CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

J. Math. Anal. Appl. 362 (2010) 90–99

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weakly K-analytic spaces and the three-space property for analyticity ☆

J. Kąkol [a], M. López Pellicer [b,*], W. Śliwa [a]

[a] Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznań, Poland
[b] Depto. de Matemática Aplicada and IMPA, Universidad Politécnica de Valencia, E-46022 Valencia, Spain

ARTICLE INFO

Article history:
Received 11 March 2008
Available online 17 September 2009
Submitted by B. Cascales

Dedicated to Professor Isaac Namioka on the occasion of his 80th birthday

Keywords:
K-analytic space
Analytic space
Three-space property

ABSTRACT

Let (E, E') be a dual pair of vector spaces. The paper studies general conditions which allow to lift analyticity (or K-analyticity) from the weak topology $\sigma(E, E')$ to stronger ones in the frame of (E, E'). First we show that the Mackey dual of a space $C_p(X)$ is analytic iff the space X is countable. This yields that for uncountable analytic spaces X the Mackey dual of $C_p(X)$ is weakly analytic but not analytic. We show that the Mackey dual E of an (LF)-space of a sequence of reflexive separable Fréchet spaces with the Heinrich density condition is analytic, i.e. E is a continuous image of the Polish space $\mathbb{N}^{\mathbb{N}}$. This extends a result of Valdivia. We show also that weakly quasi-Suslin locally convex Baire spaces are metrizable and complete (this extends a result of De Wilde and Sunyach). We provide however a large class of weakly analytic but not analytic metrizable separable Baire topological vector spaces (not locally convex!). This will be used to prove that analyticity is not a three-space property but we show that a metrizable topological vector space E is analytic if E contains a complete locally convex analytic subspace F such that the quotient E/F is analytic. Several questions, remarks and examples are included.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

A set E is said to have a *resolution* if E admits a family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets covering E such that $A_\alpha \subseteq A_\beta$ for $\alpha \leqslant \beta$. If E is a topological vector space (tvs) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a resolution is called *compact, bounded, or complete*, if every set A_α is compact, bounded or complete, respectively.

(**a**) A topological space (space) E is a *K-countable determined* (called also a Lindelöf Σ-space) if there is an upper semi-continuous (usco) map from a nonempty subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ with compact values in X whose union is X, where the set of integers $\mathbb{N}$ is discrete and $\mathbb{N}^{\mathbb{N}}$ is endowed with the product topology [39,1,31]. If the same holds for $\Sigma = \mathbb{N}^{\mathbb{N}}$, then X is called *K-analytic*. Clearly K-analytic $\Rightarrow$ Lindelöf Σ $\Rightarrow$ Lindelöf. Every K-analytic space has a compact resolution [47]; the converse fails [47,9].

(**b**) E is *quasi-Suslin* [50] if there exists a set-valued map T from $\mathbb{N}^{\mathbb{N}}$ into E covering E such that if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$.

(**c**) A continuous image of $\mathbb{N}^{\mathbb{N}}$ is called an *analytic* space. Analytic spaces admit compact resolutions consisting of metrizable sets.

(**d**) A locally convex space (lcs) E is called *weakly Lindelöf, weakly K-analytic, weakly analytic*, if the weak topology $\sigma(E, E')$ of E is Lindelöf, K-analytic, analytic, respectively.

☆ This research was supported by the project MTM 2008-01502 of the Spanish Ministry of Science and Innovation. The first and third named authors were also supported by Ministry of Science and Higher Education, Poland, grant No. N 201 2740 33.
* Corresponding author.
E-mail addresses: kakol@amu.edu.pl (J. Kąkol), mlopezpe@mat.upv.es (M. López Pellicer), sliwa@amu.edu.pl (W. Śliwa).

0022-247X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2009.09.026

K-analytic spaces and related structures listed above were intensively studied also in the frame of general theory of lcs, see [9–11,8] and also [12–14,22,24]. It is known that a weakly compactly generated (WCG) Banach space is weakly K-analytic [47] but there exist weakly K-analytic Banach spaces which are not K-analytic; for example the Hilbert space $\ell^2(\Gamma)$ for uncountable Γ. The present paper deals with the following general problem:

($\ast$) *When analyticity or K-analyticity of the weak topology $\sigma(E,E')$ of a dual pair (E,E') can be lifted to stronger topologies on E compatible with the dual pair*?

It turns out that there exist many weakly analytic lcs which are not analytic, see Corollary 3 below. Theorem 4 yields however that every weakly analytic metrizable lcs E is analytic but we show that a locally convex Baire space which is weakly quasi-Suslin is already a Fréchet space, i.e. a metrizable and complete lcs. This extends a result of De Wilde and Sunyach [51, p. 64]. We provide however a large class of weakly analytic metrizable and separable Baire tvs not analytic (clearly not locally convex!). Spaces of this type will be used to prove that analyticity is not a *three-space property* although we show that a metrizable tvs E is analytic if E contains a complete locally convex analytic subspace F such that the quotient E/F is analytic.

Since many important spaces in functional analysis are defined as certain (*LF*)-spaces, i.e. inductive limits of a sequence of Fréchet spaces, or their strong dual, see [3–5], one can ask for which (*LF*)-spaces E their Mackey dual $(E',\mu(E',E))$ or the strong dual $(E',\beta(E',E))$ is K-analytic or even analytic. It was known already by [10] that all separable (*LF*)-spaces have its precompact dual analytic. Theorem 7 below applies to show that the Mackey dual of an (*LF*)-space of a sequence of reflexive separable Fréchet spaces with the (Heinrich) density condition is analytic. This extends Valdivia's [50, Theorem 23, p. 77].

An lcs E is *unordered Baire-like* if it cannot be covered by a sequence of nowhere dense absolutely convex sets [46]. A space E has *countable tightness* if for each $A\subset X$ and $x\in\overline{A}$ there is countable $B\subset A$ whose closure contains x. For a Tichonov space X by $C_c(X)$ and $C_p(X)$ we denote the space of all continuous real-valued maps on X with the compact-open and pointwise topology, respectively. Recall [10] that a lcs E belongs to class $\mathfrak{G}$ if $E'=\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbf{N}}\}$, $A_\alpha\subseteq A_\beta$ if $\alpha\leqslant\beta$, and sequences in each A_α are equicontinuous. Class $\mathfrak{G}$ includes (*DF*)-spaces, (*LM*)-spaces, spaces of distributions $D'(\Omega)$, spaces $A(\Omega)$ of real analytic functions on open $\Omega\subset\mathbb{R}^n$.

2. A weakly analytic space need not be analytic

In this section we provide examples of weakly analytic spaces which are not analytic. It is known [1, 0.5.13] that a Tichonov space X is K-analytic (analytic) iff the weak dual of $C_p(X)$ is K-analytic (analytic). For the Mackey dual of $C_p(X)$ we prove the following theorem which extends the main result of [25].

Theorem 1. *The Mackey dual of $C_p(X)$ is analytic iff X is countable.*

Proof. Set $X:=(X,\tau)$ and assume that the Mackey dual of $C_p(X)$ is analytic. Suppose, by contradiction, that X is uncountable. For $x\in X$ the functional $\delta_x\colon C_p(X)\to\mathbb{R}$ defined by $\delta_x(f)=f(x)$ is linear and continuous. Denote by $L_p(X)$ and $L_\mu(X)$ the dual of $C_p(X)$ with the weak dual topology $\sigma=\sigma(C_p(X)',C_p(X))$ and with the Mackey topology $\mu=\mu(C_p(X)',C_p(X))$, respectively. Set $Y=\{\delta_x\colon x\in X\}$. The map $\delta\colon(X,\tau)\to(Y,\sigma|Y)$ defined by $x\to\delta_x$ is a homeomorphism and the set Y is closed in $L_p(X)$, see [1, Proposition 0.5.9]. Hence Y is also closed in $L_\mu(X)$. Thus $(Y,\mu|Y)$ is analytic. Let γ be the topology on X such that δ is a homeomorphism between (X,γ) and $(Y,\mu|Y)$. Since (X,γ) is an uncountable analytic space, it contains a set A homeomorphic to the Cantor set, see [43]. Clearly, $\gamma|A=\tau|A$. Let $(x_n)_n\subset A$ be a sequence such that $x_n\neq x_m$ if $n\neq m$, $n,m\in\mathbb{N}$, which converges to some $x_0\in A\setminus\{x_n\colon n\in\mathbb{N}\}$. It is easy to see that for every closed subspace G of (X,τ) and every $x\in X\setminus G$ there exists $f\in C(X,I)$ with $f(x)=1$ such that $G\cap\operatorname{supp}f=\emptyset$. Put $X_n=\{x_k\colon k>n\}\cup\{x_0\}$ for $n\in\mathbb{N}$. Clearly X_n is closed in X and $x_n\notin X_n$ for $n\in\mathbb{N}$. Therefore we can construct inductively a sequence $(f_n)\subset C(X,I)$, such that $f_n(x_n)=1$ and

$$\operatorname{supp}f_n\cap\left(X_n\cup\bigcup\{\operatorname{supp}f_k\colon 1\leqslant k<n\}\right)=\emptyset.$$

Then $x_0\notin\bigcup\{\operatorname{supp}f_k\colon k\in\mathbb{N}\}$ and

$$\operatorname{supp}f_n\cap\operatorname{supp}f_m=\emptyset$$

for all $n,m\in\mathbb{N}$ with $n\neq m$.

Denote by $C^b(X)$ the Banach space of all bounded real-valued continuous functions on X with the sup-norm $\|\cdot\|$. Let $g\in C^b(X)'$. For $k\in\mathbb{N}$ we put

$$\alpha_k=\left|g(f_k)\right|/g(f_k)$$

if $g(f_k)\neq 0$, and $\alpha_k=1$, otherwise. Then $|\alpha_k|=1$ and $\alpha_k g(f_k)=|g(f_k)|$ for $k\in\mathbb{N}$. Let $n\in\mathbb{N}$ and $S_n=\sum_{k=1}^n\alpha_k f_k$. Then $S_n\in C^b(X)$ and $\|S_n\|=1$. Thus

$$\sum_{k=1}^n\left|g(f_k)\right|=\left|\sum_{k=1}^n\alpha_k g(f_k)\right|=\left|g(S_n)\right|\leqslant\|g\|$$

for $n \in \mathbb{N}$, so $\sum_{k=1}^{\infty} |g(f_k)| \leqslant \|g\|$. Hence $g(f_k) \to 0$. It follows that the sequence $(f_n)_n$ converges weakly to 0 in $C^b(X)$. Thus the set

$$F_0 = \{0, f_1, -f_1, f_2, -f_2, \ldots\}$$

is weakly compact in $C^b(X)$. By the Krein–Smulian weak compactness theorem [36, Theorem 2.8.14] the closed convex hull F of F_0 in $C^b(X)$ is weakly compact. Clearly, F is the closed absolutely convex hull of the set $\{f_k\colon k \in \mathbb{N}\}$ in $C^b(X)$. The topology ϱ of the pointwise convergence in $C^b(X)$ is weaker than the weak topology of $C^b(X)$, so F is compact in $(C^b(X), \varrho)$. Hence F is compact in $C_p(X)$, since the injective map $(C^b(X), \varrho) \to C_p(X)$ is continuous. Thus the functional $p_F\colon L_\mu(X) \to [0, \infty)$, defined by $p_F(g) = \sup\{|g(f)|\colon f \in F\}$, is a continuous semi-norm. Since $(f_n)_n \subset F$ we have

$$p_F(\delta_{x_n}) \geqslant \left|f_n(x_n)\right| = 1$$

for $n \in \mathbb{N}$. It is easy to see that $f(x_0) = 0$ for all $f \in F$, so $p_F(\delta_{x_0}) = 0$. It follows that $\delta_{x_n} \not\to \delta_{x_0}$ in $(Y, \mu|Y)$, so $x_n \not\to x_0$ in (X, γ); a contradiction.

Assume now that the space X is countable. If $C_p(X)$ is finite-dimensional, then the Mackey dual $L_\mu(X)$ of $C_p(X)$ is finite-dimensional; so it is analytic. If $C_p(X)$ is infinite-dimensional, then $C_p(X)$ is a metrizable lcs isomorphic to a dense subspace of $\mathbb{R}^{\mathbb{N}}$, so $L_\mu(X)$ is algebraically isomorphic to φ, the strong dual of $\mathbb{R}^{\mathbb{N}}$. But φ with the strongest locally convex topology is the union of an increasing sequence of finite-dimensional Banach spaces, so it is an analytic space. It follows that $L_\mu(X)$ is analytic, too. □

Theorem 1 and its proof yield also the following characterization.

Corollary 2. *The strong dual of $C_p(X)$ is analytic iff X is countable.*

Recall that $L_p(X)$ is analytic iff X is analytic [1, 0.5.13]. Thus Theorem 1 provides many of concrete nonanalytic lcs whose weak topology is analytic.

Corollary 3. *Let X be an uncountable analytic space. Then the Mackey dual $L_\mu(X)$ of $C_p(X)$ is weakly analytic but not analytic.*

3. Weakly analytic and dual analytic locally convex spaces

This section studies conditions under which problem (∗) has a positive answer. First we collect a few known facts used several times in the paper.

(α) A K-analytic space is analytic iff it is a continuous image of a metric separable space [43, Theorem 5.5.1]. Hence analytic spaces have countable tightness.

(β) A regular space is analytic iff it is submetrizable and has a compact resolution, see [48], [10, Theorem 15] and [14, Corollary 4.3]. Therefore a compact space is analytic iff it is metrizable.

(γ) Metric separable spaces with a complete resolution are analytic [20, Corollary 3.2].

Let E be a vector space with vector topologies $\xi \leqslant \tau$. We say that τ is ξ-polar if τ admits a basis of ξ-closed neighbourhoods of zero. By [27, Theorem 3.2.4] it follows that if τ is ξ-polar, then a ξ-complete (sequentially complete) resolution on E is also τ-complete (sequentially complete). This situation generates the following useful fact.

(δ) If τ is a metrizable and separable vector topology and $\xi \leqslant \tau$, where τ is ξ-polar and (E, ξ) has a complete resolution, then (E, τ) is analytic. Indeed, (E, τ) has a complete resolution and (γ) applies.

(θ) If $E \in \mathfrak{G}$ is separable, then the precompact dual of E, i.e. E' with the topology $\tau_{pc}(E', E)$ of the uniform convergence on precompact sets of E, is analytic by [10, Corollary 1.15], hence the topology $\sigma(E', E)$ is analytic as well.

A space E is *angelic* if every relatively countably compact set $A \subset E$ is relatively compact and for each $x \in \overline{A}$ there exists a sequence in A converging to x. In angelic spaces (relatively) countable compact sets are (relatively) compact.

(ρ) An angelic space is K-analytic iff it is quasi-Suslin iff it has a compact resolution [9, Corollary 1.1]. Hence a submetrizable space is K-analytic iff it has a compact resolution.

The following result will be used in the sequel.

Theorem 4. *A separable tvs $E := (E, \xi)$ with a sequentially complete resolution is analytic if E satisfies one of the following conditions*:

(A) *E is covered by a sequence $(S_n)_n$ of absolutely convex metrizable subsets.*
(B) *E is a continuous linear image of a separable and metrizable tvs.*

Proof. Assume that $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a sequentially complete resolution in E.

(A) From [27, Theorem 9.2.4] it follows immediately that each $(\overline{S_n}, \xi|\overline{S_n})$ is metrizable, and then it has the complete resolution $\{\overline{S_n} \cap A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since ξ is separable, then each metrizable space $(\overline{S_n}, \xi|\overline{S_n})$ is separable as well (this fact is perhaps well known but hard to locate): Indeed, fix $n \in \mathbb{N}$ and set $F := (\overline{S_n}, \xi|\overline{S_n})$. Let $\mathfrak{F}(E)$ be the set of all circled neighbourhoods of zero in ξ and let $U_m \in \mathfrak{F}(E)$, $m \in \mathbb{N}$, such that $(F \cap [U_m + U_m])_m$ is a basis of neighbourhoods of zero in F. By separability there exists a countable set B_m such that $E \subset B_m + U_m$, and then there exists a countable subset C_m in F such that $F \subset C_m + U_m + U_m$. It is clear that $C := \bigcup_m C_m$ is a countable dense subset of F. By (γ) we deduce that $(\overline{S_n}, \xi|\overline{S_n})$ is analytic, so (E, ξ) is analytic.

(B) First note that (E, ξ) admits a stronger separable and metrizable vector topology τ. Let τ^ξ be a vector topology on E whose neighbourhoods of zero are composed by the ξ-closures of τ-neighbourhoods of zero. Then $\xi \leqslant \tau^\xi \leqslant \tau$ and τ^ξ is metrizable separable and ξ-polar. Since the space (E, τ^ξ) admits a complete resolution, we apply (γ) to get that (E, τ^ξ) and consequently (E, ξ) are analytic. □

It is known that in any ZFC-consistent system one has $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{c}$, where $\mathfrak{b}$ is the bounding cardinal [22]. If we assume the Martin axiom and negation of the continuum hypothesis (CH), then any K-analytic space in which every compact set is metrizable, is analytic [43, Theorem 5.5.3] and $\aleph_1 \neq \mathfrak{b}$, see [49]. If we assume that $\aleph_1 = \mathfrak{b}$ we have the following example providing also restrictions on possible extensions of Theorem 4.

Example 5. Set $E := C_c([0, \omega_1))$. Then $F := (E', \sigma(E', E))$ is not K-analytic but if $\aleph_1 = \mathfrak{b}$, then F has a resolution of metrizable and compact absolutely convex sets.

Proof. Since every compact set in $X := [0, \omega_1)$ is metrizable, then the polar of every neighbourhood of zero in E is $\sigma(E', E)$-metrizable by [23, Lemma 1]. E is a locally complete (DF)-space, X pseudocompact but not compact, and $\{f \in E\colon f(X) \leqslant 1\}$ generates on E a Banach topology ϑ with $\mu(E, E') \leqslant \vartheta$, see [30] for more detail. Assume that $(E', \sigma(E', E))$ is K-analytic. Then $\sigma(E, E')$ has countable tightness [12, Theorem 4.6] and $(E, \mu(E, E'))$ is quasi-barrelled [22, Theorem 1, Example 2], so $(E, \mu(E, E'))$ is barrelled (since E is locally complete). By the Mahowald closed graph theorem [42] one gets $\mu(E, E') = \vartheta$, so X is metrizable, a contradiction. Hence $(E', \sigma(E', E))$ is not K-analytic. If $\aleph_1 = \mathfrak{b}$, then by [22, Theorem 3] the space E has a basis of absolutely convex neighbourhoods of zero $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $U_\alpha \subset U_\beta$ if $\beta \leqslant \alpha$. Clearly the polars of sets U_α compose a resolution consisting of $\sigma(E', E)$-metrizable compact sets. □

Since quasi-Suslin spaces admit a countable compact resolution, Theorem 4 yields

Corollary 6. *A weakly quasi-Suslin lcs is analytic if it is a linear continuous image of a metrizable and separable tvs.*

4. Weakly compact density condition and analyticity

Let $\mathfrak{K}(E)$ be the family of all absolutely convex $\sigma(E, E')$-compact sets in a lcs E. We shall say that a metrizable lcs E with a countable basis $(U_n)_n$ of absolutely convex neighbourhoods of zero satisfies the *weakly compact density condition* (wcdc) if

($\star$) *there is a double sequence $(B_{n,k})_{n,k}$ in $\mathfrak{K}(E)$ such that for $n \in \mathbb{N}$ and $C \in \mathfrak{K}(E)$ there is $k \in \mathbb{N}$ such that $C \subset B_{n,k} + U_n$.*

If $\mathfrak{K}(E)$ is replaced by the family of all bounded sets in E, then ($\star$) defines the *density condition* (dc) for a metrizable lcs E, which characterizes the property that every $\beta(E', E)$-bounded set is metrizable [3,4]. For Fréchet–Montel spaces conditions (wcdc) and (dc) (as easily seen) are equivalent.

There is a large class of concrete reflexive separable Fréchet spaces (not Banach in general) whose strong dual is analytic being an (LB)-space of a sequence of separable Banach spaces. Let us collect a few concrete interesting cases.

For Köthe echelon spaces $\lambda_p := \lambda_p(I, A)$, where $A = (a_n)$ is any Köthe matrix on countable I with $1 < p < \infty$, the strong dual of λ_p is the (LB)-space $k_q = \operatorname{ind}_n \ell_q(I, v_n)$ with $q^{-1} + p^{-1} = 1$, $v_n = a_n^{-1}$, $n \in \mathbb{N}$, see [3], which is clearly analytic. If $A = (a_n)$ is a Köthe matrix on $\mathbb{N}$ with condition (ND), see [42], then λ_p, $p > 1$, does not satisfy the density condition [3, p. 178], although its strong dual is analytic. For $p = 1$ we note the following facts for the space λ_1.

(a) The Mackey dual $(\lambda_1', \mu(\lambda_1', \lambda_1))$ of λ_1 is analytic. Indeed, since every weakly compact set in λ_1 is compact (λ_1 is a perfect space [32]), then $\mu(\lambda_1', \lambda_1)$ equals the topology $\tau_{pc}(\lambda_1', \lambda_1)$ of the uniform convergence on λ_1-precompact sets. But the last topology is analytic by (θ). The space λ_1 satisfies (dc) iff the Köthe matrix A satisfies condition (D) [5, Theorem 4]. λ_1 satisfies (wcdc) for any A since if $(x_n)_n$ is dense in λ_1 and $(U_n)_n$ is a countable basis of neighbourhoods of zero, then for weakly compact ($=$ compact) $C \subset \lambda_1$ and $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $C \subset \{\sum_{j=1}^k a_j x_j\colon |a_j| \leqslant 1\} + U_n$.

(b) The strong dual $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is analytic iff λ_1 is Montel. Indeed, if λ_1 is Montel, then its strong dual is covered by a sequence of absolutely convex compact metrizable sets and Theorem 4 applies. The converse in (b) follows from the Diedonne–Gomes theorem [38, Theorem 27.9].

(c) If λ_1 satisfies (dc) but is not Montel, see [3, Theorem 4, Corollary 8] describing this case, then $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is even not quasi-Suslin although it admits a resolution consisting of metrizable and complete sets. Indeed, by (dc) every bounded set in the (DF)-space $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is metrizable (and complete by completeness of $\beta(\lambda_1', \lambda_1)$). Assume that $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is quasi-Suslin. Since $\mu(\lambda_1', \lambda_1)$ is analytic, then it admits a weaker metric topology by (β) and by (ρ) we deduce that $\beta(\lambda_1', \lambda_1)$ is K-analytic. Hence again by (β) the space $(\lambda_1', \beta(\lambda_1', \lambda_1))$ is analytic, which implies that λ_1 is Montel.

Since every (LF)-space belongs to class $\mathfrak{G}$, then by (θ) the precompact dual of every separable (LF)-space is analytic. Motivated by this fact and above examples we provide the following result for the Mackey dual. The following Theorem 7 extends Valdivia's [50, Theorems 20, 23, pp. 76, 77].

Theorem 7. *Let E be a metrizable lcs with (wcdc). If the weak dual $(E', \sigma(E', E))$ is separable, then the Mackey dual $(E', \mu(E', E))$ is analytic. Hence $(E', \mu(E', E))$ is analytic for an (LF)-space of a sequence $(E_n)_n$ of separable reflexive Fréchet spaces satisfying (wcdc).*

Proof. Fix $n \in \mathbb{N}$, set $S_n := U_n^0$, where $(U_n)_n$ is a decreasing basis of absolutely convex closed neighbourhoods of zero in E. By (wcdc) there is a sequence $(B_{n,k})_k$ in $\mathfrak{K}(E)$ as in ($\star$). Since polars of absolutely convex $\sigma(E, E')$-compact sets in E compose a basis of neighbourhoods of zero for $(E', \mu(E', E))$, then for a $\mu(E', E)$-neighbourhood of zero V there is $k \in \mathbb{N}$ such that $B_{n,k}^0 \cap S_n \subset 2V$. This (an adaptation of an argument due to Bierstedt and Bonet in [4, Corollary 3]) shows that $(S_n, \mu(E', E)|S_n)$ is metrizable. But sets $A_\alpha := S_{n_1}$, for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, generate a compact resolution in $\sigma(E', E)$, so the assumptions of Theorem 4 are satisfied. Since a space covered by a sequence of analytic subspaces is analytic, this yields that $(E, \mu(E', E)) = \bigcup_n S_n$ is analytic. Now assume that E is an (LF)-space with its defining sequence $(E_n)_n$ of separable reflexive Fréchet spaces satisfying (wcdc). Since each E_n is separable and reflexive, then $(E_n', \beta(E_n', E_n))$ is analytic by the previous case, and the projective limit $(E', \gamma) := \mathrm{Proj}_n(E_n', \beta(E_n', E_n))$ is a closed subspace of the analytic space $\prod_n (E_n', \beta(E_n', E_n))$. As closed subspaces of analytic spaces are analytic, so (E', γ) is analytic. Since every absolutely convex $\sigma(E, E')$-compact set is contained and bounded in some E_m by [51, Theorem 22, p. 76], then $\mu(E', E) \leqslant \gamma$. If $j_n : E_n \to E$ is the inclusion map, then the dual map

$$j_n' : (E', \mu(E', E)) \to (E_n', \mu(E_n', E_n))$$

is continuous, $n \in \mathbb{N}$. This, combined with the equality $\mu(E_n', E_n) = \beta(E_n', E_n)$ (since E_n are reflexive), yields $\gamma \leqslant \mu(E', E)$. □

5. More about analyticity in class $\mathfrak{G}$

Following Orihuela [40] a space E is *web-compact* if there exists a nonempty subset $I \subset \mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha\colon \alpha \in I\}$ of subsets of E such that if

$$C_{n_1, n_2, \ldots, n_k} := \bigcup \{A_\beta\colon \beta = (m_k) \in I,\ m_j = n_j,\ j = 1, 2, \ldots, k\},$$

for every $\alpha = (n_k) \in I$, the following holds: $\overline{\bigcup \{A_\alpha\colon \alpha \in I\}} = E$ and if $\alpha = (n_k) \in I$ and $x_k \in C_{n_1, n_2, \ldots, n_k}$ for all $k \in \mathbb{N}$, then $(x_k)_k$ has a cluster point in E. Separable spaces, K-countable determined spaces, quasi-Suslin spaces are web-compact.

Recall also that a lcs E is a (df)-space if E admits a fundamental sequence of bounded sets and $\mu(E, E')$ is c_0-quasi-barrelled, see [27]. In [30] we showed that $C_c(X)$ is a (df)-space iff each regular Borel measure on X has compact support. We provide another characterization of (df)-spaces $C_c(X)$ using the concept of quasi-Suslin spaces. Part (ii) extends the fact that every Montel (DF)-space is analytic.

Proposition 8. *Let E be a lcs in class $\mathfrak{G}$.*

(i) *If $(E, \sigma(E, E'))$ is web-compact, then $(E, \sigma(E, E'))$ has countable tightness and $(E', \tau_{pc}(E', E))$ is K-analytic. Consequently, a weakly K-countable determined space E is separable iff $(E', \sigma(E', E))$ is analytic.*
(ii) *If E has a fundamental sequence of bounded sets and $(E', \beta(E', E))$ is separable, then $(E', \beta(E', E))$ is analytic.*
(iii) *If E is separable semi-Montel (resp. E is separable Montel and $(E', \beta(E', E)) \in \mathfrak{G}$), then $(E', \beta(E', E))$ (resp. E) is analytic.*
(iv) *$(E', \beta(E', E))$ is a Fréchet space iff $(E', \beta(E', E))$ is an unordered Baire-like space. Hence $C_c(X)$ is a (df)-space iff $C_c(X)$ belongs to class $\mathfrak{G}$ and the strong dual of $C_c(X)$ is unordered Baire-like.*

Proof. By [24, Theorem 5] the space $(E', \tau_{pc}(E', E))$ is *convex quasi-Suslin*, i.e. is quasi-Suslin and admits a resolution $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of absolutely convex countably compact sets in $(E', \tau_{pc}(E', E))$ such that each sequence in A_α is equicontinuous.

(i) Since $(E, \sigma(E, E'))$ is web-compact, then by [40, Theorem 3] we know that the space $C_p(E, \sigma(E, E'))$ is angelic; therefore $(E', \sigma(E', E)) \subset C_p(E, \sigma(E, E'))$ is angelic as well. By (ρ) the space $(E', \tau_{pc}(E', E))$ is K-analytic. Now the countable tightness of $(E, \sigma(E, E'))$ follows from [12, Theorem 4.6]. If $(E, \sigma(E, E'))$ is K-countable determined and $(E', \sigma(E', E))$ is separable, then $(E, \sigma(E, E'))$ is separable by [10, Theorem 13]. Conversely, if E is separable, then (θ) yields analyticity of $(E', \tau_{pc}(E', E))$.

(ii) Since $(E', \beta(E', E))$ is a metrizable lcs and $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a complete resolution in $(E', \beta(E', E))$, then (γ) applies.

(iii) If $E \in \mathfrak{G}$ is separable semi-Montel, then $\tau_{pc}(E', E) = \beta(E', E)$. By (θ) the topology $\tau_{pc}(E', E)$ is analytic and the conclusion holds. For the case when E is a separable Montel space and $(E', \beta(E', E))$ is in class $\mathfrak{G}$, the first part of (iii) applies.

(iv) Since sequences in each A_α are equicontinuous, then A_α are also bounded in $(E', \beta(E', E))$, so $(E', \beta(E', E))$ admits a bounded resolution consisting of sequentially complete absolutely convex sets; hence $(E', \beta(E', E))$ is a *quasi-(LB)-space* in sense of [50]. But an unordered Baire-like quasi-(LB)-space is a Fréchet space [50, Corollary 1.6]. Assume that $C_c(X)$ is a (df)-space. By [7] it is ℓ^∞-quasi-barrelled, so $C_c(X)$ is a *dual metric space* and then belongs to $\mathfrak{G}$ by [10, Examples 1.2(D)]. By [30, Main theorem] the strong dual of $C_c(X)$ is Fréchet. To get the converse implication we apply the first part to show that the strong dual of $C_c(X)$ is Fréchet (unordered Baire-like quasi-(LB)-spaces are Fréchet spaces), and then again we apply [30, Main theorem] to get that $C_c(X)$ is a (df)-space. □

Part (i) applies to get many examples of lcs whose weak dual is K-analytic but not analytic. To prove Proposition 10 we need the following fact.

Lemma 9. *Let E be a metrizable tvs and F a dense Baire vector subspace of E with a resolution $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then*

$$E = \operatorname{lin} \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \overline{A_\alpha} = \bigcup \{ n_1 \overline{[A_\alpha]}\colon \alpha = (n_k) \in \mathbb{N}^{\mathbb{N}} \},$$

where the closure is taken in E and $[A]$ denotes the absolutely convex envelope of A.

Proof. For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ let K_α be the closure of A_α in E and set $G = \bigcup_\alpha K_\alpha$. Then G is a dense Baire subspace of E with a resolution consisting of closed sets in E. By [20, Theorem 3.5], or [29, Corollary 2, an alternative proof], one gets that $E \setminus G$ is of the first Baire category. Then $E \setminus \operatorname{lin} G$ $(\subset E \setminus G)$ is also of the first Baire category. We prove that $\operatorname{lin} G = E$. Assume that $a \in E \setminus \operatorname{lin} G$. Then $a + \operatorname{lin} G \subset E \setminus \operatorname{lin} G$. But the set $a + \operatorname{lin} G$ is of the second Baire category and we reach a contradiction. Hence $\operatorname{lin} G = E$ indeed. Now we prove the other equality. Fix $x \in \operatorname{lin} G$. Then there are $t_p \in \mathbb{R}$, $1 \leqslant p \leqslant n$, $\alpha_p \in \mathbb{N}^{\mathbb{N}}$ and $x_{\alpha_p} \in K_{\alpha_p}$, $1 \leqslant p \leqslant n$, such that

$$x = \sum_{p=1}^{n} t_p x_{\alpha_p} \in n|t|[K_\gamma],$$

where $|t| = \max\{|t_p|\colon p = 1, 2, \dots, n\}$ and $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha_p \leqslant \gamma$ for every $p = 1, 2, \dots, n$. Choose $m_1 \in \mathbb{N}$ and $\beta = (m_k) \in \mathbb{N}^{\mathbb{N}}$ such that $n|t| \leqslant m_1$ and $\gamma \leqslant \beta$. Then $x \in m_1 \overline{[A_\beta]}$. □

The following fact generalizes a result of De Wilde and Sunyach [51, p. 64].

Proposition 10. *Every Baire lcs which is weakly quasi-Suslin is a Fréchet space.*

Proof. Since $\sigma(E, E')$ is quasi-Suslin, then $\sigma(E, E')$ admits a resolution $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of $\sigma(E, E')$ countably compact sets, hence bounded in E. This implies metrizability of E by [29, Corollary 1]. Therefore $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a complete resolution in E and we use Lemma 9 to conclude that E is complete. □

Example 11. Proposition 10 fails for unordered Baire-like spaces.

Proof. The space $(E, \xi) := \ell^p$ for $0 < p < 1$ is a metrizable and complete separable nonlocally convex space and $\sigma(E, E')$ is Hausdorff [19]. Let ξ_c be the finest locally convex topology on E weaker than ξ; the absolute convex envelope of ξ-neighbourhoods of zero generate the topology ξ_c, so ξ_c is a metrizable locally convex topology strictly weaker than ξ. Since (E, ξ) is Baire, then (E, ξ_c) is unordered Baire-like and the completion of (E, ξ_c) is isomorphic to the space ℓ^1, see also [18, Theorem 1]. Clearly (E, ξ_c) is analytic and noncomplete. □

6. Additional remarks and examples

(i) There is an analytic lcs which is not a continuous linear image of a separable metrizable tvs. In fact, if (E, ξ) is an $\aleph_0$-dimensional vector space with the finest vector topology, then (E, ξ) is analytic (as covered by a sequence of finite-dimensional subspaces) and nonmetrizable and it is clear that (E, ξ) is as required.

(ii) Let E be a vector space with a resolution $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of finite-dimensional subspaces of E. Then $\dim E = \aleph_0$ and (E, ξ) is analytic for any vector topology ξ on E. Indeed, for a Hamel basis $\{x_t\colon t \in T\}$ of E and $\alpha \in \mathbb{N}^{\mathbb{N}}$ set $T_\alpha = \{t \in T\colon x_t \in A_\alpha\}$. Then $\{T_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of T of finite sets. Now we apply the proof of Theorem 3.3 of [20] to get that T is countable. On the other hand, there exist nonseparable uncountable-dimensional normed spaces covered by a family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of finite-dimensional subspaces: Let $\mathcal{P}(\mathbb{N})$ be the σ-algebra of all subsets of $\mathbb{N}$ and let $m_0 := m_0(\mathcal{P}(\mathbb{N}))$ be the space of $\mathcal{P}(\mathbb{N})$-simple real-valued functions on $\mathbb{R}$ with the sup-norm topology. Since $|\mathbb{N}^{\mathbb{N}}|$ coincides with the cardinality of the family of all finite subsets of $\mathcal{P}(\mathbb{N})$, the conclusion follows. A normed space may not be analytic even if it has a resolution of analytic subspaces: Under (CH) the nonseparable Banach space $c_0[0, \omega_1)$ has a resolution of closed and separable (hence analytic) subspaces of the type $c_0[0, \mu)$, $\mu < \omega_1$, see also [20, Remark 3.4].

(iii) Since Fréchet–Montel spaces satisfy (wcdc), Theorem 4 and Theorem 7 yield: $D'(\Omega)$ and $A(\Omega)$ are analytic with countable tightness, see also [13, Corollary 2.4].

(iv) Since ℓ^1 satisfies (wcdc), Theorem 7 yields analyticity of $(\ell^\infty, \mu(\ell^\infty, \ell^1))$. Moreover, $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ is a (gDF)-space not quasi-barrelled [42, Example 8.3.14]. On the other hand, in [22, Theorem 1] we showed that a (DF)-space with countable tightness is quasi-barrelled. This result fails however for (gDF)-spaces as the (analytic with countable tightness) space $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ shows.

(v) If the product $E := \prod_{t\in T} E_t$ of tvs E_t admits a bounded resolution (for example if E is quasi-Suslin), then T is countable. In fact, if T is uncountable, there is uncountable A such that the Baire space $R_0 =: \mathbb{R}^A \subset E$ admits a compact resolution and R_0 is metrizable by Proposition 10; hence A is countable, a contradiction.

7. Proof of Theorem 12 and two examples

We show how to construct a weakly analytic but not analytic metrizable separable Baire tvs starting from any infinite-dimensional separable Fréchet space. This procedure will be used to show that a three-space property fails for analyticity. By a *three-space property* (for tvs) we understand the following [44]: Suppose that E is a tvs and $F \subset E$ is a closed vector subspace such that F and the quotient E/F have certain property $\mathcal{P}$. Does E have property $\mathcal{P}$? Corson used the concept of (WCG) Banach spaces to show [16, Example 2] that the Lindelöf property is not a three-space property. Let us recall briefly Corson's approach: Let D be the subspace of $\ell^\infty[0,1]$ formed by all bounded real-valued functions on $[0,1]$ that are right continuous and have finite left limits. Since $\sigma(D,D')$ is not normal, then D is not weakly Lindelöf. But the quotient $D/C[0,1]$ is isomorphic to the (WCG) Banach space $c_0[0,1]$, so the weak topology of $c_0[0,1]$ is K-analytic. Corson's example shows also that K-analyticity is not a three-space property but since $c_0[0,1]$ is not separable, it does not cover the problem for $\mathcal{P} = $ *analytic*. If E is a tvs containing a subspace F such that F and E/F are separable Fréchet spaces, then E is separable Fréchet. This is clear, since separability, metrizability and completeness are three-space properties [44]. The two other cases are less evident: *Let F and E/F be analytic and E metrizable. Assume that F or E/F are complete. Is E analytic*?

As every metrizable tvs with a compact resolution is analytic, then part (1) of Theorem 12 below shows that a metrizable tvs E is analytic if E contains a complete locally convex analytic subspace F such that E/F is analytic. The proof of part (1) below is due to Prof. L. Drewnowski (private communication) to whom we are grateful for a permission to use this argument in this paper.

Theorem 12. (1) *Let E be a metrizable tvs containing a closed subspace F such that F and E/F have a compact resolution. If F is complete and locally convex, then E has a compact resolution.* (2) *There is a separable normed space E which is not analytic but contains a closed analytic subspace F such that E/F is a separable Banach space.*

Proof. Let G be the completion of E and let $Q: G \to G/F$ be the quotient map. By a result of Michael [37] (see also [6, Proposition 1], and [2] or [6, Corollary 7.1] for the case of Fréchet spaces) there is a continuous map $g: G/F \to G$ such that $Q \circ g$ is the identity map on G/F, i.e.

$$g(x+F) \in x+F$$

for each $x \in G$. Let $\{K_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution on $E/F \subset G/F$ and let $\{A_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution in F. Then the compact sets

$$M_\alpha := g(K_\alpha) + A_\alpha$$

form a compact resolution on E for $\alpha \in \mathbb{N}^\mathbb{N}$. Indeed, first observe that $g(K_\alpha) \subset E$, so then each compact set M_α is contained in E. Fix $x \in E$. Since $g(x+F) \in x+F$, then there exists $y \in F$ such that

$$g(x+F) + y = x.$$

For some $\alpha \in \mathbb{N}^\mathbb{N}$ we have that $x+F \in K_\alpha$ and $y \in A_\alpha$. This yields that $x \in M_\alpha$, which proves that $E = \bigcup\{M_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$, and the proof is complete.

Now we prove (2). Fix an infinite-dimensional separable Banach space $E := (E, \|.\|)$ and let $(y_n)_n$ be a sequence in E such that $\sum \|y_n\| < \infty$ whose linear span is dense in E, and if $(t_n) \in \ell^\infty$, $\sum_n t_n y_n = 0$, then $(t_n) = 0$, see [34, Theorem 1]. Define a compact injective map

$$T: \ell^1 \to E, \qquad T(x) := \sum_n x_n y_n,$$

where $x = (x_n) \in \ell^1$. Clearly its image $F := T(\ell^1)$ is dense in E and F is not barrelled by [32, 34.7(1)] or [35]. Note that $\dim(E/F) = 2^{\aleph_0}$. This follows from $\dim E = 2^{\aleph_0}$ and [42, Proposition 4.3.11]. Let τ be a normed topology defined by the norm $\|.\|$. Let $q: E \to E/F$ be the quotient map. Since the quotient topology of E/F is trivial and $\dim \ell^1 = \dim E/F$, the space E/F admits a stronger separable Banach topology α such that $(E/F, \alpha)$ is isomorphic to ℓ^1. Therefore the assumptions of [19, (2), p. 194] are satisfied and making use of this result one gets on E a coarsest vector topology ξ such that $\tau < \xi$ and the quotient topology ξ/F equals α and $\xi|F = \tau|F$. The sets $U \cap q^{-1}(V)$, where U and V run over τ- and α-neighbourhoods of zero, respectively, form a basis of neighbourhoods of zero for ξ. As separability and the property of being a normed space are three-space properties, see [45, Theorem 12.20] and [44, Theorem 3.2], the topology ξ is normed and separable. Finally, since every linear map with closed graph from a Banach space into an analytic space is continuous [15, Theorem 5.2], we deduce (E, ξ) is not analytic. Its closed subspace F is analytic and the quotient space $(E/F, \alpha)$ is isomorphic to ℓ^1. $\square$

We complete the paper with three examples (the first one uses some ideas developed by the first named author in [28]) showing that Proposition 10 fails for Baire tvs which are not locally convex.

A function f from $[0,1]$ into a vector space E is called measurable if all values of f lie in a finite-dimensional subspace $F \subset E$ (depending on f) and f is Lebesque measurable, where F is endowed with the unique Hausdorff vector topology. The set $S(E)$ of (classes) of equivalent measurable functions is a vector space. Since two constant functions agree almost everywhere iff they are identical, then there is an injective map $t_E : E \to S(E)$ that assigns to every $x \in E$ the constant function $f(t) = x$, $t \in [0,1]$. We need a result due to Peck and Porta, see [41, Theorem A].

Proposition 13. *Every* (*metrizable*) *tvs* (E,τ) *with* $\dim E \geqslant 2^{\aleph_0}$ *is linearly homeomorphic under a linear map* t_E *to a* (*metrizable*) *subspace of a tvs* $(S(E), \mu(\tau))$ *without nonzero continuous linear functionals such that* $\dim E = \operatorname{codim} t_E(E)$ (*in* $S(E)$) *and the density characters of* E *and* $S(E)$ *are the same.*

Example 14. For every infinite-dimensional separable Fréchet space (E,τ) there exist two metrizable nonanalytic but weakly analytic vector topologies ξ_1, ξ_2 such that*:*

(1) $\tau = \inf\{\xi_1, \xi_2\}$.
(2) ξ_1 is Baire and separable and ξ_2 is not separable.
(3) $(E,\tau)' = (E,\xi_1)' = (E,\xi_2)'$, i.e. the three topologies have the same weak topology.

Proof. Let $(x_t)_{t\in T}$ be a Hamel basis of E. Consider a partition $(T_n)_n$ of T such that $T = \bigcup_n T_n$ and $\operatorname{card} T = \operatorname{card} T_n$ for all $n \in \mathbb{N}$. Set

$$E_n := \operatorname{lin}\left\{x_t \colon t \in \bigcup_{i=1}^{n} T_i\right\}.$$

Then $(E_n)_n$ covers E and

$$\dim E = \dim E_n = \dim(E/E_n) = 2^{\aleph_0}$$

for $n \in \mathbb{N}$. By the Baire category theorem there exists a dense Baire subspace $F := E_m$ of E. For $0 < p < 1$ set $L^p := (L^p[0,1], \|.\|_p)$. It is known that L^p is a $2^{\aleph_0}$-dimensional metrizable complete and separable tvs with trivial topological dual [17]. Let α be a metrizable complete and separable vector topology on E/F such that $(E/F, \alpha)$ is linearly homeomorphic to L^p and let ξ_1 be a vector topology on E defined in the same manner as above, i.e.

$$\tau < \xi_1, \qquad \xi_1/F = \alpha, \qquad \xi_1|F = \tau|F.$$

Then ξ_1 is metrizable and separable. (E,ξ_1) is nonanalytic by the closed graph theorem [15, Theorem 5.2] for the identity map from (E,τ) onto (E,ξ_1). Note that (E,ξ_1) is Baire. Indeed, since $\xi_1|F$ and ξ_1/F are separable Baire topologies [26] applies, see also [44, Proposition 12.21], to deduce that (E,ξ_1) is a Baire space. Now we construct the topology ξ_2. Since (E,ξ_1) is a Baire space, the same argument as above applies to choose a ξ_1-dense subspace G of E such that $\dim G = \dim(E/G) = 2^{\aleph_0}$. By Proposition 13 there exists a $2^{\aleph_0}$-dimensional nonseparable metrizable tvs Z without nonzero continuous linear functionals and we proceed as above to define a nonseparable metrizable vector topology ξ_2 on E such that $\tau < \xi_2$, $\tau|G = \xi_2|G$, and $(E/G, \xi_2/G)$ is linearly homeomorphic to Z. Clearly $\tau \leqslant \inf\{\xi_1,\xi_2\}$ and

$$\tau|G = \inf\{\xi_1,\xi_2\}|G = \xi_2|G.$$

On the other hand the topologies $\tau/G = \xi_1/G$ are trivial, so τ/G and the topology $\inf\{\xi_1,\xi_2\}/G$ coincide. By [44, p. 23] one gets that τ and $\inf\{\xi_1,\xi_2\}$ are equal.

Finally we prove that the topologies τ, ξ_1 and ξ_2 have the same continuous linear functionals on E. This will show that the weak topologies for (E,τ), (E,ξ_1) and (E,ξ_2) are the same, consequently the weak topology of (E,ξ_i), $i = 1,2$, will be analytic. In fact, let $f \in (E,\xi_1)'$ be a ξ_1-continuous linear functional on E and let $h \in (E,\tau)'$ be an extension of $f|F$ in τ. Since $f - h \in (E,\xi_1)'$ and $(f-h)(F) = \{0\}$, we note that the map

$$x + F \to (f-h)(x)$$

belongs to $(E/F, \xi_1/F)'$, so $h(x) = f(x)$ for each $x \in E$, i.e., $f = h \in (E,\tau)'$. The same proof runs over for the topology ξ_2. □

Example 15. There exists a nonseparable metrizable and complete tvs $\lambda_0 = (\lambda_0, \xi)$ such that $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is isomorphic to a (dense) $2^{\aleph_0}$-codimensional vector subspace of $\mathbb{R}^{\mathbb{N}}$ satisfying conditions:

(i) $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is analytic unordered Baire-like but not Baire.
(ii) $\mathbb{R}^{\mathbb{N}} \setminus (\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is a Baire space.
(iii) $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ contains a homeomorphic copy of $\mathbb{N}^{\mathbb{N}}$ as a closed subset.

Proof. Very recently [21] Drewnowski and Labuda have constructed nonseparable $F\omega$-spaces λ_0 with a basis $(U_n)_n$ of balanced neighbourhoods of zero closed in $\mathbb{R}^{\mathbb{N}}$: λ_0 is the space of all sequences $x = (\varrho_n)$ of real numbers such that $\|tx\| \to 0$ as $t \to 0$, where $\|x\| := \sup \|x\|_n$, $\|x\|_n := n^{-1} \sum_{j=1}^{n} \min(1, |\varrho_j|)$. Set

$$U_n := \{x \in \lambda_0\colon \ \|x\| \leqslant n^{-1}\}$$

for $n \in \mathbb{N}$. The space λ_0 with the topology generated by the F-norm $\|.\|$ is metrizable, complete, nonseparable, whose topological dual is a $\aleph_0$-dimensional vector space, and sets U_n are closed in $\mathbb{R}^{\mathbb{N}}$, as proved in [21]. We prove claims (i), (ii) and (iii).

(i) Since for fixed $m \in \mathbb{N}$ one has $\lambda_0 = \bigcup_n nU_m$ and each nU_m is $\sigma(\lambda_0, \lambda_0')$-analytic (as complete metrizable and separable), so $E := (\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is analytic and $2^{\aleph_0}$-codimensional. Since $\sigma(\lambda_0, \lambda_0')$ is metrizable, it is equal to the finest locally convex topology ξ_c on λ_0 weaker than ξ. Clearly (λ_0, ξ_c) is unordered Baire-like, see the argument used in Example 11, and (λ_0, ξ_c) is not Baire as a simple application of the closed graph theorem.

(ii) Since $\mathbb{R}^{\mathbb{N}} \setminus E$ is a G_δ-subset of $\mathbb{R}^{\mathbb{N}}$, the Alexandrov's theorem applies to get that $\mathbb{R}^{\mathbb{N}} \setminus E$ is a Polish space, hence Baire.

(iii) Note that E cannot be covered by a sequence of bounded sets. Indeed, let F be the closure of E in $\mathbb{R}^{\mathbb{N}}$, clearly F is isomorphic to $\mathbb{R}^{\mathbb{N}}$. Assume that $\lambda_0 = \bigcup_n S_n$ is the union of bounded closed absolutely convex sets. Since E is unordered Baire-like, then some S_m is a neighbourhood of zero in E, so its closure (in F) is a neighbourhood of zero in F. Consequently $\mathbb{R}^{\mathbb{N}}$ is normed, a contradiction. Hence E is not σ-compact, so [43, Corollary 1.4.5, p. 335] applies to get the conclusion of (iii). □

Recall that $X \setminus Y$ is of the first Baire category if X is a metric space and $Y \subset X$ is its dense Baire analytic subset [33, §11.IV, Corollary 2]. Part (ii) in Example 15 shows that this fails for unordered Baire-like spaces Y when X is a separable Fréchet space.

It is known by Talagrand [47] that if E is a Banach space containing a dense weakly K-analytic subspace, then E is weakly K-analytic. For positive results of this type for K-analytic spaces and K-countable determined spaces in a more general setting we refer to [9] and [11], respectively. Example 15 applies to show that Talagrand's result fails for metrizable lcs in general.

Example 16. There exists a metrizable Baire lcs which is not weakly quasi-Suslin but contains a dense analytic unordered Baire-like subspace which is not Baire.

Proof. Choose a vector subspace H in $\mathbb{R}^{\mathbb{N}}$ such that $L := (\lambda_0, \sigma(\lambda_0, \lambda_0')) \oplus H$ (algebraic direct sum) is Baire and has codimension one in $\mathbb{R}^{\mathbb{N}}$, where $(\lambda_0, \sigma(\lambda_0, \lambda_0'))$ is dense in $\mathbb{R}^{\mathbb{N}}$ from Example 15. Note that L is separable but not analytic (L is finite-codimensional in $\mathbb{R}^{\mathbb{N}}$ and we use the closed graph theorem [15, Theorem 5.2] as in Example 14). By Corollary 6 the space L is not weakly quasi-Suslin. □

Problems. (i) Is a metrizable tvs E analytic if E contains a complete analytic vector subspace F such that E/F is analytic?

(ii) Does there exist a weakly analytic (DF)-space which is not analytic?

Acknowledgment

The authors are very grateful to the referee for very interesting comments and suggestions which led to the present form of the article.

References

[1] A.V. Arkhangel'skii, Topological Function Spaces, Math. Appl., Kluwer, 1992.
[2] R.G. Bartle, L.M. Graves, Mappings between function spaces, Trans. Amer. Math. Soc. 72 (1952) 400–413.
[3] K.D. Bierstedt, J. Bonet, Stefan Heinrich's density condition for Fréchet spaces and the characterization of the distinguished Köthe echelon spaces, Math. Nachr. 35 (1988) 149–180.
[4] K.D. Bierstedt, J. Bonet, Some aspects of the modern theory of Fréchet spaces, Rev. R. Acad. Cienc. Ser. A Mat. 97 (2003) 159–188.
[5] K.D. Bierstedt, J. Bonet, Density conditions in Fréchet and (DF)-spaces, Rev. Mat. Complut. 2 (1989) 59–75.
[6] Cz. Bessaga, A. Pelczyński, Selected Topics in Infinite-Dimensional Topology, Monogr. Mat., vol. 58, PWN, 1978.
[7] H. Buchwalter, J. Schmets, Sur quelques propriétés de l'espace $C_s(T)$, J. Math. Pures Appl. 52 (1973) 337–352.
[8] M.A. Canela, Operator and function spaces which are K-analytic, Port. Math. 42 (1983) 203–218.
[9] B. Cascales, On K-analytic locally convex spaces, Arch. Math. 49 (1987) 232–244.
[10] B. Cascales, J. Orihuela, On compactness in locally convex spaces, Math. Z. 195 (1987) 365–381.
[11] B. Cascales, J. Orihuela, A sequential property of set-valued maps, J. Math. Anal. Appl. 156 (1991) 86–100.
[12] B. Cascales, J. Kąkol, S.A. Saxon, Weight of precompact subsets and tightness, J. Math. Anal. Appl. 269 (2002) 500–518.
[13] B. Cascales, J. Kąkol, S.A. Saxon, Metrizability vs. Fréchet–Urysohn property, Proc. Amer. Math. Soc. 269 (2002) 500–518.
[14] B. Cascales, L. Oncina, Compactoid filter and USCO maps, J. Math. Anal. Appl. 282 (2003) 826–845.
[15] J.P.R. Christensen, Topology and Borel Structure, vol. 10, North-Holland, 1974.
[16] H.H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. 101 (1961) 1–15.
[17] M.M. Day, The spaces L^p, $0 < p < 1$, Bull. Amer. Math. Soc. 46 (1940) 816–823.
[18] P. Dierolf, S. Dierolf, L. Drewnowski, Remarks and examples concerning unordered Baire-like and ultrabarrelled spaces, Colloq. Math. 39 (1978) 109–116.

[19] S. Dierolf, A note on the lifting of linear and locally convex topologies on a quotient space, Collect. Math. 31 (1980) 193–198.
[20] L. Drewnowski, Resolutions of topological linear spaces and continuity of linear maps, J. Math. Anal. Appl. 335 (2007) 1177–1194.
[21] L. Drewnowski, I. Labuda, Sequence F-spaces of L_o-type over submeasures of $\mathbb{N}$, Illinois J. Math., in press.
[22] J.C. Ferrando, J. Kąkol, M. López Pellicer, S.A. Saxon, Tightness and distinguished Fréchet spaces, J. Math. Anal. Appl. 324 (2006) 862–881.
[23] J.C. Ferrando, J. Kąkol, M. López Pellicer, Necessary and sufficient conditions for precompact sets to be metrizable, Bull. Austral. Math. Soc. 74 (2006) 7–13.
[24] J.C. Ferrando, J. Kąkol, M. López Pellicer, S.A. Saxon, Quasi-Suslin weak duals, J. Math. Anal. Appl. 339 (2008) 1253–1263.
[25] J.C. Ferrando, A weakly analytic locally convex space which is not K-analytic, Bull. Austral. Math. Soc. 79 (2009) 31–35.
[26] Z. Frolik, Remarks concerning the invariants of Baire spaces and the mappings, Czechoslovak Math. J. 11 (1961) 381–385.
[27] K. Jarchow, Locally Convex Spaces, Teubner, 1981.
[28] J. Kąkol, A note on compatible vector topologies, Proc. Amer. Math. Soc. 99 (1987) 690–692.
[29] J. Kąkol, M. López Pellicer, Compact coverings for Baire locally convex spaces, J. Math. Anal. Appl. 332 (2007) 965–974.
[30] J. Kąkol, S. Saxon, A.T. Todd, Pseudocompact spaces X and df-spaces $C_c(X)$, Proc. Amer. Math. Soc. 132 (2004) 1703–1712.
[31] W. Kubiś, O. Okunev, P.J. Szeptycki, On some classes of Lindelöf Σ-spaces, Topology Appl. 153 (2006) 2574–2590.
[32] G. Köthe, Topological Vector Spaces II, Springer-Verlag, 1979.
[33] K. Kuratowski, Topology, vol. I, Academic Press, PWN, New York, London, Warszawa, 1966.
[34] I. Labuda, Z. Lipecki, On subseries convergent series and m-quasi-bases in topological linear spaces, Manuscripta Math. 38 (1982) 87–98.
[35] M. Mahowald, Barrelled spaces and the closed graph theorem, J. London Math. Soc. 36 (1961) 108–110.
[36] R. Megginson, An Introduction to Banach Space Theory, Springer, New York, Berlin, 1988.
[37] E. Michael, A short proof of the Arens–Eells embedding theorem, Proc. Amer. Math. Soc. 15 (1964) 415–416.
[38] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxf. Grad. Texts Math., 1997.
[39] K. Nagami, Σ-spaces, Fund. Math. 61 (1969) 169–192.
[40] J. Orihuela, Pointwise compactness in spaces of continuous functions, J. London Math. Soc. 36 (1987) 143–152.
[41] N.T. Peck, N. Porta, Linear topologies which are suprema of duals-less topologies, Studia Math. 67 (1978) 63–73.
[42] P. Perez Carreras, J. Bonet, Barrelled Locally Convex Spaces, vol. 131, North-Holland, 1987.
[43] C.A. Rogers, J.E. Jayne, C. Dellacherie, F. Topsøe, J. Hoffman-Jørgensen, D.A. Martin, A.S. Kechris, A.H. Stone, Analytic Sets, Academic Press, 1980.
[44] W. Roelcke, S. Dierolf, On the three space property for topological vector spaces, Collect. Math. 32 (1981) 3–25.
[45] W. Roelcke, S. Dierolf, Uniform Structures on Topological Groups and Their Quotients, McGraw–Hill Publ. Co., New York, 1981.
[46] S.A. Saxon, A.R. Todd, A property of locally convex Baire spaces, Math. Ann. 206 (1973) 23–34.
[47] M. Talagrand, Espaces de Banach faiblement K-analytiques, Ann. of Math. 110 (1979) 407–438.
[48] M. Talagrand, Sur la structure borélienne des espaces analitiques, Bull. Sci. Math. 101 (1977) 415–422.
[49] E.K. van Douwen, The integers and topology, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretical Topology, North-Holland, 1984, pp. 111–168.
[50] M. Valdivia, Quasi-(LB)-spaces, J. London Math. Soc. 35 (1987) 149–168.
[51] M. Valdivia, Topics in Locally Convex Spaces, North-Holland, Amsterdam, 1982.