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The attractors for the nonhomogeneous nonautonomous Navier–Stokes equations ☆

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Abstract

In this paper, we consider the attractors for the two-dimensional nonautonomous Navier–Stokes equations in nonsmooth bounded domain Ω with nonhomogeneous boundary condition $u = \varphi$ on $\partial\Omega$. Assuming $f = f(x,t) \in L^2_{\mathrm{loc}}((0,T); D(A^{\alpha/4}))$, which is translation compact and $\varphi \in L^\infty(\partial\Omega)$, we establish the existence of the uniform attractor in $L^2(\Omega)$ and $D(A^{1/4})$.

Keywords: Attractor; Measure of noncompactness; Uniformly ω-limit compact; Uniform condition (C); Nonhomogeneous boundary; Navier–Stokes equation

1. Introduction

Let Ω be nonsmooth bounded domain in R^2. We consider two-dimensional Navier–Stokes equations in a bounded Lipschitz domain Ω with nonhomogeneous boundary condition:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0, \\ u = \varphi \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $f = f(x,t) \in L^2_{\mathrm{loc}}((0,T); E)$, where $E = D(A^{\alpha/4})$, $\alpha = -1$ or -2, and $\varphi \in L^\infty(\partial\Omega)$ is time-independent functions. We consider this equation in an appropriate Hilbert space and show that there is an attractor $\mathfrak{A}$ which all solutions approach as $t \to \infty$. The main interest of this work lies in our assumptions on the domain Ω occupied by the fluid as well as on the nonhomogeneous boundary data φ. Indeed, we will only assume that Ω is a (simply connected) Lipschitz domain in R^2 and

$$\varphi \in L^\infty(\partial\Omega), \quad \varphi \cdot n = 0 \quad \text{a.e. on } \partial\Omega, \tag{1.2}$$

where n is the outward unit normal to $\partial\Omega$. Such assumptions are much more physically realistic than the ones in the existing estimates.

In this paper, we reduce the problem (1.1) to the Navier–Stokes equations with homogeneous boundary condition. This will be done by constructing a function ψ (background flow) such that

$$\operatorname{div}\psi = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \psi = \varphi \quad \text{on } \partial\Omega. \tag{1.3}$$

The basic idea of our construction, which is motivated by the works of Miranville and Wang [17] and Brown et al. [3], is to localize the solution of the Stokes system with boundary data φ to a ε-neighborhood of $\partial\Omega$.

In addition, we assume that the function $f(\cdot,t) =: f(t) \in L^2_{\mathrm{loc}}(R; E)$ is translation bounded. This property implies that

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(R;E)} = \sup_{t\in R}\int_t^{t+1}\big\|f(s)\big\|^2_E\,ds < \infty. \tag{1.4}$$

In the last decade the study of the nonautonomous infinite-dimensional dynamical systems has been paid much attention and fast developed. In the book [11] Haraux considers some special classes of such systems and studies systematically the notion of uniform attractor paralleling to that of global attractor for autonomous systems. Later on, Chepyzhov and Vishik [7,8] present a general approach that is well suited to study equations arising in mathematical physics. In this approach, to construct the uniform (or trajectory) attractors, instead of the associated process $\{U_\sigma(t,\tau) \mid t \geqslant \tau,\ \tau \in R\}$ one should consider a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, in some Banach space E, where the functional parameter $\sigma_0(s)$, $s \in R$ is called the symbol and Σ is the symbol space including $\sigma_0(s)$. Naturally from the applications, there is some invariant semigroup acting on Σ and satisfying the so-called translation identity. If the family of processes is $(E \times \Sigma, E)$ continuous, i.e., the mappings $(u,\sigma) \to U_\sigma(t,\tau)u$ are continuous from $E \times \Sigma$ to E, it can be reduced to semigroup by constructing skew product flow. The approach preserves the leading concept of invariance which implies the structure of uniform attractor described by the representation as a union of sections of all kernels of the family of processes. The kernel is the set of all complete trajectories of a process. Moreover, the methods of autonomous systems are applicable. For example, Moise et al. [19] formulate in a systematic way the energy method (the idea belongs to Ball [1]) for the noncompact semiprocesses which extends their earlier work [18] on noncompact semigroup. Following these ways, the strongly compact uniform attractors are obtained for the systems with symbols of strongly compact hulls. In Chepyzhov and Vishik [4,6], a different approach based on the concept of trajectory attractor is developed and has many applications (cf. Bona and Dougalis [2], Chepyzhov and Vishik [5,7,8], Karch [12], Ladyzhenskaya [14], Lu et al. [15], Ma et al. [16], Robinson [20], Temam [22]). For further applications to the nonautonomous systems on unbounded domain, we refer to Efendiev and Zelik [9], Karachlios and Stavrakakis [13], Zelik [23].

In the paper, we study the existence of compact uniform attractor for the nonautonomous Navier–Stokes equations in nonsmooth bounded domain Ω with nonhomogeneous boundary condition $u=\varphi$ on $\partial\Omega$. We apply a new method to nonautonomous Navier–Stokes equation with external forces $f(x,t)$ in $L^2_{\mathrm{loc}}(R;E)$ which is translation compact. To this end, some abstract results are established in Section 4. We give a characterization by the concept of measure of noncompactness as well as a method to verify it.

Throughout this paper we introduce the spaces

$$H=\{L^2(\Omega)\mid \operatorname{div}u=0 \text{ in } \Omega,\ u\cdot n=0 \text{ on } \partial\Omega\},$$
$$V=\{H_0^1(\Omega)\mid \operatorname{div}u=0 \text{ in } \Omega\},$$
$$|\cdot|_p,\ \text{the } L^p(\Omega) \text{ norm},$$
$$\|\cdot\|,\ \text{the norm in } V,$$
$$(\,,\,) \text{ the inner product in } H \text{ or the dual product between } V \text{ and } V',$$
$$((\,,\,)) \text{ the inner product in } V.$$

We can define the powers A^s of A for $s\in R$. The space $V_s=D(A^{s/2})$ turns out to be a Hilbert space with the inner product and the norm

$$(u,v)_{V_s}=\left(A^{s/2}u,A^{s/2}v\right),\qquad \|u\|^2_{V_s}=(u,u)_{V_s}.$$

Here V' is the dual of $V=V_1$. The constants $C_i(c_i)$, $i\in N$, are considered in a generic sense.

2. Setting of the problem

Let Ω be a bounded domain in R^d. We say that Ω is a Lipschitz domain if its boundary $\partial\Omega$ can be covered by finite many balls $B_j=B(Q_j,r_0)$ centered at $Q_j\in\partial\Omega$ such that for each B_j, there exists a rectangular coordinate system and a Lipschitz function $\psi_j: R^{d-1}\to R$ with

$$B(Q_j,3r_0)\cap\Omega=\{(x_1,\dots,x_d)\mid x_d>\psi_j(x_1,\dots,x_{d-1})\}\cap\Omega.$$

Throughout this paper we will assume that Ω is a simply connected Lipschitz domain in R^2.

For a function u on Ω, we define its nontangential maximal function $(u)^*$ by

$$(u)^*(Q)=\sup\{|u(x)|\mid x\in\Omega,\ |x-Q|\leqslant 2\operatorname{dist}(x,\partial\Omega)\},\quad Q\in\partial\Omega. \tag{2.1}$$

As is mentioned in Brown et al. [3], if $\varphi\in L^2(\partial\Omega)$ and $\int_{\partial\Omega}\varphi\cdot n\,d\varsigma=0$, our background flow will be constructed using the solution to the Stokes system:

$$\begin{cases}-\Delta u+\nabla q=0 & \text{in } \Omega,\\ \operatorname{div}u=0 & \text{in } \Omega,\\ u=\varphi \text{ a.e.} & \text{on } \partial\Omega \text{ in the sense of nontangential convergence.}\end{cases} \tag{2.2}$$

There exists a unique u and a unique (up to a constant) q satisfying (2.2) and $(u)^*\in L^2(\partial\Omega)$. In fact, the solution (u,q) will satisfy

$$\int_{\partial\Omega}|(u)^*|^2\,d\varsigma+\int_{\Omega}|\nabla u(x)|^2\operatorname{dist}(x,\partial\Omega)\,dx+\int_{\Omega}|q(x)|^2\operatorname{dist}(x,\partial\Omega)\,dx\leqslant C_0\int_{\partial\Omega}|\varphi|^2\,d\varsigma. \tag{2.3}$$

If, in addition, $\varphi\in L^\infty(\partial\Omega)$, then

$$\sup_{x\in\Omega}\left|u(x)\right| + \sup_{x\in\Omega}\left|\nabla u(x)\right| \operatorname{dist}(x,\partial\Omega) \leqslant C_0\|\varphi\|_{L^\infty(\partial\Omega)}. \tag{2.4}$$

Let $u=(u_1,u_2)$ be the solution of (2.2) with $\varphi\in L^\infty(\partial\Omega)$ and $\varphi\cdot n=0$. Fix $P\in\partial\Omega$. We define

$$g(x)=\int_P^x(-u_2,u_1)\cdot T\,ds, \tag{2.5}$$

where T denotes the unit tangent vector to the path from P to $x=(x_1,x_2)$. Since Ω is simply connected and $\operatorname{div} u=0$ in Ω, g is well defined by Green's theorem, and

$$u=\left(\frac{\partial g}{\partial x_2},-\frac{\partial g}{\partial x_1}\right). \tag{2.6}$$

Moreover, since $u=\varphi$ on $\partial\Omega$ and $\varphi\cdot n=0$ a.e., we have

$$g=0 \quad \text{on } \partial\Omega.$$

Next let $\varepsilon\in(0,c_0\operatorname{diam}(\Omega))$ be a constant. Let $\eta_\varepsilon\in C_0^\infty(R^2)$ such that, $0\leqslant\eta\leqslant1$,

$$\begin{cases}\eta_\varepsilon=1 & \text{in } \{x\in R^2 \mid \operatorname{dist}(x,\partial\Omega)\leqslant c_1\varepsilon\},\\ \eta_\varepsilon=0 & \text{in } \{x\in R^2 \mid \operatorname{dist}(x,\partial\Omega)\geqslant c_2\varepsilon\},\end{cases} \tag{2.7}$$

and

$$\left|\nabla^s\eta_\varepsilon\right|\leqslant c_s/\varepsilon^{|s|}. \tag{2.8}$$

We remark that η_ε can be found in the form $f(\frac{\rho(x)}{\varepsilon})$ where $\rho\in C^\infty$ is a regularized distance function to $\partial\Omega$ and f is a standard bump function.

Finally, we define the background flow

$$\psi=\psi_\varepsilon=\left(\frac{\partial}{\partial x_2}(g\eta_\varepsilon),-\frac{\partial}{\partial x_1}(g\eta_\varepsilon)\right). \tag{2.9}$$

Clearly, $\operatorname{div}\psi=0$ in Ω, $\psi=u$ in $\{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)<c_1\varepsilon\}$. Hence, $\psi=\varphi$ on $\partial\Omega$ in the sense of nontangential convergence. Also note that

$$\operatorname{supp}\psi\subset\left\{x\in\overline{\Omega} \mid \operatorname{dist}(x,\partial\Omega)\leqslant c_2\varepsilon\right\}. \tag{2.10}$$

Therefore, we have from Brown et al. [3]:

Lemma 2.1. *With φ and ψ as above, we have*

$$\|\psi\|_{L^\infty(\Omega)}\leqslant C_1\|\varphi\|_{L^\infty(\partial\Omega)}. \tag{2.11}$$

Lemma 2.2. *Let $2\leqslant p\leqslant\infty$. Then*

$$\left\||\nabla\psi|\operatorname{dist}(\cdot,\Omega)^{1-1/p}\right\|_{L^p(\Omega)}\leqslant C_2\|\varphi\|_{L^p(\partial\Omega)}. \tag{2.12}$$

Lemma 2.3. *Let ψ be defined by* (2.9). *Then*

$$\Delta\psi=\nabla(q\eta_\varepsilon)+F, \tag{2.13}$$

where $\operatorname{supp}F\subset\{x\in\Omega \mid c_1\varepsilon\leqslant\operatorname{dist}(x,\partial\Omega)\leqslant c_2\varepsilon\}$ *and*

$$\|F\|_{L^2(\Omega)}\leqslant\frac{C_3}{\varepsilon^{3/2}}\|\varphi\|_{L^2(\partial\Omega)}. \tag{2.14}$$

We now set $v = u - \psi$ where u is a solution of (1.1). Using (2.13), we see that

$$\begin{cases} \frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + \nabla(p + \nu q \eta_\varepsilon) \\ \quad = f + \nu F - (\psi \cdot \nabla)\psi; \\ \operatorname{div} v = 0; \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

3. Preliminary results

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau) \mid t \geqslant \tau,\ \tau \in R\}$ act on E:

$$U(t,\tau): E \to E, \quad t \geqslant \tau,\ \tau \in R.$$

Definition 3.1. A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in E if

$$U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t \geqslant s \geqslant \tau,\ \tau \in R, \tag{3.1}$$

$$U(\tau,\tau) = \mathrm{Id}, \quad \tau \in R. \tag{3.2}$$

By $\mathcal{B}(E)$ we denote the collection of the *bounded* sets of E. We consider a family of processes $\{U_\sigma(t,\tau)\}$ depending on a parameter $\sigma \in \Sigma$. The parameter σ is said to be the *symbol* of the process $\{U_\sigma(t,\tau)\}$ and the set Σ is said to be the *symbol space*. In the sequel Σ is assumed to be a complete metric space.

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to be *uniformly* (with respect to (*w.r.t.*) $\sigma \in \Sigma$) *bounded* if for any $B \in \mathcal{B}(E)$ the set

$$\bigcup_{\sigma \in \Sigma} \bigcup_{\tau \in R} \bigcup_{t \geqslant \tau} U_\sigma(t,\tau)B \in \mathcal{B}(E). \tag{3.3}$$

A set $B_0 \subset E$ is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *absorbing* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in R$ and every $B \in \mathcal{B}(E)$ there exists $t_0 = t_0(\tau, B) \geqslant \tau$ such that $\bigcup_{\sigma \in \Sigma} U_\sigma(t,\tau)B \subseteq B_0$ for all $t \geqslant t_0$.

A set $P \subset E$ is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for an arbitrary fixed $\tau \in R$,

$$\lim_{t \to +\infty} \Big(\sup_{\sigma \in \Sigma} \operatorname{dist}_E\big(U_\sigma(t,\tau)B, P\big)\Big) = 0. \tag{3.4}$$

A family of processes possessing a compact uniformly absorbing set is called *uniformly compact* and a family of processes possessing a compact uniformly attracting set is called *uniformly asymptotically compact*.

Definition 3.2. A closed set $\mathcal{A}_\Sigma \subset E$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if it is uniformly (w.r.t. $\sigma \in \Sigma$) attracting and it is contained in any closed uniformly (w.r.t. $\sigma \in \Sigma$) attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$: $\mathcal{A}_\Sigma \subseteq \mathcal{A}'$.

Let us return to general families of processes.

A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, acting in E is said to be $(E \times \Sigma, E)$-*continuous*, if for all fixed t and τ, $t \geqslant \tau$, $\tau \in R$ the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E \times \Sigma$ into E.

A curve $u(s)$, $s \in R$ is said to be *a complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t), \quad \forall t \geqslant \tau,\ \tau \in R. \tag{3.5}$$

The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \bigl\{u(\cdot) \bigm| u(\cdot) \text{ satisfies (3.5) and } \bigl\|u(s)\bigr\|_E \leqslant M_u \text{ for } s \in R\bigr\}.$$

The set

$$\mathcal{K}(s) = \bigl\{u(s) \mid u(\cdot) \in \mathcal{K}\bigr\} \subseteq E$$

is said to be the *kernel section* at a time moment $t = s$, $s \in R$.

We consider two projectors Π_1 and Π_2 from $E \times \Sigma$ onto E and Σ, respectively:

$$\Pi_1(u,\sigma) = u, \qquad \Pi_2(u,\sigma) = \sigma.$$

Now we recall the basic results in Chepyzhov and Vishik [5,7].

Theorem 3.1. *Let a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ acting in the space E be uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact and $(E \times \Sigma, E)$-continuous. Also let Σ be a compact metric space and let $\{T(t)\}$ be a continuous invariant $(T(t)\Sigma = \Sigma)$ semigroup on Σ satisfying translation identity*

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma,\ t \geqslant \tau,\ \tau \in R,\ s \geqslant 0. \tag{3.6}$$

Then the semigroup $\{S(t)\}$ corresponding to the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ and acting on $E \times \Sigma$:

$$S(t)(u,\sigma) = \bigl(U_\sigma(t,0)u, T(t)\sigma\bigr), \quad t \geqslant 0,\ (u,\sigma) \in E \times \Sigma,$$

possesses the compact attractor $\mathcal{A}$ which is strictly invariant with respect to $\{S(t)\}$: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$. Moreover,

(i) $\Pi_1\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_\Sigma$ *is the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$;*
(ii) $\Pi_2\mathcal{A} = \mathcal{A}_2 = \Sigma$;
(iii) *the global attractor satisfies*

$$\mathcal{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\};$$

(iv) *the uniform attractor satisfies*

$$\mathcal{A}_\Sigma = \mathcal{A}_1 = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0).$$

Here $\mathcal{K}_\sigma(0)$ is the section at $t = 0$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \Sigma$.

4. Existence and structure of uniform attractor

For convenience, let $B_t = \bigcup_{\sigma\in\Sigma}\bigcup_{s\geqslant t} U_\sigma(s,t)B$, the closure $\overline{B}$ of the set B and $R_\tau = \{t \in R \mid t \geqslant \tau\}$. Define the uniform (*w.r.t.* $\sigma \in \Sigma$) ω-limit set $\omega_{\tau,\Sigma}(B)$ of B by $\omega_{\tau,\Sigma}(B) = \bigcap_{t\geqslant\tau} \overline{B}_t$ which can be characterized, analogously to that for semigroup, by the following:

$$\begin{cases} y \in \omega_{\tau,\Sigma}(B) \;\Leftrightarrow\; \text{there are sequences } \{x_n\} \subset B,\ \{\sigma_n\} \subset \Sigma,\ \{t_n\} \subset R_\tau \\ \quad \text{such that } t_n \to +\infty \text{ and } U_{\sigma_n}(t_n,\tau)x_n \to y\ (n\to\infty). \end{cases} \tag{4.1}$$

We will characterize the existence of uniform attractor for a family of processes satisfying (3.6) in term of the concept of measure of noncompactness that is put forward first by Kuratowski.

Let $B \in \mathcal{B}(E)$. Its Kuratowski measure of noncompactness $\kappa(B)$ is defined by

$$\kappa(B) = \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets of diameter} \leqslant \delta\}.$$

It has following properties (see Hale [10], Sell and You [21]).

Lemma 4.1. *Let $B, B_1, B_2 \in \mathcal{B}(E)$. Then*

(1) $\kappa(B) = 0 \Leftrightarrow \kappa(\mathcal{N}(B,\varepsilon)) \leqslant 2\varepsilon \Leftrightarrow \overline{B}$ *is compact*;
(2) $\kappa(B_1 + B_2) \leqslant \kappa(B_1) + \kappa(B_2)$;
(3) $\kappa(B_1) \leqslant \kappa(B_2)$ *whenever* $B_1 \subset B_2$;
(4) $\kappa(B_1 \cup B_2) \leqslant \max\{\kappa(B_1), \kappa(B_2)\}$;
(5) $\kappa(\overline{B}) = \kappa(B)$;
(6) *if B is a ball of radius ε then $\kappa(B) \leqslant 2\varepsilon$.*

Lemma 4.2. *Let $\cdots \supset F_n \supset F_{n+1} \supset \cdots$ be a sequence of nonempty closed subsets of E such that $\kappa(F_n) \to 0$ as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.*

Definition 4.1. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to be uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact if for any $\tau \in R$ and $B \in \mathcal{B}(E)$ the set B_t is bounded for every t and $\lim_{t\to\infty} \kappa(B_t) = 0$.

Proposition 4.1. *If $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact, then for any $\{x_n\} \subset B \in \mathcal{B}(E)$, $\{\sigma_n\} \subset \Sigma$, $\{t_n\} \subset R_\tau$, $t_n \to +\infty$ as $n \to \infty$, there exists a convergent subsequence of $\{U_{\sigma_n}(t_n,\tau)x_n\}$ whose limit lies in $\omega_{\tau,\Sigma}(B)$.*

Proof. For any $\varepsilon > 0$, it derives from Definition 4.1 and (3)–(4) of Lemma 4.1 that for a sufficiently large N_0,

$$\kappa\big(\big\{U_{\sigma_n}(t_n,\tau)x_n \mid n \in N\big\}\big) = \kappa\big(\big\{U_{\sigma_n}(t_n,\tau)x_n \mid n \geqslant N_0\big\}\big) \leqslant \varepsilon. \tag{4.2}$$

Let $\varepsilon \to 0$, then by (1) of Lemma 4.1 $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact. (4.1) informs all limits of the convergent subsequences lie in $\omega_{\tau,\Sigma}(B)$. □

Proposition 4.2. *If $\{U_\sigma(t,\tau)\}$ is uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact, then for any $\tau \in R$ and $B \in \mathcal{B}(E)$,*

(i) $\omega_{\tau,\Sigma}(B)$ *is nonempty and compact*;

(ii) $\lim_{t\to+\infty}(\sup_{\sigma\in\Sigma}\operatorname{dist}(U_\sigma(t,\tau)B,\omega_{\tau,\Sigma}(B)))=0$;
(iii) *if Y is a closed set uniformly* (*w.r.t.* $\sigma\in\Sigma$) *attracting B then* $\omega_{\tau,\Sigma}(B)\subseteq Y$.

Also let Σ be a compact metric space and let $\{T(t)\}$ be a continuous invariant $T(t)\Sigma=\Sigma$ on Σ satisfying translation identity (3.6). *Then*

(iv) $\omega_{\tau,\Sigma}(B)=\omega_{0,\Sigma}(B)$, *that is, the set $\omega_{\tau,\Sigma}(B)$ is independent on $\tau\in R$.*

Proof. (i) Obviously, for any increasing sequence $\{t_n\}\subset R_\tau$ such that $t_n\to+\infty$ as $n\to\infty$, $\omega_{\tau,\Sigma}(B)=\bigcap_{n=1}^{\infty}\overline{B}_{t_n}$. Since $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, is uniformly (*w.r.t.* $\sigma\in\Sigma$) ω-limit compact and $B\in\mathcal{B}(E)$, we can find such a sequence of $\{t_n\}$ that $\kappa(\overline{B}_{t_n})\leqslant 1/n$. Thanks to Lemma 4.2, $\omega_{\tau,\Sigma}(B)$ is nonempty and compact.

(ii) and (iii) Noticing Proposition 4.1, the proofs are similar to those of Proposition VII.1.1 in Chepyzhov and Vishik [7]. So we omit here.

(iv) If $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, satisfies (3.6), then its uniformly (*w.r.t.* $\sigma\in\Sigma$) absorbing set B_0 is independent of τ. In fact, let B_0 be the one for $\tau=0$. Then for any fixed $\tau\in R$ and $B\in\mathcal{B}(E)$, by $\bigcup_{\sigma\in\Sigma}U_\sigma(t,\tau)B=\bigcup_{\sigma\in\Sigma}U_\sigma(t-\tau,0)B$ which implies $T_0(\tau,B)=\tau+T_0(0,B)$. Similarly from (4.1), we find $\omega_{\tau,\Sigma}(B)=\omega_{0,\Sigma}(B)$ for all $\tau\in R$. □

Theorem 4.1. *Let Σ be a compact metric space and let $\{T(t)\}$ be a continuous invariant $T(t)\Sigma=\Sigma$ on Σ satisfying translation identity* (3.6). *A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$, acting in E is* $(E\times\Sigma,E)$-(*weakly*) *continuous and possesses compact uniform* (*w.r.t.* $\sigma\in\Sigma$) *attractor $\mathcal{A}_\Sigma$ satisfying*

$$\mathcal{A}_\Sigma=\omega_{0,\Sigma}(B_0)=\omega_{\tau,\Sigma}(B_0),\quad \forall\tau\in R, \tag{4.3}$$

if and only if it

(i) *has a bounded uniformly* (*w.r.t.* $\sigma\in\Sigma$) *absorbing set* B_0; *and*
(ii) *is uniformly* (*w.r.t.* $\sigma\in\Sigma$) *ω-limit compact.*

Proof. The sufficiency follows immediately from Proposition 4.2.

We now prove the necessity. First, any ε-neighborhood of A_Σ is a uniformly (*w.r.t.* $\sigma\in\Sigma$) absorbing set. Second, for any $\tau\in R$, $B\in\mathcal{B}(E)$ and $\varepsilon>0$, there exists $t_\varepsilon=t(\tau,B,\varepsilon)\geqslant\tau$ such that $B_{t_\varepsilon}\subset\mathcal{N}(\mathcal{A}_\Sigma,\varepsilon/2)$. Since A_Σ is compact, by Lemma 4.1 $\kappa(B_{t_\varepsilon})\leqslant\kappa(\mathcal{N}(\mathcal{A}_\Sigma,\varepsilon/2))\leqslant\varepsilon$ which implies the uniform ω-limit compactness. □

We present now a method to verify the uniform (*w.r.t.* $\sigma\in\Sigma$) ω-limit compactness.

Definition 4.2. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ is said to be satisfying uniform (*w.r.t.* $\sigma\in\Sigma$) Condition (C) if for any fixed $\tau\in R$, $B\in\mathcal{B}(E)$ and $\varepsilon>0$, there exist $t_0=t(\tau,B,\varepsilon)\geqslant\tau$ and a finite-dimensional subspace E_1 of E such that

(i) $P(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geqslant t_0}U_\sigma(t,\tau)B)$ is bounded; and
(ii) $\|(I-P)(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geqslant t_0}U_\sigma(t,\tau)x)\|\leqslant\varepsilon$, $\forall x\in B$,

where $P:E\to E_1$ is a bounded projector.

Proposition 4.3. *A family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *satisfies uniform* (*w.r.t.* $\sigma \in \Sigma$) *condition* (C) *implies uniform* (*w.r.t.* $\sigma \in \Sigma$) ω*-limit compactness. Moreover, if E is a uniformly convex Banach space then the converse is true.*

Proof. From (2), (3) and (6) of Lemma 4.1, for any $\tau \in R$, $B \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exists $t_0 = t(\tau, B, \varepsilon) \geqslant \tau$ such that

$$\kappa(B_{t_0}) \leqslant \kappa(PB_{t_0}) + \kappa\big((I-P)B_{t_0}\big) \leqslant \kappa\big(\mathcal{N}(0,\varepsilon)\big) = 2\varepsilon, \tag{4.4}$$

where $P : E \to E_1$ and dimension of E_1 is finite. This means $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is uniformly (*w.r.t.* $\sigma \in \Sigma$) ω-limit compact.

On the other hand, there exists $t_0 = t(\tau, B, \varepsilon) \geqslant \tau$ such that B_{t_0} is covered by some finite number of subsets $A_1, A_2, \ldots, A_n$ with diameters less than ε. Let $x_i \in A_i$ and $E_1 = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$. Since E is uniformly convex, there exists a projection $P : E \to E_1$ such that for any $x \in E$, $\|x - Px\| = \operatorname{dist}(x, E_1)$. Hence

$$\big\|(I-P)x\big\| \leqslant \operatorname{dist}\big(x, \{x_1, x_2, \ldots, x_n\}\big) \leqslant \varepsilon, \quad \forall x \in B_{t_0}. \tag{4.5}$$

Namely $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is satisfying uniform (*w.r.t.* $\sigma \in \Sigma$) condition (C). □

It follows from Theorem 4.1 and Proposition 4.3 that

Theorem 4.2. *Let* Σ *be a compact metric space and let* $\{T(t)\}$ *be a continuous invariant* $T(t)\Sigma = \Sigma$ *on* Σ *satisfying translation identity* (3.6). *A family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *acting in E is* $(E \times \Sigma, E)$-(*weakly*) *continuous and possesses compact uniform* (*w.r.t.* $\sigma \in \Sigma$) *attractor* A_Σ *satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0) = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0), \quad \forall \tau \in R, \tag{4.6}$$

if it

(i) *has a bounded uniformly* (*w.r.t.* $\sigma \in \Sigma$) *absorbing set* B_0; *and*
(ii) *satisfies uniform* (*w.r.t.* $\sigma \in \Sigma$) *condition* (C).

Moreover, if E is a uniformly convex Banach space then the converse is true.

5. Translation compact functions

Let us describe a typical symbol space Σ for a particular problem. We are given some fixed symbol $\sigma_0(s)$, $s \in R$. We choose an appropriate enveloping topological space $\Xi = \{\zeta(s) \mid s \in R\}$ such that $\sigma_0(s) \in \Xi$. Consider the closure in Ξ of the following set:

$$\big\{T(h)\sigma_0(s) \mid h \in R\big\} = \big\{\sigma_0(h+s) \mid h \in R\big\}.$$

This closure is said to be the hull of the function $\sigma_0(s)$ in Ξ and is denoted by

$$\mathcal{H}(\sigma_0) = \big[\big\{T(h)\sigma_0 \mid h \in R\big\}\big]_\Xi.$$

Here $[\cdot]_\Xi$ denotes the closure in Ξ. Evidently, $T(h)\mathcal{H}(\sigma_0) = \mathcal{H}(\sigma_0)$ for any $h \in R$.

Definition 5.1. The function $\sigma_0(s) \in \Xi$ is said to be translation compact in Ξ if the hull $\mathcal{H}(\sigma_0)$ is compact in Ξ.

Now recall the following facts which can be found in Chepyzhov and Vishik [7].

Lemma 5.1. *A set $\Sigma \subset L^p_{\mathrm{loc}}(R; E)$ is precompact in $L^p_{\mathrm{loc}}(R; E)$ if and only if the set $\Sigma_{[t_1,t_2]}$ is precompact in $L^p(t_1, t_2; E)$ for every segment $[t_1, t_2] \subset R$. Here $\Sigma_{[t_1,t_2]}$ denotes the restriction of the set Σ to the segment $[t_1, t_2]$.*

Proposition 5.1. *Assume that $f(s) \in L^2_c(R; E)$ is translation compact, then for any $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\sup_{t\in R} \int_t^{t+\eta} \|f(s)\|_E^2 \, ds \leqslant \varepsilon. \tag{5.1}$$

Proof. $f(s) \in L^2_c(R; E)$ means that $\{f(s+t) \mid t \in R\}$ is precompact in $L^2_{\mathrm{loc}}(R; E)$ which is equivalent to that, from Lemma 5.1, $\{f(s+t) \mid t \in R\}|_{s\in[0,1]}$ is precompact in $L^2(0, 1; E)$. So for any $\varepsilon > 0$ there exist finite number $g_1(s), \dots, g_N(s) \in L^2(0, 1; E)$ such that

$$\big\{f(s+t) \mid t \in R\big\}\big|_{s\in[0,1]} \subset \bigcup_{i=1}^{N} B_{L^2(0,1;E)}\left(g_i, \frac{\varepsilon}{4}\right). \tag{5.2}$$

Then there exists $0 < \eta = \eta(\varepsilon) < 1$ satisfying

$$\max_{i=1,\dots,N} \int_0^{\eta} \|g_i(s)\|_E^2 \, ds \leqslant \frac{\varepsilon}{4}. \tag{5.3}$$

From (5.2) and (5.3), for any $t \in R$ there exists $i \in \{1, \dots, N\}$ such that

$$\int_0^{\eta} \|f(s+t)\|_E^2 \, ds \leqslant 2\int_0^{\eta} \|f(s+t) - g_i(s)\|_E^2 \, ds + 2\int_0^{\eta} \|g_i(s)\|_E^2 \, ds \leqslant \varepsilon, \tag{5.4}$$

which implies

$$\int_t^{t+\eta} \|f(s)\|_E^2 \, ds \leqslant \varepsilon. \qquad \square$$

6. Uniform attractor of nonautonomous Navier–Stokes equations

This section deals with the existence of the attractor for the two-dimensional nonautonomous Navier–Stokes equations in a bounded Lipschitz domain Ω with nonhomogeneous boundary condition (see Brown et al. [3]).

Let $A = -P\Delta$ denote the Stokes operator and $B(u, v) = P[(u \cdot \nabla)v]$, where P is the orthogonal projector in $L^2(\Omega)$ on the space H. We may rewrite the Navier–Stokes equations (2.15) for v in the form

$$\frac{dv}{dt} + \nu Av + B(v, v) + B(v, \psi) + B(\psi, v) = P(f + \nu F) - B(\psi, \psi), \tag{6.1}$$

$$v(x, \tau) = v_\tau(x) \in H. \tag{6.2}$$

We first establish the existence of solution of (6.1) and (6.2) by the standard Faedo–Galerkin method.

Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \quad \lambda_j \to +\infty \text{ as } j \to \infty, \tag{6.3}$$

and let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of H such that $Aw_j = \lambda_j w_j$. Fix $m \geqslant 1$, let

$$v_m = \sum_{j=1}^{m} g_{jm}(t) w_j.$$

We solve the system of ODE's

$$\begin{cases} \left(\frac{\partial v_m}{\partial t}, w_j\right) + \nu((v_m, w_j)) + b(v_m, v_m, w_j) + b(\psi, v_m, w_j) + b(v_m, \psi, w_j) \\ \quad = (\bar{f}, w_j) - b(\psi, \psi, w_j), \quad j = 1, 2, \ldots, m; \\ v_m(0) = P_m v_0, \end{cases} \tag{6.4}$$

where $b(u, v, w) = (B(u, v), w)$, $\bar{f} = P(f + \nu F)$, and $P_m : H \to \operatorname{span}\{w_1, \ldots, w_m\}$ is the projector. We claim that $b(\psi, v_m, w_j)$, $b(v_m, \psi, w_j)$ and $b(\psi, \psi, w_j)$ are well defined. This follows easily from the estimate (see Brown et al. [3]).

Here the forcing functions f and ψ satisfy (1.3) and (1.4) for the nonhomogeneous boundary condition as is constructed in Brown et al. [3] and we have the following inequalities:

$$\left|\psi(x)\right| + \left|\nabla\psi(x)\right| \operatorname{dist}(x, \partial\Omega) \leqslant C_4, \quad \forall x \in \Omega, \tag{6.5}$$

$$\int_{\Omega} \left|\nabla\psi(x)\right|^2 \operatorname{dist}(x, \partial\Omega)\, dx \leqslant C_5. \tag{6.6}$$

In Brown et al. [3], the authors have shown that the semigroup $S(t) : H \to H$ $(t \geqslant 0)$ associated with the autonomous systems (6.1) and (6.2) possesses a global attractor in H and a bounded absorbing set in $D(A^{1/4})$. The main objective of this section is to prove that the nonautonomous systems (6.1) and (6.2) have uniform attractors in H and $D(A^{1/4})$.

To this end, we first state some results selected from Brown et al. [3].

Lemma 6.1 *(Hardy's inequality). There exists a constant C_6 such that for any $u \in H_0^1(\Omega)$,*

$$\int_{\Omega} \frac{|u(x)|^2}{[\operatorname{dist}(x, \partial\Omega)]^2}\, dx \leqslant C_6 \int_{\Omega} \left|\nabla u(x)\right|^2 dx. \tag{6.7}$$

Lemma 6.2. *There exists a constant C_7 such that for any $u \in D(A^{1/4})$,*

$$\int_{\Omega} \frac{|u(x)|^2}{\operatorname{dist}(x, \partial\Omega)}\, dx \leqslant C_7 \int_{\Omega} \left|A^{1/4}u(x)\right|^2 dx, \tag{6.8}$$

$$|u|_4 \leqslant C_7 \left|A^{1/4}u\right|_2. \tag{6.9}$$

Proposition 6.1. *Let $f \in D(A^{\alpha/4})$ where $\alpha = -1$ or -2 and let $v_0 \in H$. ψ satisfies* (6.5) *and* (6.6). *Then the problem* (6.1) *and* (6.2) *has a unique solution $v(t)$ such that for any $T > 0$,*

$$v \in C\big([0, T]; H\big) \cap L^2\big([0, T]; V\big), \quad \frac{dv}{dt} \in L^2\big((0, T), V'\big), \tag{6.10}$$

and such that for almost all $t \in [0, T]$ and for any $w \in V$,

$$\left(\frac{\partial v}{\partial t}, w\right) + \nu\big(\big(v(t), w\big)\big) + b\big(v(t), v(t), w\big) + b\big(\psi, v(t), w\big) + b\big(v(t), \psi, w\big)$$
$$= (\bar{f}, w) - b(\psi, \psi, w). \tag{6.11}$$

Proof. The proof of Proposition 6.1 is similar to the autonomous Navier–Stokes in Brown et al. [3]. □

Recall that the power of the Stokes operator A are defined for $z \in C$ by

$$A^z g = \sum_j \lambda_j^z a_j w_j \quad \text{for } g = \sum_j a_j w_j$$

and

$$D\big(A^z\big) = \big\{g \mid A^z g \in H\big\} = \left\{g = \sum a_j w_j \;\Big|\; \sum_j \lambda_j^{2\operatorname{Re} z} |a_j|^2 < \infty\right\}.$$

Now we will write (6.1), (6.2) in the operator form

$$\partial_t v = A_{\sigma(t)}(v), \quad v|_{t=\tau} = v_\tau, \tag{6.12}$$

where $\sigma(s) = f(x, s)$ is the symbol of Eq. (6.12). Thus, if $v_\tau \in H$, then problem (6.12) has a unique solution $v(t) \in C([0, T]; H) \cap L^2([0, T]; V)$. This implies that the process $\{U_\sigma(t, \tau)\}$ given by the formula $U_\sigma(t, \tau)v_\tau = v(t)$ is defined in H.

We now define the *symbol space* $\mathcal{H}(\sigma_0)$ for (6.12). Let a fixed symbol $\sigma_0(s) = f_0(s) = f_0(\cdot, s)$ be translation compact in $L^2_{\mathrm{loc}}(R; E)$; that is, the family of translation $\{f_0(s + h), h \in R\}$ forms a precompact set in $L^2_{\mathrm{loc}}([T_1, T_2]; E)$, where $[T_1, T_2]$ is an arbitrary interval of the time axis R.

As $f_0(x, s)$ is translation compact in $L^2_{\mathrm{loc}}(R; E)$, the hull

$$\mathcal{H}(\sigma_0) = \mathcal{H}(f_0) = \big[f_0(x, s + h) \mid h \in R\big]_{L^2_{\mathrm{loc}}(R; E)}$$

is compact in $\Xi = L^2_{\mathrm{loc}}(R; E)$.

Now, for any $f(x, t) \in \mathcal{H}(f_0)$, the problem (6.12) with f instead of f_0 possesses a corresponding process $\{U_f(t, \tau)\}$ acting on H. As is proved in Chepyzhov and Vishik [7], the family $\{U_f(t, \tau) \mid f \in \mathcal{H}(f_0)\}$ of processes is $(H \times \mathcal{H}(f_0); H)$-continuous.

Let

$$\mathcal{K}_f = \big\{v_f(x, t) \text{ for } t \in R \;\big|\; v_f(x, t) \text{ is solution of (6.12) satisfying } \big\|v_f(\cdot, t)\big\|_H \leqslant M_f \text{ for all } t \in R\big\}$$

be the so-called kernel of the process $\{U_f(t, \tau)\}$.

Proposition 6.2. *The process* $\{U_f(t,\tau)\}: H \to H(D(A^{1/4}))$ *associated with Eq.* (6.12) *possesses absorbing sets*

$$\mathcal{B}_0 = \{v \in H \mid |v|_2 \leqslant \rho_0\} \quad \text{and} \quad \mathcal{B}_1 = \{v \in D(A^{1/4}) \mid |A^{1/4}v|_2 \leqslant \rho_1\}$$

which absorb all bounded sets of H. *Moreover,* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *absorb all bounded sets of* H *and* $D(A^{1/4})$ *in the norms of* H *and* $D(A^{1/4})$*, respectively.*

Proof. The proof of Proposition 6.2 is similar to that of the autonomous Navier–Stokes equation. We can obtain absorbing sets in H and $D(A^{1/4})$ following Brown et al. [3], Chepyzhov and Vishik [7], and Temam [22]. □

The main results in this section are as follows.

Now we prove the existence of compact uniform (*w.r.t.* $f \in \mathcal{H}(f_0)$) attractors in H and $D(A^{1/4})$ by applying the method established in Section 4.

Theorem 6.1. *If* $f_0(x,s)$ *is translation compact in* $L^2_{\text{loc}}(R;V')$*, then the processes* $\{U_{f_0}(t,\tau)\}$ *corresponding to problem* (6.12) *possesses compact uniform* (*w.r.t.* $\tau \in R$) *attractor* $\mathfrak{A}_0$ *in* H *which coincides with the uniform* (*w.r.t.* $f \in \mathcal{H}(f_0)$) *attractor* $\mathfrak{A}_{\mathcal{H}(f_0)}$ *of the family of processes* $\{U_f(t,\tau) \mid f \in \mathcal{H}(f_0)\}$:

$$\mathfrak{A}_0 = \mathfrak{A}_{\mathcal{H}(f_0)} = \omega_{0,\mathcal{H}(f_0)}(\mathcal{B}_0) = \bigcup_{f \in \mathcal{H}(f_0)} \mathcal{K}_f(0), \tag{6.13}$$

where $\mathcal{B}_0$ *is the uniformly* (*w.r.t.* $f \in \mathcal{H}(f_0)$) *absorbing set in* H *and* $\mathcal{K}_f$ *is the kernel of the process* $\{U_f(t,\tau)\}$. *Furthermore, the kernel* $\mathcal{K}_f$ *is nonempty for all* $f \in \mathcal{H}(f_0)$.

Proof. As in the previous section, for fixed N, let H_1 be the subspace spanned by $w_1,\dots,w_N$, and H_2 the orthogonal complement of H_1 in H. We write

$$v = v_1 + v_2, \quad v_1 \in H_1, \quad v_2 \in H_2 \quad \text{for any } v \in H.$$

Now, we only have to verify condition (C). Namely, we need to estimate $|v_2(t)|_2$, where $v(t) = v_1(t) + v_2(t)$ is a solution of Eqs. (6.1) and (6.2) given in Proposition 6.1.

Multiplying Eq. (6.1) by v_2, we have

$$\begin{aligned}&\left(\frac{dv}{dt}, v_2\right) + (\nu Av, v_2) + \big(B(v,v), v_2\big) + \big(B(v,\psi), v_2\big) + \big(B(\psi,v), v_2\big)\\&\quad = (\bar{f}, v_2) - \big(B(\psi,\psi), v_2\big).\end{aligned} \tag{6.14}$$

It follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_2|_2^2 + \nu|A^{1/2}v_2|_2^2 &\leqslant \big|\big(B(v,v),v_2\big)\big| + \big|\big(B(v,\psi),v_2\big)\big| + \big|\big(B(\psi,v),v_2\big)\big|\\&\quad + \big|(f,v_2)\big| + \nu\big|(F,v_2)\big| + \big|\big(B(\psi,\psi),v_2\big)\big|.\end{aligned} \tag{6.15}$$

We have to estimate each term in the right-hand side of (6.15).

First, by Hölder's inequality, Lemma 6.2 and Proposition 6.2,

$$\begin{aligned}\left|\left(B(v,v),v_2\right)\right| &\leqslant \int_\Omega |v||\nabla v||v_2|\,dx \leqslant |v|_4\left|A^{1/2}v\right|_2|v_2|_4\\ &\leqslant C_7^2\left|A^{1/4}v\right|_2\left|A^{1/2}v\right|_2\left|A^{1/4}v_2\right|_2 \leqslant \frac{C_7^2\rho_1}{\lambda_{m+1}^{1/4}}\left|A^{1/2}v\right|_2\left|A^{1/2}v_2\right|_2\\ &\leqslant \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2+\frac{3C_7^4\rho_1^2}{\nu\lambda_{m+1}^{1/2}}\left|A^{1/2}v\right|_2^2. \end{aligned}\tag{6.16}$$

Next, using (6.5), (2.10), (6.7) and the Cauchy inequality,

$$\begin{aligned}\left|\left(B(v,\psi),v_2\right)\right| &\leqslant \int_\Omega |v||\nabla \psi||v_2|\,dx \leqslant C_4\int_{\mathrm{dist}(x,\partial\Omega)\leqslant c_2\varepsilon}\frac{|v|}{\mathrm{dist}(x,\partial\Omega)}|v_2|\,dx\\ &\leqslant C_4c_3\left(\int_\Omega \frac{|v|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx\right)^{1/2}|v_2|_2 \leqslant \frac{C_4C_6}{\lambda_{m+1}^{1/2}}c_3\left|A^{1/2}v\right|_2\left|A^{1/2}v_2\right|_2\\ &\leqslant \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2+\frac{3c_3^2C_4^2C_6^2}{\nu\lambda_{m+1}}\left|A^{1/2}v\right|_2^2. \end{aligned}\tag{6.17}$$

Similarly by (6.5),

$$\begin{aligned}\left|\left(B(\psi,v),v_2\right)\right| &\leqslant \int_\Omega |\psi||\nabla v||v_2|\,dx \leqslant C_4\int_\Omega |\nabla v||v_2|\,dx \leqslant C_4|\nabla v|_2|v_2|_2\\ &\leqslant \frac{C_4}{\lambda_{m+1}^{1/2}}\left|A^{1/2}v\right|_2\left|A^{1/2}v_2\right|_2 \leqslant \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2+\frac{3C_4^2}{\nu\lambda_{m+1}}\left|A^{1/2}v\right|_2^2. \end{aligned}\tag{6.18}$$

We now estimate $|(B(\psi,\psi),v_2)|$ by (6.5), (6.7), (2.10), and (2.11),

$$\begin{aligned}\left|\left(B(\psi,\psi),v_2\right)\right| &\leqslant \int_\Omega |\psi||\nabla \psi||v_2|\,dx \leqslant C_4\int_\Omega \frac{|v_2|}{\mathrm{dist}(x,\partial\Omega)}|\psi|\,dx\\ &\leqslant C_4\left\{\frac{|v_2|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx\right\}^{1/2}\left\{\int_{\mathrm{dist}(x,\partial\Omega)\leqslant c_2\varepsilon}|\psi|^2\,dx\right\}^{1/2}\\ &\leqslant c_4C_1C_4^2C_6|\partial\Omega|^{1/2}\left|A^{1/2}v_2\right|_2\cdot\sqrt{\varepsilon}\\ &\leqslant \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2+\frac{3c_4^2C_1^2C_4^4C_6^2|\partial\Omega|\varepsilon}{\nu}. \end{aligned}\tag{6.19}$$

Finally, we estimate $|(\bar f,v_2)|$ by

$$\left|(f,v_2)\right| \leqslant |f|_{V'}\left|A^{1/2}v_2\right|_2 \leqslant \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2+\frac{3|f|_{V'}^2}{\nu}.\tag{6.20}$$

Since $\operatorname{supp} F \subset \{x\in\Omega \mid c_1\varepsilon \leqslant \mathrm{dist}(x,\partial\Omega)\leqslant c_2\varepsilon\}$, it then follows from Lemma 2.3 that

$$\nu\left|(F,v_2)\right| \leqslant \nu\cdot|F|_2\left\{\int_\Omega \frac{|v_2|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}\,dx\right\}^{1/2}\cdot c_5\varepsilon \leqslant \nu\cdot\frac{C_3\|\varphi\|_{L^2(\partial\Omega)}}{\varepsilon^{3/2}}\cdot\|v_2\|\cdot c_5\varepsilon$$

$$\leqslant \frac{3c_5^2C_3^2\nu}{\varepsilon}\|\varphi\|_{L^2(\partial\Omega)}^2 + \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2$$
$$\leqslant \frac{3c_5^2C_3^2\nu}{\varepsilon}|\partial\Omega|\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \frac{\nu}{12}\left|A^{1/2}v_2\right|_2^2, \tag{6.21}$$

where $\|\varphi\|_{L^2(\partial\Omega)} \leqslant |\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}$.

Putting (6.16)–(6.21) together, there exist constant C_8, C_9 such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_2|_2^2 + \nu\|v_2\|^2\\
&\quad\leqslant \frac{\nu}{2}\|v_2\|^2 + \frac{C_8}{\nu\lambda_{m+1}}\left|A^{1/2}v\right|_2^2 + \frac{C_8}{\nu\lambda_{m+1}^{1/2}}\left|A^{1/2}v\right|_2^2 + C_9 + \frac{3}{\nu}|f|_{V'}^2\\
&\quad\leqslant \frac{\nu}{2}\|v_2\|^2 + \frac{4C_8}{\nu\lambda_{m+1}}\left(\left|A^{1/2}v_1\right|_2^2 + \left|A^{1/2}v_2\right|_2^2\right)\\
&\qquad + \frac{4C_8}{\nu\lambda_{m+1}^{1/2}}\left(\left|A^{1/2}v_1\right|_2^2 + \left|A^{1/2}v_2\right|_2^2\right) + C_9 + \frac{3}{\nu}|f|_{V'}^2\\
&\quad\leqslant \frac{\nu}{2}\|v_2\|^2 + \frac{4C_8\lambda_m^{1/2}}{\nu\lambda_{m+1}}\rho_1^2 + \frac{4C_8}{\nu\lambda_{m+1}}\left|A^{1/2}v_2\right|_2^2 + \frac{4C_8\lambda_m^{1/2}}{\nu\lambda_{m+1}^{1/2}}\rho_1^2\\
&\qquad + \frac{4C_8}{\nu\lambda_{m+1}^{1/2}}\left|A^{1/2}v_2\right|_2^2 + C_9 + \frac{3}{\nu}|f|_{V'}^2,
\end{aligned}\tag{6.22}$$

where we use

$$\left|A^{1/2}v_1\right|_2^2 \leqslant \lambda_m^{1/2}\left|A^{1/4}v_1\right|_2^2 \leqslant \lambda_{m+1}^{1/2}\left|A^{1/4}v_1\right|_2^2. \tag{6.23}$$

Therefore, we deduce that

$$\frac{d}{dt}|v_2|_2^2 + \frac{1}{2}\nu\lambda_{m+1}|v_2|_2^2 \leqslant M + \frac{3}{\nu}|f|_{V'}^2. \tag{6.24}$$

Here M depends on λ_{m+1}, is not increasing as λ_{m+1} increasing.

By the Gronwall inequality, the above inequality implies

$$\left|v_2(t)\right|_2^2 \leqslant \left|v_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2} + \frac{2M}{\nu\lambda_{m+1}} + \frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)/2}|f|_{V'}^2\,ds. \tag{6.25}$$

Applying Proposition 5.1 and Lemma II 1.3 in Chepyzhov and Vishik [7] for any ε,

$$\frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)/2}|f|_{V'}^2\,ds < \frac{\varepsilon}{3}.$$

Using (6.3) and letting $t_1 = t_0 + 1 + \frac{2}{\nu\lambda_{m+1}}\ln\frac{3\rho_0^2}{\varepsilon}$, then $t \geqslant t_1$ implies

$$\frac{2M}{\nu\lambda_{m+1}} < \frac{\varepsilon}{3};$$
$$\left|v_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2} \leqslant \rho_0^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2} < \frac{\varepsilon}{3}.$$

Therefore, we deduce from (6.25) that

$$|v_2|_2^2 \leqslant \varepsilon, \quad \forall t \geqslant t_1,\ f \in \mathcal{H}(f_0), \tag{6.26}$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, satisfying uniform (*w.t.r.* $f \in \mathcal{H}(f_0)$) condition (C) in H. □

According to Propositions 6.1 and 6.2, we can now regard that the families of processes $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, are defined in $D(A^{1/4})$ and $\mathcal{B}_1$ is a uniformly (*w.r.t.* $f \in \mathcal{H}(f_0)$) absorbing set in $D(A^{1/4})$.

Theorem 6.2. *If $f_0(x,s)$ is translation compact in $L^2_{\mathrm{loc}}(R; D(A^{-1/4}))$, then the processes $\{U_{f_0}(t,\tau)\}$ corresponding to problem* (6.12) *possesses compact uniform* (*w.r.t.* $\tau \in R$) *attractor $\mathfrak{A}_1$ in $D(A^{1/4})$ which coincides with the uniform* (*w.r.t.* $f \in \mathcal{H}(f_0)$) *attractor $\mathfrak{A}_{\mathcal{H}(f_0)}$ of the family of processes $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$:*

$$\mathfrak{A}_1 = \mathfrak{A}_{\mathcal{H}(f_0)} = \omega_{0,\mathcal{H}(f_0)}(\mathcal{B}_1) = \bigcup_{f \in \mathcal{H}(f_0)} \mathcal{K}_f(0), \tag{6.27}$$

where $\mathcal{B}_1$ is the uniformly (*w.r.t.* $f \in \mathcal{H}(f_0)$) *absorbing set in $D(A^{1/4})$ and $\mathcal{K}_f$ is the kernel of the process $\{U_f(t,\tau)\}$. Furthermore, the kernel $\mathcal{K}_f$ is nonempty for all $f \in \mathcal{H}(f_0)$.*

Proof. Using Proposition 6.2, we have the family of processes $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, corresponding to (6.12) possesses the uniformly (*w.r.t.* $f \in \mathcal{H}(f_0)$) absorbing set in $D(A^{1/4})$.

Now we prove the existence of compact uniform (*w.r.t.* $f \in \mathcal{H}(f_0)$) attractor in $D(A^{1/4})$ by applying the method established in Section 4, that is, we testify that the family of processes $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$ corresponding to (6.12) satisfies uniform (*w.r.t.* $f \in \mathcal{H}(f_0)$) condition (C).

Multiplying Eq. (6.1) by $A^{1/2}v_2(t)$, similarly to Theorem 6.1, we have

$$\begin{aligned}
&\left(\frac{dv}{dt}, A^{1/2}v_2\right) + \left(\nu Av, A^{1/2}v_2\right) + \left(B(v,v), A^{1/2}v_2\right) \\
&\quad + \left(B(v,\psi), A^{1/2}v_2\right) + \left(B(\psi,v), A^{1/2}v_2\right) \\
&\quad = \left(\bar f, A^{1/2}v_2\right) - \left(B(\psi,\psi), A^{1/2}v_2\right).
\end{aligned} \tag{6.28}$$

It follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|A^{1/4}v_2\right|_2^2 + \nu\left|A^{3/4}v_2\right|_2^2 &\leqslant \left|\left(B(v,v), A^{1/2}v_2\right)\right| + \left|\left(B(v,\psi), A^{1/2}v_2\right)\right| \\
&\quad + \left|\left(B(\psi,v), A^{1/2}v_2\right)\right| + \left|\left(\bar f, A^{1/2}v_2\right)\right| \\
&\quad + \left|\left(B(\psi,\psi), A^{1/2}v_2\right)\right|.
\end{aligned} \tag{6.29}$$

We have to estimate each term in the right-hand side of (6.29).

First, by Hölder's inequality and Lemma 6.2,

$$\begin{aligned}
\left|\left(B(v,v), A^{1/2}v_2\right)\right| &\leqslant \int_\Omega |v||\nabla v|\left|A^{1/2}v_2\right| dx \leqslant |v|_4\left|A^{1/2}v\right|_4\left|A^{1/2}v_2\right|_2 \\
&\leqslant C_7^2\left|A^{1/4}v\right|_2\left|A^{3/4}v\right|_2 \frac{1}{\lambda_{m+1}^{1/4}}\left|A^{3/4}v_2\right|_2 \leqslant \frac{C_7^2\rho_1}{\lambda_{m+1}^{1/4}}\left|A^{3/4}v\right|_2\left|A^{3/4}v_2\right|_2
\end{aligned}$$

$$\leqslant \frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2 + \frac{3C_7^4\rho_1^2}{\nu\lambda_{m+1}^{1/2}}\left|A^{3/4}v\right|_2^2. \tag{6.30}$$

Next, using (6.5), (6.7) and the Cauchy inequality,

$$\begin{aligned}
\left|\left(B(v,\psi), A^{1/2}v_2\right)\right| &\leqslant \int_\Omega |v||\nabla\psi|\left|A^{1/2}v_2\right|dx \leqslant C_4\int_\Omega \frac{|v|}{\operatorname{dist}(x,\partial\Omega)}\left|A^{1/2}v_2\right|dx \\
&\leqslant C_4\left(\int_\Omega \frac{|v|^2}{[\operatorname{dist}(x,\partial\Omega)]^2}\,dx\right)^{1/2}\left|A^{1/2}v_2\right|_2 \\
&\leqslant C_4C_6|A^{1/2}v|_2\left|A^{1/2}v_2\right|_2 \leqslant \frac{C_4C_6}{\lambda_1^{1/4}\lambda_{m+1}^{1/4}}\left|A^{3/4}v\right|_2\left|A^{3/4}v_2\right|_2 \\
&\leqslant \frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2 + \frac{3C_4^2C_6^2}{\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\left|A^{3/4}v\right|_2^2.
\end{aligned} \tag{6.31}$$

Similarly by (6.5),

$$\begin{aligned}
\left|\left(B(\psi,v), A^{1/2}v_2\right)\right| &\leqslant \int_\Omega |\psi||\nabla v|\left|A^{1/2}v_2\right|dx \leqslant C_4\int_\Omega |\nabla v|\left|A^{1/2}v_2\right|dx \\
&\leqslant C_4|\nabla v|_2\left|A^{1/2}v_2\right|_2 \leqslant \frac{C_4}{\lambda_1^{1/4}\lambda_{m+1}^{1/4}}\left|A^{3/4}v\right|_2\left|A^{3/4}v_2\right|_2 \\
&\leqslant \frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2 + \frac{3C_4^2}{\nu\lambda_1^{1/2}\lambda_{m+1}^{1/2}}\left|A^{3/4}v\right|_2^2.
\end{aligned} \tag{6.32}$$

We now estimate $|(B(\psi,\psi), A^{1/2}v_2)|$ by (6.5), (6.6) and Lemma 6.2,

$$\begin{aligned}
&\left|\left(B(\psi,\psi), A^{1/2}v_2\right)\right| \\
&\quad\leqslant \int_\Omega |\psi||\nabla\psi|\left|A^{1/2}v_2\right|dx \\
&\quad\leqslant C_4\left\{\int_\Omega |\nabla\psi|^2\operatorname{dist}(x,\partial\Omega)\,dx\right\}^{1/2}\left\{\int_\Omega \left|A^{1/2}v_2\right|^2\frac{1}{\operatorname{dist}(x,\partial\Omega)}\,dx\right\}^{1/2} \\
&\quad\leqslant C_4C_5^{1/2}C_7^{1/2}\left|A^{3/4}v_2\right|_2 \leqslant \frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2 + \frac{3C_4^2C_5C_7}{\nu}.
\end{aligned} \tag{6.33}$$

Finally, we estimate $|(\bar{f}, A^{1/2}u_2)|$ by

$$\left|\left(f, A^{1/2}v_2\right)\right| \leqslant |f|_2\left|A^{1/2}v_2\right|_2 \leqslant \frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2 + \frac{3}{\nu}|f|_{D(A^{-1/4})}^2. \tag{6.34}$$

Similarly (6.21) by Lemma 2.3,

$$\nu\left|\left(F, A^{1/2}v_2\right)\right| \leqslant \nu\int_\Omega |F|\left|A^{1/2}v_2\right|dx \leqslant \nu\int_\Omega |F|\frac{|A^{1/2}v_2|}{[\operatorname{dist}(x,\partial\Omega)]^{1/2}}\,dx\cdot c_6\sqrt{\varepsilon}$$

$$\begin{aligned}
&\leqslant c_6\nu\sqrt{\varepsilon}|F|_2\left\{\int_\Omega \frac{|A^{1/2}v_2|^2}{\operatorname{dist}(x,\partial\Omega)}\right\}^{1/2}dx\\
&\leqslant c_6\nu\sqrt{\varepsilon}|F|_2\left|A^{3/4}v_2\right|_2 \leqslant 3c_6^2\nu\varepsilon|F|_2^2+\frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2\\
&\leqslant 3c_6^2\nu\varepsilon\frac{C_3^2}{\varepsilon^3}\|\varphi\|^2_{L^2(\partial\Omega)}+\frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2\\
&\leqslant 3c_6^2\nu\frac{C_3^2}{\varepsilon^2}|\partial\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{\nu}{12}\left|A^{3/4}v_2\right|_2^2. \qquad (6.35)
\end{aligned}$$

Putting (6.30)–(6.35) together, using (6.23), there exists a constant C_{10} such that

$$\begin{aligned}
&\frac{d}{dt}\left|A^{1/4}v_2\right|_2^2+\nu\left|A^{3/4}v_2\right|_2^2\\
&\quad\leqslant \frac{C_{10}}{\lambda_{m+1}^{1/2}}\left|A^{3/4}v\right|_2^2+C_{10}+\frac{3}{\nu}|f|^2_{D(A^{-1/4})}\\
&\quad\leqslant \frac{4C_{10}}{\lambda_{m+1}^{1/2}}\left(\left|A^{3/4}v_1\right|_2^2+\left|A^{3/4}v_2\right|_2^2\right)+C_{10}+\frac{3}{\nu}|f|^2_{D(A^{-1/4})}\\
&\quad\leqslant 4C_{10}\lambda_{m+1}^{1/2}\left|A^{1/4}v_1\right|_2^2+\frac{M_1}{\lambda_{m+1}^{1/2}}\left|A^{3/4}v_2\right|_2^2+M_1+\frac{3}{\nu}|f|^2_{D(A^{-1/4})}. \qquad (6.36)
\end{aligned}$$

Here M_1 depends on λ_{m+1}, is not increasing as λ_{m+1} increasing. Therefore, we deduce that

$$\frac{d}{dt}\left|A^{1/4}v_2\right|_2^2+\frac{1}{2}\nu\lambda_{m+1}\left|A^{1/4}v_2\right|_2^2\leqslant C_{11}\lambda_{m+1}^{1/2}+C_{11}+\frac{3}{\nu}|f|^2_{D(A^{-1/4})}. \qquad (6.37)$$

By the Gronwall inequality, the above inequality implies

$$\begin{aligned}
\left|A^{1/4}v_2(t)\right|_2^2 &\leqslant \left|A^{1/4}v_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2}+\frac{2C_{12}}{\nu\lambda_{m+1}}\\
&\quad+\frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)/2}|f|^2_{D(A^{-1/4})}\,ds. \qquad (6.38)
\end{aligned}$$

Applying Proposition 5.1 and Lemma II 1.3 in Chepyzhov and Vishik [7] for any ε,

$$\frac{3}{\nu}\int_{t_0+1}^{t} e^{-\nu\lambda_{m+1}(t-s)/2}|f|^2_{D(A^{-1/4})}\,ds<\frac{\varepsilon}{3}.$$

Using (6.3) and let $t_1=t_0+1+\frac{2}{\nu\lambda_{m+1}}\ln\frac{3\rho_1^2}{\varepsilon}$, then $t\geqslant t_1$ implies

$$\begin{aligned}
&\frac{2C_{12}}{\nu\lambda_{m+1}}<\frac{\varepsilon}{3};\\
&\left|A^{1/4}v_2(t_0+1)\right|_2^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2}\leqslant \rho_1^2 e^{-\nu\lambda_{m+1}(t-(t_0+1))/2}<\frac{\varepsilon}{3}.
\end{aligned}$$

Therefore, we deduce from (6.38) that

$$\left|A^{1/4}v_2\right|_2^2\leqslant\varepsilon,\quad \forall t\geqslant t_1,\ f\in\mathcal{H}(f_0), \qquad (6.39)$$

which indicates $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, satisfying uniform (*w.t.r.* $f \in \mathcal{H}(f_0)$) condition (C) in $D(A^{1/4})$. Applying Theorem 4.2 the proof is complete. □

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