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Fractional Poincaré inequalities for general measures

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Abstract

We prove a fractional version of Poincaré inequalities in the context of $\mathbb{R}^n$ endowed with a fairly general measure. Namely we prove a control of an L^2 norm by a non-local quantity, which plays the role of the gradient in the standard Poincaré inequality. The assumption on the measure is the fact that it satisfies the classical Poincaré inequality, so that our result is an improvement of the latter inequality. Moreover we also quantify the tightness at infinity provided by the control on the fractional derivative in terms of a weight growing at infinity. The proof goes through the introduction of the generator of the Ornstein–Uhlenbeck semigroup and some careful estimates of its powers. To our knowledge this is the first proof of fractional Poincaré inequality for measures more general than Lévy measures.

Résumé

On montre une version fractionnaire des inégalités de Poincaré dans $\mathbb{R}^n$ muni d'une mesure assez générale. Plus précisément, on dèmontre le contrôle d'une norme L^2 par une quantité non locale, qui joue le rôle du gradient dans l'inégalité de Poincaré standard. L'hypothèse sur la mesure est qu'elle vérifie l'inégalité de Poincaré classique, de sorte que notre résultat améliore cette inégalité. De plus, on quantifie la tension à l'infini donnée par le contrôle par les dérivées fractionnaires à l'aide d'un poids dont on connaît la croissance à l'infini. La démonstration utilise le générateur d'un semi-groupe d'Ornstein–Uhlenbeck et des estimations précises de ses puissances. Il s'agit, à notre connaissance, de la première démonstration d'une inégalité de Poincaré fractionnaire pour des mesures plus générales que les mesures de Lévy.

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1. Introduction

The aim of this paper is to prove a Poincaré inequality on $\mathbb{R}^n$, endowed with a measure $M(x)\,dx$, involving non-local quantities in the right-hand side in the spirit of Gagliardo semi-norms for Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ with fractional order $s \in (0,1)$ (see e.g. [1]).

Fractional diffusions naturally appear in many models, ranging from plasma turbulence [2] or geostrophic flows [3] in fluid dynamics, grazing collisions in kinetic theory (cf. the Boltzmann equation for long-range interactions [4–7]), all the way to stockmarket modeling based on Lévy processes [8]. They also appear naturally in mathematics: in probability they appear in the important class of infinitely divisible Markov processes given (cf. the Lévy–Khinchin representation [9]); in analysis they naturally appear in the study of singular integral operators (e.g. for the Boltzmann equation, cf. references above) as well as in the so-called "Dirichlet-to-Neuman" boundary value problem and in the Signorini (obstacle) problem [10] (see for instance among other references [11] and [12]). The search for a Poincaré inequality for the non-local fractional energy associated with such fractional diffusion is therefore a natural and interesting question since this inequality governs the spectral gap of the underlying operator and the speed of (fractional) diffusion towards an equilibrium.

Throughout this paper, we denote by M a positive weight in $L^1(\mathbb{R}^n)$. In the sequel, by $L^2(\mathbb{R}^n, M)$, we mean the space of measurable functions on $\mathbb{R}^n$ which are square integrable with respect to the measure $M(x)\,dx$, by $L^2_0(\mathbb{R}^n, M)$ the subspace of functions of $L^2(\mathbb{R}^n, M)$ such that $\int_{\mathbb{R}^n} f(x)M(x)\,dx = 0$, and by $H^1(\mathbb{R}^n, M)$, the Sobolev space of functions in $L^2(\mathbb{R}^n, M)$, the weak derivative of which belongs to $L^2(\mathbb{R}^n, M)$. Finally for any measurable subset $A \subset \mathbb{R}^n$ by $L^2(A, M)$ we mean the obvious restriction of the definition above to the set A.

We assume that M is a C^2 function and that this measure M satisfies the usual Poincaré inequality: there exists a constant $\lambda(M) > 0$ such that $\forall f \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} \left|\nabla f(y)\right|^2 M(y)\,dy \geqslant \lambda(M) \int_{\mathbb{R}^n} \left| f(y) - \int_{\mathbb{R}^n} f(x)M(x)\,dx \right|^2 M(y)\,dy. \tag{1}$$

If the measure M can be written $M = e^{-V}$, this inequality is known to hold (see [13], or also [14], Appendix A.19, Theorem 1.2, see also [15], proof of Theorem 6.2.21 for related criteria) whenever there exist $a \in (0,1)$, $c > 0$ and $R > 0$ such that

$$\forall |x| \geqslant R, \quad a\left|\nabla V(x)\right|^2 - \Delta V \geqslant c. \tag{2}$$

In particular, the inequality (1) holds, for instance, when $M = (2\pi)^{-n/2}\exp(-|x|^2/2)$ is the Gaussian measure, but also when $M(x) = e^{-|x|}$, and more generally when $M(x) = e^{-|x|^\alpha}$ with $\alpha \geqslant 1$. Note that, when V is convex, and

$$\mathrm{Hess}(V) \geqslant \mathrm{cst}\,\mathrm{Id}$$

on the set where $|V| < +\infty$, the measure $M(x)\,dx$ satisfies the log-Sobolev inequality, which in turn implies (1) (see [16]).

As it shall be proved to be useful later on, remark that, under a slightly stronger assumption than (2), the Poincaré inequality (1) admits the following self-improvement:

Proposition 1.1. *Assume that there exists $\varepsilon > 0$ such that*

$$\frac{(1-\varepsilon)|\nabla V|^2}{2} - \Delta V \xrightarrow[x\to\infty]{} +\infty, \qquad M = e^{-V}. \tag{3}$$

Then there exists $\lambda'(M) > 0$ such that, for all functions $f \in L^2_0(\mathbb{R}^n, M) \cap H^1(\mathbb{R}^n, M)$:

$$\iint_{\mathbb{R}^n} \left|\nabla f(x)\right|^2 M(x)\,dx \geqslant \lambda'(M) \int_{\mathbb{R}^n} \left|f(x)\right|^2 \left(1 + \left|\nabla \ln M(x)\right|^2\right) M(x)\,dx. \tag{4}$$

The proof of Proposition 1.1 is classical and will be given in Appendix A for the sake of completeness.

We want to generalize the inequality (1) in the strengthened form of Proposition 1.1, replacing, in the left-hand side, the H^1 semi-norm by a non-local expression in the flavour of the Gagliardo semi-norms.

We establish the following theorem:

Theorem 1.2. *Assume that $M = e^{-V}$ is a C^2 positive L^1 function which satisfies* (3). *Let $\alpha \in (0, 2)$. Then there exist $\lambda_\alpha(M) > 0$ and $\delta(M)$ (constructive from our proof and the usual Poincaré constant $\lambda'(M)$) such that, for any function f belonging to a dense subspace of $L^2_0(\mathbb{R}^n, M)$, we have*:

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x) e^{-\delta(M)|x-y|}\,dx\,dy \geqslant \lambda_\alpha(M) \int_{\mathbb{R}^n} \left|f(x)\right|^2 \left(1 + \left|\nabla \ln M(x)\right|^\alpha\right) M(x)\,dx. \tag{5}$$

Remark 1.3. Inequality (5) could as usual be extended to any function f with zero average such that both sides of the inequality make sense. In particular it is satisfied for any function f with zero average belonging to the domain of the operator $L = -\Delta - \nabla V \cdot \nabla$ that we shall introduce later on. Functions of this domain with zero integral with respect to $M(x)\,dx$ are dense in $L^2_0(\mathbb{R}^n, M)$.

Observe that the right-hand side of (5) involves a fractional moment of order α related to the homogeneity of the semi-norm appearing in the left-hand side. One could expect in the left-hand side of (5) the Gagliardo semi-norm for the fractional Sobolev space $H^{\alpha/2}(\mathbb{R}^n, M)$, namely

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)M(y)\,dx\,dy.$$

Notice that, instead of this semi-norm, we obtain a "non-symmetric" expression. However, our norm is more natural: one should think of the measure over y as the Lévy measure, and the measure over x as the ambient measure. We emphasize on the fact that our measure is rather general and in particular, as a corollary of Theorem 1.2, we obtain an automatic improvement of the Poincaré inequality (1) by:

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy \geqslant \lambda_\alpha(M) \int_{\mathbb{R}^n} \left|f(x)\right|^2 M(x)\,dx.$$

The question of obtaining Poincaré-type inequalities (or more generally entropy inequalities) for Lévy operators was studied in the probability community in the last decades. For instance it was proved by Wu [17] and Chafaï [18] that

$$\mathrm{Ent}^{\Phi}_{\mu}(f) \leqslant \int \Phi''(f)\nabla f \cdot \sigma \cdot \nabla f\,d\mu + \iint D_\Phi\big(f(x), f(x+z)\big)\,d\nu_\mu(z)\,d\mu(x)$$

(see also the use of this inequality in [19]) with

$$\mathrm{Ent}^{\Phi}_{\mu}(f) = \int \Phi(f)\,d\mu - \Phi\left(\int f\,d\mu\right),$$

and D_Φ is the so-called Bregman distance associated to Φ:

$$D_\Phi(a,b) = \Phi(a) - \Phi(b) - \Phi'(b)(a-b),$$

where Φ is some well-suited functional with convexity properties, σ the matrix of diffusion of the process, μ a rather general measure, and ν_μ the (singular) Lévy measure associated to μ. Choosing $\Phi(x) = x^2$ and $\sigma = 0$ yields a Poincaré inequality for this choice of measure (μ, ν_μ). The improvement of our approach is that we do not impose any link between our measure M on x and the singular measure $|z|^{-n-\alpha}$ on $z = x - y$. This is to our knowledge the first result that gets rid of this strong constraint.

Remark 1.4. Note that the exponentially decaying factor $e^{-\delta(M)|x-y|}$ in (5) also *improves* the inequality as compared to what is expected from Poincaré inequality for Lévy measures. This decay on the diagonal could most probably be further improved, as shall be studied in future works. Other extensions in progress are to allow more general singularities than the Martin Riesz kernel $\frac{1}{|x-y|^{n+\alpha}}$ (see the book [20]) and to develop an L^p theory of the previous inequalities.

Our proof heavily relies on fractional powers of a (suitable generalization of the) Ornstein–Uhlenbeck operator, which is defined by:

$$Lf = -M^{-1}\operatorname{div}(M\nabla f) = -\Delta f - \nabla \ln M \cdot \nabla f,$$

for all $f \in \mathcal{D}(L) := \{g \in H^1(\mathbb{R}^n, M);\ (1/\sqrt{M})\operatorname{div}(M\nabla g) \in L^2(\mathbb{R}^n)\}$. One therefore has, for all $f \in \mathcal{D}(L)$ and $g \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} Lf(x)g(x)M(x)\,dx = \int_{\mathbb{R}^n} \nabla f(x)\cdot\nabla g(x)M(x)\,dx.$$

It is obvious that L is symmetric and nonnegative on $L^2(\mathbb{R}^n, M)$, which allows to define the usual power L^β for any $\beta \in (0,1)$ by means of spectral theory. Note that $L^{\alpha/2}$ is *not* the symmetric operator associated to the Dirichlet form $\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy$.

We now describe the strategy of our proofs. The proof of Theorem 1.2 goes in three steps. We first establish L^2 off-diagonal estimates of Gaffney type on the resolvent of L on $L^2(\mathbb{R}^n, M)$. These estimates are needed in our context since we do not have Gaussian pointwise estimates on the kernel of the operator L.

Then, we bound the quantity,

$$\int_{\mathbb{R}^n} \big|f(x)\big|^2\big(1 + \big|\nabla \ln M(x)\big|^\alpha\big)M(x)\,dx,$$

in terms of $\|L^{\alpha/4} f\|^2_{L^2(\mathbb{R}^n,M)}$. This will be obtained by an abstract argument of functional calculus based on rewriting in a suitable way the conclusion of Proposition 1.1. Finally, using the L^2 off-diagonal estimates for the kernel of L, we establish that

$$\big\|L^{\alpha/4} f\big\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy,$$

which concludes the proof.

As can be seen from the rough sketch previously described, we borrow methods from harmonic analysis. This seems not so common in the field of Poincaré and log-Sobolev inequalities (to the knowledge of the authors), where standard techniques rely on global functional inequalities, see for instance the powerful so-called Γ_2-calculus of Bakry and Émery [21]. We hope this paper will stimulate further exchanges between these two fields.

2. Off-diagonal L^2 estimates for the resolvent of L

We recall that for every $f \in \mathcal{D}(L)$, we define

$$Lf = -M^{-1}\operatorname{div}(M\nabla f) = -\Delta f - \nabla \ln M \cdot \nabla f. \tag{6}$$

From the fact that L is self-adjoint and nonnegative on $L^2(\mathbb{R}^n, M)$ we have:

$$\big\|(L-\mu)^{-1}\big\|_{L^2(\mathbb{R}^n,M)} \leqslant \frac{1}{\operatorname{dist}(\mu, \Sigma(L))}$$

where $\Sigma(L)$ denotes the spectrum of L, and $\mu \notin \Sigma(L)$. Then we deduce that $(\mathrm{I}+tL)^{-1}$ is bounded with norm less than 1 for all $t > 0$. Since $tL(\mathrm{I}+tL)^{-1} = \mathrm{I} - (\mathrm{I}+tL)^{-1}$, the same is true for $tL(\mathrm{I}+tL)^{-1} = \mathrm{I} - (\mathrm{I}+tL)^{-1}$ with a norm less than 2. Moreover, $(\mathrm{I}+tL)^{-1} f \in H^1(\mathbb{R}^n, M)$.

Actually, when $f \in L^2(\mathbb{R}^n, M)$ is supported in a closed set $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n$ is a closed subset disjoint from E, a much more precise estimate on the L^2 norm of $(\mathrm{I}+tL)^{-1} f$ and $tL(\mathrm{I}+tL)^{-1} f$ on F can be given. Here are these L^2 off-diagonal estimates for the resolvent of L:

Lemma 2.1. *There exists $C_1 = C_1(M) > 0$ (constructive from our proof) with the following property: for all compact disjoint subsets $E, F \subset \mathbb{R}^n$, F bounded, with* $\operatorname{dist}(E,F) =: d > 0$*, all functions $f \in L^2(\mathbb{R}^n, M)$ supported in E and all $t > 0$,*

$$\left\|(\mathrm{I}+tL)^{-1} f\right\|_{L^2(F,M)} + \left\|tL(\mathrm{I}+tL)^{-1} f\right\|_{L^2(F,M)} \leqslant 8e^{-C_1 \frac{d}{\sqrt{t}}} \|f\|_{L^2(E,M)}.$$

Note that, in different contexts, this kind of estimate, originating in [22], turns out to be a powerful tool, especially when no pointwise upper estimate on the kernel of the semigroup generated by L is available (see for instance [23–25]). Since we found no reference for these off-diagonal estimates for the resolvent of L, we give here a proof.

Proof of Lemma 2.1. We argue as in [24, Lemma 1.1]. Since $(\mathrm{I}+tL)^{-1}$ is bounded with norm less than 1 for all $t > 0$ it is clearly enough to restrict to $0 < t < d$.

Define $u_t = (\mathrm{I}+tL)^{-1} f$, so that, for all functions $v \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} u_t(x)v(x)M(x)\,dx + t\int_{\mathbb{R}^n} \nabla u_t(x)\cdot\nabla v(x)M(x)\,dx = \int_{\mathbb{R}^n} f(x)v(x)M(x)\,dx. \tag{7}$$

Fix now a nonnegative function $\eta \in \mathcal{D}(\mathbb{R}^n)$ vanishing on E. Since f is supported in E, applying (7) with $v = \eta^2 u_t$ (remember that $u_t \in H^1(\mathbb{R}^n, M)$) yields,

$$\int_{\mathbb{R}^n} \eta^2(x)\left|u_t(x)\right|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \nabla u_t(x)\cdot\nabla\left(\eta^2 u_t\right)M(x)\,dx = 0,$$

which implies:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta^2(x)\left|u_t(x)\right|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \eta^2(x)\left|\nabla u_t(x)\right|^2 M(x)\,dx \\
&\quad = -2t\int_{\mathbb{R}^n} \eta(x)u_t(x)\nabla\eta(x)\cdot\nabla u_t(x)M(x)\,dx \\
&\quad \leqslant t\int_{\mathbb{R}^n} \left|u_t(x)\right|^2\left|\nabla\eta(x)\right|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \eta^2(x)\left|\nabla u_t(x)\right|^2 M(x)\,dx,
\end{aligned}$$

hence

$$\int_{\mathbb{R}^n} \eta^2(x)\left|u_t(x)\right|^2 M(x)\,dx \leqslant t\int_{\mathbb{R}^n} \left|u_t(x)\right|^2\left|\nabla\eta(x)\right|^2 M(x)\,dx. \tag{8}$$

Let ζ be such that $\zeta = 0$ on E and ζ nonnegative so that $\eta := e^{\alpha\zeta} - 1 \geqslant 0$ and η vanishes on E for some $\alpha > 0$ to be chosen. Choosing this particular η in (8) with $\alpha > 0$ gives:

$$\int_{\mathbb{R}^n} \left|e^{\alpha\zeta(x)} - 1\right|^2\left|u_t(x)\right|^2 M(x)\,dx \leqslant \alpha^2 t\int_{\mathbb{R}^n} \left|u_t(x)\right|^2\left|\nabla\zeta(x)\right|^2 e^{2\alpha\zeta(x)} M(x)\,dx.$$

Taking $\alpha = 1/(2\sqrt{t}\|\nabla\zeta\|_\infty)$, one obtains:

$$\int_{\mathbb{R}^n} \left|e^{\alpha\zeta(x)} - 1\right|^2\left|u_t(x)\right|^2 M(x)\,dx \leqslant \frac{1}{4}\int_{\mathbb{R}^n} \left|u_t(x)\right|^2 e^{2\alpha\zeta(x)} M(x)\,dx.$$

Using the fact that the norm of $(I+tL)^{-1}$ is bounded by 1 uniformly in $t > 0$, this gives:

$$\begin{aligned}
\left\|e^{\alpha\zeta} u_t\right\|_{L^2(\mathbb{R}^n,M)} &\leqslant \left\|\left(e^{\alpha\zeta} - 1\right)u_t\right\|_{L^2(\mathbb{R}^n,M)} + \|u_t\|_{L^2(\mathbb{R}^n,M)} \\
&\leqslant \frac{1}{2}\left\|e^{\alpha\zeta} u_t\right\|_{L^2(\mathbb{R}^n,M)} + \|f\|_{L^2(\mathbb{R}^n,M)},
\end{aligned}$$

therefore

$$\int_{\mathbb{R}^n} \left|e^{\alpha\zeta(x)}\right|^2 \left|u_t(x)\right|^2 M(x)\,dx \leqslant 4 \int_{\mathbb{R}^n} \left|f(x)\right|^2 M(x)\,dx.$$

We choose now ζ such that $\zeta = 0$ on E as before and additionally that $\zeta = 1$ on F (η is then compactly supported from the fact that F is bounded). It can trivially be chosen with $\|\nabla\zeta\|_\infty \leqslant C/d$, which yields the desired conclusion for the L^2 norm of $(I + tL)^{-1} f$ with a factor 4 in the right-hand side. Since $tL(\mathrm{I} + tL)^{-1} f = f - (\mathrm{I} + tL)^{-1} f$, the desired inequality with a factor 8 readily follows. □

Remark 2.2. Arguing similarly, we could also obtain analogous gradient estimates for $\|\sqrt{t}\nabla(\mathrm{I} + tL)^{-1} f\|_{L^2(F,M)}$.

3. Control of $\|L^{\alpha/4} f\|_{L^2(\mathbb{R}^n,M)}$

This section is devoted to the control of the L^2 norm of fractional powers of L. This is the cornerstone of the proof of Theorem 1.2. In the functional calculus theory of sectorial operators L, fractional powers (for the particular powers we are interested in) are defined as follows (see for instance [26, p. 24]):

$$\forall\beta \in (0, 1), \quad L^\beta f = \frac{1}{\Gamma(1-\beta)} \int_0^\infty t^{-\beta} L e^{-Lt} f\,dt. \tag{9}$$

They can also be defined in terms of the resolvent by the Balakrishnan formulation (see for instance [26, p. 25]):

$$\forall\beta \in (0, 1), \quad L^\beta f = \frac{\sin(\pi(1-\beta))}{\pi} \int_0^\infty \lambda^{\beta-1} L(L + \lambda\mathrm{I})^{-1} f\,d\lambda. \tag{10}$$

We shall in fact not need any of the representations (9) or (10); instead we shall rely on the powerful tool of the so-called "quadratic estimates" obtained in the functional calculus. This is the object of the next lemma.

Lemma 3.1. *Let $\alpha \in (0, 2)$. There exists $C_3 = C_3(M) > 0$ such that, for all $f \in \mathcal{D}(L)$,*

$$\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C_3 \int_0^{+\infty} t^{-1-\alpha/2} \left\|tL(\mathrm{I} + tL)^{-1} f\right\|^2_{L^2(\mathbb{R}^n,M)}\,dt. \tag{11}$$

Proof. Let $\mu \in (0, \frac{\pi}{2})$, and

$$\Sigma_{\mu^+} = \left\{z \in \mathbb{C}^*;\ |\arg z| < \mu\right\}.$$

Let ψ be a holomorphic function in $H^\infty(\Sigma_{\mu^+})$ such that for some $C, \sigma, \tau > 0$,

$$\left|\psi(z)\right| \leqslant C \inf\left\{|z|^\sigma;\ |z|^{-\tau}\right\},$$

for any $z \in \Sigma_{\mu^+}$. Since L is positive self-adjoint operator on $L^2(\mathbb{R}^n, M)$ and L is one-to-one on $L^2_0(\mathbb{R}^n, M)$ by (1), one has by the spectral theorem,

$$\|F\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C \int_0^{+\infty} \left\|\psi(tL)F\right\|^2_{L^2(\mathbb{R}^n,M)} \frac{dt}{t}$$

whenever $F \in L^2_0(\mathbb{R}^n, M)$. Choosing $\psi(z) = z^{1-\alpha/4}/(1+z)$ yields,

$$\|F\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C \int_0^{+\infty} \left\|(tL)^{1-\alpha/4}(\mathrm{I} + tL)^{-1} F\right\|^2_{L^2(\mathbb{R}^n,M)} \frac{dt}{t}, \tag{12}$$

whenever $F \in L^2_0(\mathbb{R}^n, M)$.

Let $f \in L^2(\mathbb{R}^n, M)$. Since

$$\int_{\mathbb{R}^n} Lf(x)M(x)\,dx = 0,$$

it follows from (9) that the same is true with $L^{\alpha/4}f$. Applying now (12) with $F = L^{\alpha/4}f$ gives the conclusion of Lemma 3.1. □

Let us draw a simple corollary of Lemma 3.1:

Corollary 3.2. *For any $\alpha \in (0,2)$ and $\varepsilon > 0$, there is $A = A(M,\varepsilon)$ such that*

$$\|L^{\alpha/4}f\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C_3 \int_0^A t^{-1-\alpha/2}\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\,dt + \varepsilon\|f\|^2_{L^2(\mathbb{R}^n,M)}. \tag{13}$$

Proof. The proof is straightforward since

$$\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)} \leqslant C\|f\|^2_{L^2(\mathbb{R}^n,M)}$$

and

$$\int_A^{+\infty} t^{-1-\alpha/2}\,dt \xrightarrow[A\to+\infty]{} 0. \qquad \square$$

We now come to the desired estimate.

Lemma 3.3. *Let $\alpha \in (0,2)$ and ε and A given by Corollary 3.2. There exist $C_4 = C_4(M,A) > 0$ and $c' = c'(M,A) > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)$,*

$$\int_0^A t^{-1-\alpha/2}\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\,dt \leqslant C_4 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)e^{-c'|x-y|}\,dx\,dy.$$

Proof. Throughout this proof, for all $x \in \mathbb{R}^n$ and all $s > 0$, denote by $Q(x,s)$ the closed cube centered at x with side length s. For fixed $t \in (0,A)$, following Lemma 3.1, we shall look for an upper bound for $\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}$ involving first order differences for f. Pick up a countable family of points $x_j^t \in \mathbb{R}^n$, $j \in \mathbb{N}$, such that the cubes $Q(x_j^t, \sqrt{t})$ have pairwise disjoint interiors, and

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} Q\big(x_j^t, \sqrt{t}\big). \tag{14}$$

By Lemma B.1 in Appendix B, there exists a constant $\tilde{C} > 0$ such that for all $\theta > 1$ and all $x \in \mathbb{R}^n$, there are at most $\tilde{C}\theta^n$ indexes j such that $|x - x_j^t| \leqslant \theta\sqrt{t}$.

For fixed j, one has

$$tL(\mathrm{I}+tL)^{-1}f = tL(\mathrm{I}+tL)^{-1}g^{j,t},$$

where, for all $x \in \mathbb{R}^n$,

$$g^{j,t}(x) := f(x) - m^{j,t}$$

and $m^{j,t}$ is defined by:

$$m^{j,t} := \frac{1}{|Q(x_j^t, 2\sqrt{t})|} \int_{Q(x_j^t, 2\sqrt{t})} f(y)\,dy.$$

Note that, here, the mean value of f is computed with respect to the Lebesgue measure on $\mathbb{R}^n$. Since (14) holds and the cubes $Q(x_j^t, \sqrt{t})$ have pairwise disjoint interiors, one clearly has:

$$\begin{aligned}\left\|tL(\mathrm{I}+tL)^{-1} f\right\|^2_{L^2(\mathbb{R}^n, M)} &= \sum_{j\in\mathbb{N}} \left\|tL(\mathrm{I}+tL)^{-1} f\right\|^2_{L^2(Q(x_j^t, \sqrt{t}), M)} \\ &= \sum_{j\in\mathbb{N}} \left\|tL(\mathrm{I}+tL)^{-1} g^{j,t}\right\|^2_{L^2(Q(x_j^t, \sqrt{t}), M)},\end{aligned}$$

and we are left with the task of estimating,

$$\left\|tL(\mathrm{I}+tL)^{-1} g^{j,t}\right\|^2_{L^2(Q(x_j^t, \sqrt{t}), M)}.$$

To that purpose, set

$$C_0^{j,t} = Q\big(x_j^t, 2\sqrt{t}\big) \quad \text{and} \quad C_k^{j,t} = Q\big(x_j^t, 2^{k+1}\sqrt{t}\big) \setminus Q\big(x_j^t, 2^k\sqrt{t}\big), \quad \forall k \geqslant 1,$$

and $g_k^{j,t} := g^{j,t}\mathbf{1}_{C_k^{j,t}}$, $k \geqslant 0$, where, for any subset $A \subset \mathbb{R}^n$, $\mathbf{1}_A$ is the usual characteristic function of A. Since $g^{j,t} = \sum_{k\geqslant 0} g_k^{j,t}$ one has:

$$\left\|tL(\mathrm{I}+tL)^{-1} g^{j,t}\right\|_{L^2(Q(x_j^t, \sqrt{t}), M)} \leqslant \sum_{k\geqslant 0} \left\|tL(\mathrm{I}+tL)^{-1} g_k^{j,t}\right\|_{L^2(Q(x_j^t, \sqrt{t}), M)} \tag{15}$$

and, using Lemma 2.1, one obtains (for some constants $C, c > 0$):

$$\left\|tL(\mathrm{I}+tL)^{-1} g^{j,t}\right\|_{L^2(Q(x_j^t, \sqrt{t}), M)} \leqslant C\bigg(\left\|g_0^{j,t}\right\|_{L^2(C_0^{j,t}, M)} + \sum_{k\geqslant 1} e^{-c2^k} \left\|g_k^{j,t}\right\|_{L^2(C_k^{j,t}, M)}\bigg). \tag{16}$$

By Cauchy–Schwarz's inequality, we deduce (for another constant $C' > 0$):

$$\left\|tL(\mathrm{I}+tL)^{-1} g^{j,t}\right\|^2_{L^2(Q(x_j^t, \sqrt{t}), M)} \leqslant C'\bigg(\left\|g_0^{j,t}\right\|^2_{L^2(C_0^{j,t}, M)} + \sum_{k\geqslant 1} e^{-c2^k} \left\|g_k^{j,t}\right\|^2_{L^2(C_k^{j,t}, M)}\bigg). \tag{17}$$

As a consequence, we have:

$$\begin{aligned}\int_0^A t^{-1-\alpha/2} \left\|tL(\mathrm{I}+tL)^{-1} f\right\|^2_{L^2(\mathbb{R}^n, M)}\,dt &\leqslant C' \int_0^A t^{-1-\alpha/2} \sum_{j\geqslant 0} \left\|g_0^{j,t}\right\|^2_{L^2(C_0^{j,t}, M)}\,dt \\ &\quad + C' \int_0^A t^{-1-\alpha/2} \sum_{k\geqslant 1} e^{-c2^k} \sum_{j\geqslant 0} \left\|g_k^{j,t}\right\|^2_{L^2(C_k^{j,t}, M)}\,dt.\end{aligned} \tag{18}$$

We claim that

Lemma 3.4. *There exists $\bar{C} > 0$ such that, for all $t > 0$ and all $j \in \mathbb{N}$:*

A. *For the first term*:

$$\left\|g_0^{j,t}\right\|^2_{L^2(C_0^{j,t}, M)} \leqslant \frac{\bar{C}}{t^{n/2}} \int_{Q(x_j^t, 2\sqrt{t})} \int_{Q(x_j^t, 2\sqrt{t})} \left|f(x) - f(y)\right|^2 M(x)\,dx\,dy.$$

B. *For all* $k \geqslant 1$,

$$\|g_k^{j,t}\|^2_{L^2(C_k^{j,t},M)} \leqslant \frac{\bar{C}}{(\sqrt{t})^n} \int\limits_{x\in Q(x_j^t,2^{k+1}\sqrt{t})} \int\limits_{y\in Q(x_j^t,2^{k+1}\sqrt{t})} \left|f(x)-f(y)\right|^2 M(x)\,dx\,dy.$$

We postpone the proof to the end of the section and finish the proof of Lemma 3.3. Using Assertion A in Lemma 3.4, summing up on $j \geqslant 0$ and integrating over $(0, A)$, we get:

$$\begin{aligned}
&\int_0^A t^{-1-\alpha/2} \sum_{j\geqslant 0} \|g_0^{j,t}\|^2_{L^2(C_0^{j,t},M)}\,dt \\
&\quad = \sum_{j\geqslant 0} \int_0^A t^{-1-\alpha/2} \|g_0^{j,t}\|^2_{L^2(C_0^{j,t},M)}\,dt \\
&\quad \leqslant \bar{C} \sum_{j\geqslant 0} \int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}} \left(\int\limits_{Q(x_j^t,2\sqrt{t})} \int\limits_{Q(x_j^t,2\sqrt{t})} \left|f(x)-f(y)\right|^2 M(x)\,dx\,dy \right) dt \\
&\quad \leqslant \bar{C} \sum_{j\geqslant 0} \iint\limits_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \left|f(x)-f(y)\right|^2 M(x) \left(\int\limits_{t\geqslant \max\{\frac{|x-x_j^t|^2}{n};\frac{|y-x_j^t|^2}{n}\}}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\,dt \right) dx\,dy.
\end{aligned}$$

The Fubini theorem now shows:

$$\sum_{j\geqslant 0} \int\limits_{t\geqslant \max\{\frac{|x-x_j^t|^2}{n};\frac{|y-x_j^t|^2}{n}\}}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\,dt = \int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}} \sum_{j\geqslant 0} \mathbf{1}_{(\max\{\frac{|x-x_j^t|^2}{n};\frac{|y-x_j^t|^2}{n}\},+\infty)}(t)\,dt.$$

Observe that, by Lemma B.1, there is a constant $N \in \mathbb{N}$ such that, for all $t > 0$, there are at most N indexes j such that $|x - x_j^t|^2 < nt$ and $|y - x_j^t|^2 < nt$. If such an index j exists, one has $|x - y| < 2\sqrt{nt}$. It therefore follows that

$$\sum_{j\geqslant 0} \mathbf{1}_{(\max\{\frac{|x-x_j^t|^2}{n};\frac{|y-x_j^t|^2}{n}\},+\infty)}(t) \leqslant N \mathbf{1}_{(|x-y|^2/4n,+\infty)}(t),$$

so that

$$\begin{aligned}
\int_0^A t^{-1-\alpha/2} \sum_j \|g_0^{j,t}\|^2_{L^2(C_0^{j,t},M)}\,dt &\leqslant \bar{C}N \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \left|f(x)-f(y)\right|^2 M(x) \left(\int\limits_{|x-y|^2/4n}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\,dt \right) dx\,dy \\
&\leqslant \bar{C}N \iint\limits_{|x-y|\leqslant 2\sqrt{nA}} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy. \qquad (19)
\end{aligned}$$

Using now Assertion B in Lemma 3.4, we obtain, for all $j \geqslant 0$ and all $k \geqslant 1$,

$$\begin{aligned}
&\int_0^A t^{-1-\alpha/2} \sum_{j\geqslant 0} \|g_k^{j,t}\|_2^2\,dt \\
&\quad \leqslant \bar{C} \sum_{j\geqslant 0} \int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}} \left(\iint\limits_{Q(x_j^t,2^{k+1}\sqrt{t})\times Q(x_j^t,2^{k+1}\sqrt{t})} \left|f(x)-f(y)\right|^2 M(x)\,dx\,dy \right) dt
\end{aligned}$$

$$\leqslant \bar{C}\sum_{j\geqslant 0}\iint_{x,y\in\mathbb{R}^n}\left|f(x)-f(y)\right|^2 M(x)\left(\int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\mathbf{1}_{(\max\{\frac{|x-x_j^t|^2}{4^k n},\frac{|y-x_j^t|^2}{4^k n}\},+\infty)}(t)\,dt\right)dx\,dy.$$

But, given $t>0$, $x,y\in\mathbb{R}^n$, by Lemma B.1 again, there exist at most $\tilde{C}2^{kn}$ indexes j such that

$$\left|x-x_j^t\right|\leqslant 2^k\sqrt{nt}\quad\text{and}\quad\left|y-x_j^t\right|\leqslant 2^k\sqrt{nt},$$

and for these indexes j, $|x-y|\leqslant 2^{k+1}\sqrt{nt}$. As a consequence we have:

$$\begin{aligned}\int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\sum_{j\geqslant 0}\mathbf{1}_{(\max\{\frac{|x-x_j^t|^2}{4^k n},\frac{|x-x_j^t|^2}{4^k n}\},+\infty)}(t)\,dt &\leqslant \tilde{C}2^{kn}\int_{t\geqslant\frac{|x-y|^2}{4^{k+1}n}}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\,dt\\ &\leqslant \tilde{C}'2^{k(\alpha+n)}|x-y|^{-n-\alpha}\mathbf{1}_{|x-y|\leqslant 2^{k+1}\sqrt{nA}},\end{aligned}\tag{20}$$

for some other constant $\tilde{C}'>0$, and therefore

$$\int_0^A t^{-1-\alpha/2}\sum_j\left\|g_k^{j,t}\right\|^2_{L^2(C_0^{j,t},M)}dt\leqslant \bar{C}\tilde{C}'2^{k(\alpha+n)}\iint_{|x-y|\leqslant 2^{k+1}\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy.$$

We can now conclude the proof of Lemma 3.3, using Lemma 3.1, (16), (19) and (20). We have proved, by reconsidering (18):

$$\begin{aligned}\int_0^A t^{-1-\alpha/2}\left\|tL(\mathrm{I}+tL)^{-1}f\right\|^2_{L^2(\mathbb{R}^n,M)}dt &\leqslant C'\bar{C}N\iint_{|x-y|\leqslant 2\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy\\ &\quad+\sum_{k\geqslant 1}C'\bar{C}\tilde{C}'2^{k\alpha}e^{-c2^k}\iint_{|x-y|\leqslant 2^{k+1}\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy\end{aligned}\tag{21}$$

and we deduce that

$$\int_0^A t^{-1-\alpha/2}\left\|tL(\mathrm{I}+tL)^{-1}f\right\|^2_{L^2(\mathbb{R}^n,M)}dt\leqslant C_4\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)e^{-c'|x-y|}\,dx\,dy$$

for some constants C_4 and $c'>0$ as claimed in the statement. □

Proof of Lemma 3.4. Observe first that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned}g_0^{j,t}(x) &= f(x)-\frac{1}{|Q(x_j^t,2\sqrt{t})|}\int_{Q(x_j^t,2\sqrt{t})}f(y)\,dy\\ &=\frac{1}{|Q(x_j^t,2\sqrt{t})|}\int_{Q(x_j^t,2\sqrt{t})}\bigl(f(x)-f(y)\bigr)\,dy.\end{aligned}$$

By Cauchy–Schwarz inequality, it follows that

$$\left|g_0^{j,t}(x)\right|^2\leqslant\frac{C}{t^{n/2}}\int_{Q(x_j^t,2\sqrt{t})}\left|f(x)-f(y)\right|^2dy.$$

Therefore,

$$\left\|g_0^{j,t}\right\|^2_{L^2(C_0^{j,t},M)} \leqslant \frac{C}{t^{n/2}} \int\limits_{Q(x_j^t,2\sqrt{t})} \int\limits_{Q(x_j^t,2\sqrt{t})} \left|f(x)-f(y)\right|^2 M(x)\,dx\,dy,$$

which shows Assertion A. We argue similarly for Assertion B and obtain:

$$\left\|g_k^{j,t}\right\|^2_{L^2(C_k^{j,t},M)} \leqslant \frac{C}{t^{n/2}} \int\limits_{x\in Q(x_j^t,2^{k+1}\sqrt{t})} \int\limits_{y\in Q(x_j^t,2^{k+1}\sqrt{t})} \left|f(x)-f(y)\right|^2 M(x)\,dx\,dy,$$

which ends the proof of Lemma 3.4. $\square$

We end up this section with a few comments on Lemma 3.4. It is a well-known fact [27] that, when $0<\alpha<2$, for all $p\in(1,+\infty)$,

$$\left\|(-\Delta)^{\alpha/4} f\right\|_{L^p(\mathbb{R}^n)} \leqslant C_{\alpha,p}\|S_\alpha f\|_{L^p(\mathbb{R}^n)}, \tag{22}$$

where

$$S_\alpha f(x) = \left(\int\limits_0^{+\infty}\left(\int\limits_B \left|f(x+ry)-f(x)\right|dy\right)^2 \frac{dr}{r^{1+\alpha}}\right)^{\frac{1}{2}},$$

and also [28]

$$\left\|(-\Delta)^{\alpha/4} f\right\|_{L^p(\mathbb{R}^n)} \leqslant C_{\alpha,p}\|D_\alpha f\|_{L^p(\mathbb{R}^n)} \tag{23}$$

where

$$D_\alpha f(x) = \left(\int\limits_{\mathbb{R}^n} \frac{|f(x+y)-f(x)|^2}{|y|^{n+\alpha}}\,dy\right)^{\frac{1}{2}}.$$

In [29], these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-laplacian Δ, relying on semigroups techniques and Littlewood–Paley–Stein functionals. In particular, in [29], we use *pointwise* estimates of the kernel of the semigroup generated by Δ. The conclusion of Lemma 3.4 means that the norm of $L^{\alpha/4}f$ in $L^2(\mathbb{R}^n,M)$ is bounded from above by the $L^2(\mathbb{R}^n,M)$ norm of an appropriate version of D_α. Note that this does not require pointwise estimates for the kernel of the semigroup generated by L, and that the L^2 off-diagonal estimates given by Lemma 2.1, which hold for a general measure M, are enough for our argument to hold. However, we do not know if an L^p version of Lemma 3.4 still holds. Note also that we do not compare the $L^2(\mathbb{R}^n,M)$ norm of $L^{\alpha/4}f$ with the $L^2(\mathbb{R}^n,M)$ norm of a version of $S_\alpha f$. Finally, the converse inequalities to (22) and (23) hold in $\mathbb{R}^n$ and also on a unimodular Lie group [29], and we did not consider the corresponding inequalities in the present paper.

4. Control of the moment of f by $\|L^{\alpha/4}f\|_{L^2(\mathbb{R}^n,M)}$ and proof of Theorem 1.2

Observe first that, by the definition of L, we have

$$\int\limits_{\mathbb{R}^n}\left|\nabla f(x)\right|^2 M(x)\,dx = \int\limits_{\mathbb{R}^n} Lf(x)f(x)M(x)\,dx,$$

for all $f\in\mathcal{D}(L)$. The inequality (4) can therefore be rewritten, in terms of operators, as

$$L \geqslant \lambda'\mu, \tag{24}$$

where μ is the multiplication operator by $x\mapsto 1+|\nabla\ln M(x)|^2$. Since μ is a nonnegative operator on $L^2(\mathbb{R}^n,M)$, using a functional calculus argument (see [30, p. 110]), one deduces from (24) that, for any $\alpha\in(0,2)$,

$$L^{\alpha/2} \geqslant \left(\lambda'\right)^{\alpha/2}\mu^{\alpha/2},$$

which implies, thanks to the fact $L^{\alpha/2}=(L^{\alpha/4})^2$ and the symmetry of $L^{\alpha/4}$ on $L^2(\mathbb{R}^n,M)$, that

$$(\lambda')^{\alpha/2} \int_{\mathbb{R}^n} \left|f(x)\right|^2 \left(1 + \left|\nabla \ln M(x)\right|^2\right)^{\alpha/2} M(x)\,dx \leqslant \int_{\mathbb{R}^n} \left|L^{\alpha/4} f(x)\right|^2 M(x)\,dx = \left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n, M)}.$$

The conclusion of Theorem 1.2 readily follows by using the previous inequality in conjunction with Corollary 3.2 and Lemma 3.3, and picking ε small enough.

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Appendix A. Improved Poincaré inequality

In this section, we prove Proposition 1.1, namely:

Proposition A.1. *Assume that $M = e^{-V}$ satisfies* (3). *Then there exists $\lambda'(M) > 0$ such that, for all functions $f \in L^2_0(\mathbb{R}^n, M) \cap H^1(\mathbb{R}^n, M)$*:

$$\int_{\mathbb{R}^n} \left|\nabla f(x)\right|^2 M(x)\,dx \geqslant \lambda'(M) \int_{\mathbb{R}^n} \left|f(x)\right|^2 \left(1 + \left|\nabla \ln M(x)\right|^2\right) M(x)\,dx. \tag{A.1}$$

Note that of course in general the constants $\lambda(M)$ and $\lambda'(M)$ in (1) and (4) are different.

Proof of Proposition 1.1. Let f be as in the statement of Proposition 1.1 and let $g := f M^{\frac{1}{2}}$. Since

$$\nabla f = M^{-\frac{1}{2}} \nabla g - \frac{1}{2} g M^{-\frac{3}{2}} \nabla M,$$

assumption (3) yields two positive constants β, γ such that

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\nabla f(x)\right|^2(x) M(x)\,dx &= \int_{\mathbb{R}^n} \left(\left|\nabla g(x)\right|^2 + \frac{1}{4} g^2(x) \left|\nabla \ln M(x)\right|^2 - g(x) \nabla g(x) \cdot \nabla \ln M(x) \right) dx \\ &= \int_{\mathbb{R}^n} \left(\left|\nabla g(x)\right|^2 + \frac{1}{4} g^2(x) \left|\nabla \ln M(x)\right|^2 - \frac{1}{2} \nabla g^2(x) \cdot \nabla \ln M(x) \right) dx \\ &\geqslant \int_{\mathbb{R}^n} g^2(x) \left(\frac{1}{4} \left|\nabla \ln M(x)\right|^2 + \frac{1}{2} \Delta \ln M(x) \right) dx \\ &\geqslant \int_{\mathbb{R}^n} f^2(x) \left(\beta \left|\nabla \ln M(x)\right|^2 - \gamma\right) M(x)\,dx. \end{aligned} \tag{A.2}$$

The conjunction of (1) (which holds because (2) is satisfied), and (A.2) yields the desired conclusion. □

Appendix B. Technical lemma

We prove the following lemma.

Lemma B.1. *There exists a constant $\tilde{C} > 0$ with the following property*: *for all $\theta > 1$ and all $x \in \mathbb{R}^n$, there are at most $\tilde{C}\theta^n$ indexes j such that $|x - x^t_j| \leqslant \theta\sqrt{t}$.*

Proof. The argument is very simple (see [31]) and we give it for the sake of completeness. Let $x \in \mathbb{R}^n$ and $I(x) := \{j \in \mathbb{N};\ |x - x^t_j| \leqslant \theta\sqrt{t}\}$. Since, for all $j \in I(x)$,

$$Q\left(x_j^t, \sqrt{t}\right) \subset B\left(x, \left(\theta + \frac{1}{2}\right)\sqrt{nt}\right),$$

one has

$$C\left(\left(\theta + \frac{1}{2}\right)\sqrt{nt}\right)^n \geqslant \sum_{j \in I(x)} \left|Q\left(x_j^t, \sqrt{t}\right)\right| = \left|I(x)\right|\sqrt{t}^n,$$

we get the desired conclusion. □

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