties and then sum, because the averages themselves are directly available from the sieve formalism.

Let Ω be a set of n objects, and let S be a set of properties of these objects. We let $N(\supseteq S)$ (*resp.* $N(= S)$) denote the number of objects whose set of properties contains S (*resp.* is equal to S). Further we let

$$N(m) = \sum_{|S|=m} N(\supseteq S) \qquad (m \geqslant 0), \tag{1}$$

$$E(m) = \sum_{|S|=m} N(= S) \qquad (m \geqslant 0), \tag{2}$$

so that $E(m)$ is the number of objects with exactly m properties.

Finally, for each object $\omega \in \Omega$ we let $j(\omega)$ denote the number of properties which ω has.

Now consider a fixed object $\omega \in \Omega$. Its contribution to the right side of (1) is $\binom{j(\omega)}{m}$ and so

$$N(m) = \sum_{j \geqslant 0} \binom{j}{m} E(j). \tag{3}$$

Before proceeding with our discussion of averages, we pause to note that (3) gives a quick proof of the extended principle of inclusion–exclusion. Indeed, if we define

$$\tilde{N}(t) = \sum N(m)\, t^m, \qquad \tilde{E}(t) = \sum E(m)\, t^m$$

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then (3) reads

$$\tilde{N}(t) = \tilde{E}(t + 1)$$

whence

$$\tilde{E}(t) = \tilde{N}(t - 1). \tag{4}$$

Equation (4) is the extended sieve. With $t = 0$ it is the usual PIE, and by matching coefficients of powers of t we have at once the $E(m)$ in terms of the $N(m)$.

Now, for any function $f: \Omega \to \mathbb{R}$ we define the expectation of f as

$$\langle f \rangle = \frac{1}{n} \sum_{\Omega} f(\omega).$$

In Eq. (3), divide by n and obtain

$$\left\langle \binom{j(\omega)}{m} \right\rangle = \frac{N(m)}{n} \tag{5}$$

for each fixed $m = 1, 2, 3, \ldots$. Taking $m = 1$, for example, we find that the average number of properties which an object has is

$$\langle j(\omega) \rangle = N(1)/n \tag{6}$$

while with $m = 2$ we find that the mean square is

$$\langle j(\omega)^2 \rangle = [N(1) + 2N(2)]/n \tag{7}$$

and so the standard deviation is

$$\sigma = (1/n)\{nN(1) + 2nN(2) - N(1)^2\}^{1/2}. \tag{8}$$

In fact, however, (5) provides an explicit formula for all of the power means

$$\mu_p = \langle j(\omega)^p \rangle \qquad (p = 1, 2, \ldots) \tag{9}$$

which is obtained by inverting the binomial series (5), yielding the formula

$$\mu_p = \langle j(\omega)^p \rangle = \frac{1}{n} \sum_{k=0}^{p} \left\{ {p \atop k} \right\} k!\, N(k) \tag{10}$$

where the { } are the Stirling numbers of the first kind.

So far the results are general, but now we give some examples, to fixed points of permutations, subwords of words, balls in cells, and m-cycles of permutations.

First, for n given, let the objects and properties be permutations and fixed points, as usual. Then $N(\supseteq S) = (n - |S|)!$ if $|S| \leqslant n$ and so

$$N(m) = n!/m! \qquad (0 \leqslant m \leqslant n). \tag{11}$$

Now from (5), the average number of fixed points is 1, while $\sigma = 1$ from (8). Finally, substitution of (11) into (10) yields for the power means

$$\langle j(\omega)^p \rangle = \sum_{k=0}^{\min(n,p)} \left\{ {p \atop k} \right\} \qquad (p = 1, 2, \ldots). \tag{12}$$

If $p \leqslant n$, the right side is the number b_p of partitions of a set of p elements. We have therefore the

THEOREM. *For $p \leqslant n$, the mean pth power of the number of fixed points of a permutation of n letters is equal to the Bell number b_p, and* (12) *holds in any case.*

This result was derived by Goldman [2] by a different argument (see also [1]).

As a second application, let $\nu_1, \ldots, \nu_n$ be given nonnegative integers with $N = \nu_1 + \cdots + \nu_n$. Suppose we are given ν_1 copies of the letter $1, \ldots, \nu_n$ copies of n. Among the $\binom{N}{\vec{\nu}}$ distinct words which can be made from all of these letters, what is the average number of appearances of the subword 12? More generally, given a distinguished subword W, what is the average number of appearances of W? (For a related problem see [3].)

Suppose W has μ_1 1's, μ_2 2's,..., and M letters altogether. Suppose further that a word has property i if the distinguished subword W appears in positions $i, i+1, \ldots, i+M-1$. Then

$$N(\supseteq\{i\}) = \begin{cases} \binom{N-M}{\vec{\nu}-\vec{\mu}} & \text{if } 1 \leqslant i \leqslant N - M + 1 \\ 0 & \text{otherwise} \end{cases}$$

whence

$$N(1) = (N - M + 1)\binom{N-M}{\vec{\nu}-\vec{\mu}}.$$

It follows from (6) that W appears an average of

$$(N - M + 1)\binom{N-M}{\vec{\nu}-\vec{\mu}} \Big/ \binom{N}{\vec{\nu}} \tag{14}$$

times per word. In particular, the subword 12 occurs an average of $\nu_1\nu_2/N$ times per word.

Next, in the classical occupancy problem, where r different balls are arranged in n different cells, let property i hold if cell i is empty. Then

$$N(\supseteq S) = (n - |S|)^r,$$

$$N(k) = \binom{n}{k}(n-k)^r \qquad (k \leqslant n).$$

Hence from (6) there are on the average

$$\frac{N(1)}{n^r} = \left(1 - \frac{1}{n}\right)^r n$$

empty cells. The power moments of the number of empty cells are from (10),

$$\mu_p = n! \sum_{k=0}^{\min(p,n)} \left\{ {p \atop k} \right\} \frac{(1 - k/n)^r}{(n-k)!} \qquad (p = 1, 2, \ldots). \tag{15}$$

As a final example of the method, consider, for fixed $m \leqslant n$, the distribution of m-cycles in permutations of n letters. Our objects ω are the $n!$ permutations. For each m-subset $T \subseteq \{1, 2, \ldots, n\}$ we say that a permutation ω has property T if ω has an m-cycle whose (unordered) elements are precisely those of T.

For a fixed collection $T_1, \ldots, T_l$ of m-subsets, the number of objects which have at least this collection of properties is 0 if the subsets fail to be pairwise disjoint and is easily seen to be

$$\frac{(m-1)!^l\,(n - lm)!}{l!} \qquad \left(l \leqslant \frac{n}{m}\right)$$

otherwise.

It follows that

$$N(l) = \frac{n!}{m!^l\,(n-lm)!} \frac{(m-1)!^l\,(n-lm)!}{l!}$$

$$= \frac{n!}{m^l l!} \qquad \left(l \leqslant \frac{n}{m}\right).$$

The average number of m-cycles in an n-permutation is, from (6),

$$\frac{N(1)}{n!} = \frac{1}{m} \tag{16}$$

while from (8) we find the standard deviation

$$\sigma = \frac{1}{m^{1/2}}. \tag{17}$$

Indeed, from (10) we obtain all of the power moments

$$\mu_p = \sum_{k=0}^{\min(p,\lfloor n/m \rfloor)} \left\{ {p \atop k} \right\} m^{-k} \qquad (p = 1, 2, \ldots) \tag{18}$$

which generalizes the case $m = 1$, of Eq. (12).

References

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