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Spanners of additively weighted point sets ☆

Prosenjit Bose, Paz Carmi *, Mathieu Couture

School of Computer Science, Carleton University, Herzberg Building, 1125 Colonel By Drive, Ottawa, Ontario, Canada

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ABSTRACT

We study the problem of computing geometric spanners for (additively) weighted point sets. A weighted point set is a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_i p_j| - r_i - r_j$. We show that in the case where all r_i are positive numbers and $|p_i p_j| \geqslant r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane), a variant of the Yao graph is a $(1+\epsilon)$-spanner that has a linear number of edges. We also show that the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) has a spanning ratio bounded by a constant. The straight-line embedding of the Additively Weighted Delaunay graph may not be a plane graph. Given the Additively Weighted Delaunay graph, we show how to compute a plane straight-line embedding that also has a spanning ratio bounded by a constant in $O(n \log n)$ time.

1. Introduction

Let G be a complete weighted graph where edges have positive weight. Given two vertices u, v of G, we denote by $\delta_G(u, v)$ the length of a shortest path in G between u and v. A spanning subgraph H of G is a *t-spanner* of G if $\delta_H(u, v) \leqslant t\delta_G(u, v)$ for all pairs of vertices u and v. The smallest t having this property is called the *spanning ratio* of the graph H with respect to G. Thus, a graph with spanning ratio t approximates the $\binom{n}{2}$ distances between the vertices of G within a factor of t. Let P be a set of n points in the plane. A *geometric graph* with vertex set P is an undirected graph whose edges are line segments that are weighted by their length. The problem of constructing t-spanners of geometric graphs with $O(n)$ edges for any given point set has been studied extensively; see the book by Narasimhan and Smid [6] for an overview.

In this paper, we address the problem of computing geometric spanners with additive constraints on the points. More precisely, we define a weighted point set as a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_i p_j| - r_i - r_j$. The problem we address is to compute a spanner of a complete graph on a weighted point set. To the best of our knowledge, the problem of constructing a geometric spanner in this context has not been previously addressed. We show how the Yao [9] graph can be adapted to compute a $(1+\epsilon)$-spanner in the case where all r_i are positive real numbers and $|p_i p_j| \geqslant r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane). In the same case, we also how the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) provides a plane spanner that has the same spanning ratio as the Delaunay graph of a set of points (see [7] for a comprehensive survey of Voronoi diagrams and Delaunay graphs). Since $|p_i p_j| < r_i + r_j$ implies that the distance is negative, we believe that the restriction $|p_i p_j| \geqslant r_i + r_j$ is reasonable because the t-spanner problem does not make sense when there are negative distances.

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* Corresponding author.
E-mail address: paz@cg.scs.carleton.ca (P. Carmi).

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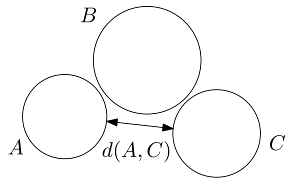

Fig. 1. The additively weighted distance is not a metric.

The remainder of the paper is organized as follows. In the next section, we review related work. In Section 3, we provide an overview of the notation and definitions used throughout the paper. Then, in Section 4, we show how to modify the standard Yao graph such that it is an Additively Weighted $(1+\epsilon)$-spanner for any $\epsilon > 0$. In Section 5, we define how to take the quotient of a graph (or a minor of a graph), and summarize what properties are sufficient in our setting such that the quotient is still a spanner when the original graph is a spanner. We use this property in Section 6 to show that a variant of the Additively Weighted Delaunay triangulation is a plane spanner with spanning ratio bounded by a constant. Conclusions and open problems are discussed in Section 8.

2. Related work

Well known examples of geometric t-spanners include the Yao graph [9], the Delaunay graph [4], and the Well-Separated Pair Decomposition [1]. Let $\theta < \pi/4$ be an angle such that $2\pi/\theta = k$, where k is an integer. The Yao graph with angle θ is defined as follows. For every point p, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. Then, there is an oriented edge from p to q if and only if q is the closest point to p in some cone $C_{p,i}$. For Yao graphs [9], the spanning ratio is at most $1/(\cos\theta - \sin\theta)$ provided that $\theta < \pi/4$, and for θ-graphs, the spanning ratio is at most $1/(1 - 2\sin\frac{\theta}{2})$ provided that $\theta < \pi/3$ [8].

Given a set of points in the plane, there is an edge between p and q in the Delaunay graph if and only if there is an empty circle with p and q on its boundary [4]. The spanning ratio of the Delaunay triangulation is at most 2.42 [4]. The *Voronoi diagram* [2] of a finite set of points P is a partition of the plane into $|P|$ regions such that each region contains exactly those points having the same nearest neighbor in P. The points in P are also called *sites*. It is well known that the Voronoi diagram of a set of points is the face-dual of the Delaunay graph of that set of points [2], i.e. two points have adjacent Voronoi regions if and only if they share an edge in the Delaunay graph.

3. Definitions and notation

Definition 1. A set $P = \{(p_1, r_1), \ldots, (p_n, r_n)\}$ of ordered pairs, where each p_i is a point in the plane and each r_i is a real number, is called a *weighted point set*. The notation $p_i \in P$ means that there exists an ordered pair (p_i, r_i) such that $(p_i, r_i) \in P$. The *additive distance* from a point $p \notin P$ in the plane to a point $p_i \in P$, denoted $d(p, p_i)$, is defined as $|pp_i| - r_i$, where $|pp_i|$ is the Euclidean distance from p to p_i. The *additive distance* between two points $p_i, p_j \in P$, noted $d(p_i, p_j)$, is defined as $|p_i p_j| - r_i - r_j$, where $|p_i p_j|$ is the Euclidean distance from p_i to p_j.

The problem we address in this paper is the following:

Problem 1. Let P be a weighted point set and let $K(P)$ be the complete weighted graph with vertex set P and edges weighted by the additive distance between their endpoints. Compute a t-spanner with $O(n)$ edges of $K(P)$ for a fixed constant $t > 1$.

Notice that in the case where all r_i are positive numbers, the pairs (p_i, r_i) can be viewed as disks D_i in the plane. If, for all i, j we also have $d(p_i, p_j) \geqslant 0$, then the disks are disjoint. In that case, the distance $d(D_i, D_j) = d(p_i, p_j) = |p_i p_j| - r_i - r_j$ is also equal to $\min\{|q_i q_j|\colon\ q_i \in D_i \text{ and } q_j \in D_j\}$, where the notation $q_i \in D_i$ means $|p_i q_i| \leqslant r_i$. To compute a spanner of an additively weighted point set is then equivalent to computing a spanner of a set of disks in the plane. **From now to the end of this paper, it is assumed that all r_i are positive numbers and $d(p_i, p_j) \geqslant 0$ for all i, j.** If $\mathcal{D}$ is a set of disks in the plane, then a *spanner* of $\mathcal{D}$ is a spanner of the complete graph whose vertex set is $\mathcal{D}$ and whose edges (D_i, D_j) are given weights equal to $d(D_i, D_j)$.

Notice also that the additive distance may not be a metric since the triangle inequality does not necessarily hold (see Fig. 1). Although this may seem counter-intuitive, this makes sense in some networks, since a direct communication is not always easier than routing through a common neighbor. For example, in wireless networks, the amount of energy that is needed to transmit a message is a power of the Euclidean distance between the sender and the receiver. Therefore, using several small hops can be more energy efficient than a direct communication over one long-distance link.

Fig. 2 shows how the Yao graph can be generalized using the additive distance: for each cone, a disk keeps an outgoing edge with the closest disk whose center is contained in that cone. However, this graph is not a spanner. Fig. 3 shows how

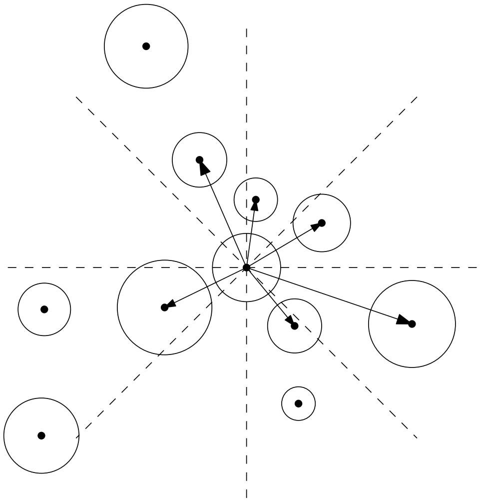

Fig. 2. A straight forward generalization of the Yao graph.

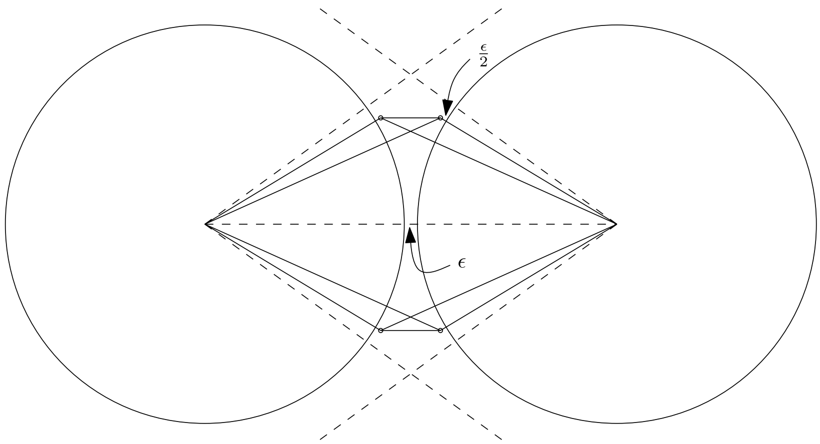

Fig. 3. The straight forward generalization of the Yao graph does not have constant spanning ratio.

to construct an example with six disks that has spanning ratio of $(1+\epsilon)/\epsilon$ for any $\epsilon > 0$. Notice that the additive distance between the centers of the two large circles is ϵ. They are not joined by an edge in the Yao graph because the distance from the center of the large circle to one of the tiny circles is $\epsilon/2$. As a result, the path between the centers of the two large circles in the Yao graph has a length of $(1+\epsilon)$ whereas their additive distance is only ϵ. Nonetheless, in Section 4, we see that a minor adjustment to the Yao graph can be made in order to compute a $(1+\epsilon)$-spanner of a set of disjoint disks that has $O(n)$ edges.

The Delaunay graph in the additively weighted setting is computable in time $O(n \log n)$ [3]. To the best of our knowledge, its spanning properties have not been previously studied. In Section 6, we show that it is a spanner and that its spanning ratio is the same as that of the standard Delaunay graph.

4. The Additively Weighted Yao graph

As we saw in the previous section, a straight forward generalization of the Yao graph fails to provide a graph with bounded spanning ratio. In this section, we show how a few subtle modifications to the construction, provide an approach to build a $(1+\epsilon)$-spanner. We define the modified Yao construction below.

Definition 2. Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta < \sin^{-1}(1/3)$ be an angle such that $2\pi/\theta = k$, where k is an integer. The Yao$(\theta, \mathcal{D})$ graph is defined as follows. For every disk $D = (p, r)$, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. A disk *blocks* a cone $C_{p,i}$ when the disk intersects both rays of $C_{p,i}$. Let $F \in \mathcal{D}$ be a disk (different from D) whose center c_F is in $C_{p,j}$. Add an edge from D to F in Yao$(\theta, \mathcal{D})$ if and only if one of the two following conditions is met:

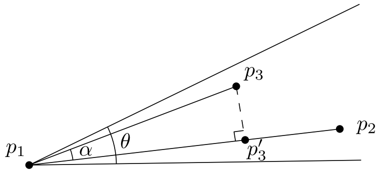

Fig. 4. Illustration of the proof of Lemma 1.

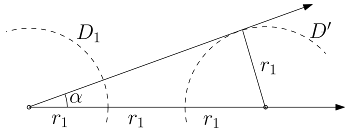

Fig. 5. A cone of angle $\alpha = \sin^{-1}(1/3)$ with apex at D_1's center, and a disk D' at distance r_1 from D_1, such that $r_1 \leqslant r'$, D' is blocking.

1. among all blocking disks B_m with center in $C_{p,j}$, F is the one that is the closest to D, i.e. $\forall m,\ d(F, D) \leqslant d(B_m, D)$;
2. among all disks B_m with center in $C_{p,j}$ and $d(B_m, D) \geqslant r$, F is the one that is closest to D, i.e. $\forall m,\ d(F, D) \leqslant d(B_m, D)$.

Notice that there are two main changes. Within each cone, we now add potentially two edges as opposed to only one edge in the case of unweighted points. Next, in the second condition to add an edge, we do not add an edge to the closest disk within a cone but to the closest disk whose distance is at least r from the disk centered at the apex with radius r. The reason why the angle θ cannot exceed $\sin^{-1}(1/3)$ will be made explicit in the proof of Theorem 1. Notice that given this modified definition, in Fig. 3 we no longer have a counter-example since when disks are not blocking, ϵ cannot be arbitrarily small but must be at least r. We now prove that these two modifications imply that the resulting graph is a $(1+\epsilon)$-spanner.

Lemma 1. *Let p_1, p_2, p_3 be such that the angle $\angle p_3 p_1 p_2 = \alpha \leqslant \theta < \pi/4$ and $|p_1 p_3| \leqslant |p_1 p_2|$. Then $|p_2 p_3| \leqslant |p_1 p_2| - (\cos\theta - \sin\theta)|p_1 p_3|$.*

Proof. Let p'_3 be the projection of p_3 on the line through p_1 and p_2 (see Fig. 4). Then

$$\begin{aligned}
|p_2 p_3| &\leqslant |p_2 p'_3| + |p'_3 p_3| \\
&= |p_1 p_2| - |p_1 p'_3| + |p'_3 p_3| \\
&= |p_1 p_2| - |p_1 p_3|(\cos\alpha - \sin\alpha) \\
&\leqslant |p_1 p_2| - |p_1 p_3|(\cos\theta - \sin\theta) \qquad \square
\end{aligned}$$

Theorem 1. *Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta < \sin^{-1}(1/3)$. Then $Y(\theta, \mathcal{D})$ is a t-spanner of $\mathcal{D}$, where $t = 1/(\cos 2\theta - \sin 2\theta)$.*

Proof. We proceed by induction on the rank of the distances between the pairs of disks $D_1 = (p_1, r_1)$ and $D_2 = (p_2, r_2)$.

Base case: The disks D_1 and D_2 form a closest pair. In that case, the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$. To see this, let $r_1 \leqslant r_2$. If D_2 is blocking the cone centered at p_1 that contains p_2, then it is in Yao$(\theta, \mathcal{D})$ by Case 1 of Definition 2. If D_2 is not blocking, then D_2 is at distance at least r_1 from D_1, since otherwise D_2 would be blocking. This is a consequence of θ being less than $\sin^{-1}(1/3)$ and $r_1 \leqslant r_2$ (see Fig. 5). Therefore, the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$ by Case 2 of Definition 2.

Induction step: Let $D_1 = (p_1, r_1)$ and $D_2 = (p_2, r_2)$. Without loss of generality, $r_1 \leqslant r_2$. If the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$, then there is nothing to prove. Otherwise, there are two cases to consider depending on whether the shortest path from D_1 to D_2 in the complete graph on $\mathcal{D}$ is the edge (D_1, D_2). If the shortest path is not the edge (D_1, D_2), then all edges on the shortest path must have length less than $d(D_1, D_2)$. By applying the inductive hypothesis on each of these edges, we conclude that the distance from D_1 to D_2 in Yao$(\theta, \mathcal{D})$ is at most t times the length of the shortest path from D_1 to D_2 in the complete graph on $\mathcal{D}$, as required.

We now consider the case when the edge (D_1, D_2):

1. is not in Yao$(\theta, \mathcal{D})$ and
2. is the shortest path from D_1 to D_2 in the complete graph.

Observe that the conjunction of those two facts implies that the disk D_2 does not block the cone whose apex is p_1 and contains p_2: If D_2 was blocking the cone, then since (D_1, D_2) is not an edge in Yao$(\theta, \mathcal{D})$, there must be a disk D_3 that is also blocking the cone and is closer to D_1 than D_2. However, this implies that the shortest path from D_1 to D_2 in the complete graph is not the edge (D_1, D_2) (see Fig. 6).

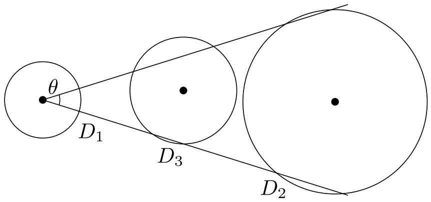

Fig. 6. If D_2 blocks the cone but the edge (D_1, D_2) is not in $\mathrm{Yao}(\theta, \mathcal{D})$, then there exists D_3 such that $d(D_1, D_3) + d(D_3, D_2) < d(D_1, D_2)$.

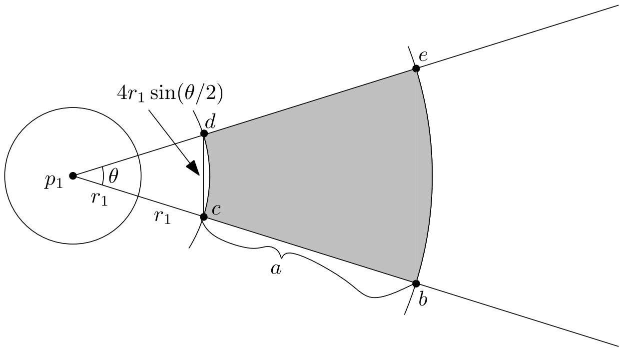

Fig. 7. Illustration to show why $d(D_2, D_3) \leqslant a + 4r_1 \sin(\theta/2)$.

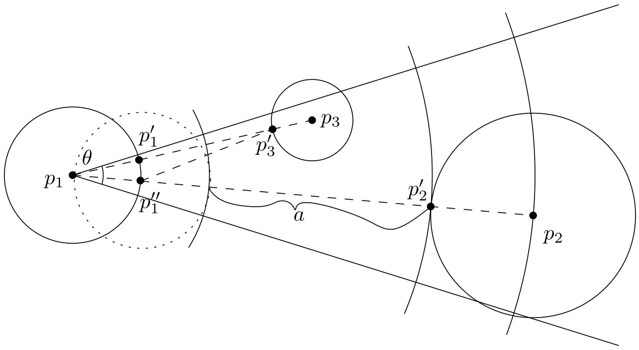

Fig. 8. Illustration of the proof of Theorem 1.

The conjunction of the three following facts:

1. $r_1 \leqslant r_2$;
2. $\theta < \sin^{-1}(1/3)$; and
3. D_2 does not block the cone,

imply that $d(D_1, D_2) > r_1$. Since (D_1, D_2) is not an edge, there is another disk whose distance is at least r that is closer to D_1. Let $D_3 = (p_3, r_3)$ be the closest disk to D_1 such that p_3 is in the same θ-cone with apex at p_1 as p_2 and $d(D_1, D_3) \geqslant r_1$. By definition, the edge (D_1, D_3) is in $\mathrm{Yao}(\theta, \mathcal{D})$. Observe that $d(D_2, D_3) < d(D_1, D_2)$. To see this, let $a := d(D_1, D_2) - r_1$ and refer to Fig. 7. We have that $d(D_2, D_3) \leqslant a + 4r_1 \sin(\theta/2)$. This is because the center of D_3 must lie in the shaded region and D_2 is tangent to the circular arc (with center p_1 and radius $a + 2r_1$) from b to e by construction. Furthermore, bd is a diameter of the shaded region and $|bd| < a + 4r_1 \sin(\theta/2)$. Therefore, we have

$$d(D_2, D_3) \leqslant a + 4r_1 \sin(\theta/2) < a + r_1 = d(D_1, D_2).$$

The first inequality follows from the fact that $d(D_2, D_3)$ is less than the diameter of the shaded region. The last inequality follows from the fact that $4\sin(\theta/2) < 1$ when $\theta < \sin^{-1}(1/3)$.

Let p'_1 be the point of D_1 that is the closest to D_3, p''_1 be the point of D_1 that is the closest to D_2, p'_2 be the point of D_2 that is the closest to D_1, and p'_3 be the point of D_3 that is the closest to D_1 (see Fig. 8). Notice that $|p'_1 p'_3| \leqslant |p''_1 p'_2|$. Since $d(D_1, D_2) \geqslant d(D_1, D_3) \geqslant r_1$, the angle $\angle p'_2 p''_1 p'_3$ is at most $2\theta < 2\sin^{-1}(1/3) < \pi/4$. Therefore, we can apply Lemma 1 to conclude that

$$|p_2'p_3'| \leqslant |p_1''p_2'| - (\cos 2\theta - \sin 2\theta)|p_1''p_3'|.$$

Since $d(D_2, D_3) \leqslant |p_2'p_3'|$ and $d(D_1, D_3) \leqslant |p_1''p_3'|$, we conclude that

$$d(D_2, D_3) \leqslant d(D_1, D_2) - (\cos 2\theta - \sin 2\theta) d(D_1, D_3).$$

Finally, since $d(D_2, D_3) < d(D_1, D_2)$, the inductive hypothesis tells us that $\mathrm{Yao}(\theta, \mathcal{D})$ contains a path from D_2 to D_3 whose length is at most $td(D_2, D_3)$. This means that the distance from D_1 to D_2 in $\mathrm{Yao}(\theta, \mathcal{D})$ is at most

$$d(D_1, D_3) + td(D_2, D_3) \leqslant d(D_1, D_3) + t\left(d(D_1, D_2) - \frac{1}{t} d(D_1, D_3)\right) = td(D_1, D_2).$$

We note that $\theta < \sin^{-1}(1/3)$ is the most stringent constraint on θ such that $t > 0$. □

Corollary 1. *For any $\epsilon > 0$ and any set $\mathcal{D}$ of n disjoint disks, it is possible to compute a $(1+\epsilon)$-spanner of $\mathcal{D}$ that has $O(n)$ edges.*

Proof. The bound on the number of edges comes from the fact that one can direct the edges, such that in each cone there are at most two outgoing edges, and the spanning ratio of $1+\epsilon$ comes from the fact that $\lim_{\theta \to 0} 1/(\cos 2\theta - \sin 2\theta) = 1$. □

5. Quotient graphs and quotient spanners

The main idea in the remainder of this paper is the following: we show how to compute a set of points from each D_i such that the (standard) Delaunay graph of those points is *equivalent* to the Additively Weighted Delaunay graph. By choosing the appropriate equivalence relation as well as the appropriate point set, we can then show that the spanning ratio of the Additively Weighted Delaunay graph is bounded by the spanning ratio of the standard Delaunay graph. The reduction of one graph to another is done by means of a quotient. This operation is quite standard in the graph theory and group theory literature. In graph theory, it is also referred to as a *contraction* and the graphs that result after a number of contractions are *minors*.

Definition 3. Let P_1 and P_2 be non-empty sets of points in the plane. The *distance* between P_1 and P_2, denoted by $|P_1P_2|$, is defined as the minimum $|p_1p_2|$ over all pairs of points such that $p_1 \in P_1$ and $p_2 \in P_2$.

Definition 4. Let $G = (V, E)$ be a geometric graph and $\mathcal{V}$ be a partition of V. The *quotient graph* of G by $\mathcal{V}$, denoted by $G/\mathcal{V}$, is the graph having $\mathcal{V}$ as vertices and there is an edge (U, W) (where U and W are in $\mathcal{V}$) if and only if there exists an edge $(u, w) \in E$ with $u \in U$ and $w \in W$. The weight of the edge (U, W) is equal to $|UW|$.

If P is a (non-weighted) point set and $\mathcal{P}$ is a partition of P, then the notation $P/\mathcal{P}$ designates the quotient of the complete Euclidean graph on P by $\mathcal{P}$. If $\mathcal{S}$ is a set of pairwise disjoint sets of points in the plane such that $P \subseteq \bigcup \mathcal{S}$, then the notation $P/\mathcal{S}$ designates the quotient of the complete Euclidean graph on P by the partition of P induced by $\mathcal{S}$.

Lemma 2. *Let $G = (V, E)$ be a complete geometric graph, $\mathcal{V}$ be a partition of V and S be a t-spanner of G. Then $S/\mathcal{V}$ is a t-spanner of $G/\mathcal{V}$.*

Proof. Let (U, W) be an edge of $G/\mathcal{V}$ and (u, w) be an edge of G such that $|uw| = |UW|$ (see Fig. 9). Since G is complete, the edge (u, w) is in G, and since S is a t-spanner of G, there is a path $\psi = u_1, \ldots, u_k$ in S such that $u_1 = u, u_k = w$ and the length of ψ is at most $t|uw|$. For each u_i of ψ, let $U_i \in \mathcal{V}$ be such that $u_i \in U_i$. Notice that it is possible that $U_i = U_{i+1}$ for some i. Let Ψ be the subsequence of $U = U_1, \ldots, U_k = W$ that consists of those U_i such that $i < k$ and $U_i \neq U_{i+1}$. By definition, the sequence Ψ is a path in $S/\mathcal{V}$ and it consists of at most $k' \leqslant k$ nodes. The length of Ψ is at most

$$\sum_{i=1}^{k'-1} |U_iU_{i+1}| \leqslant \sum_{i=1}^{k-1} |u_iu_{i+1}| \leqslant t|uw| = t|UW|$$

which means that Ψ is a t-spanning path for (U, W) in $S/\mathcal{V}$. □

6. The Additively Weighted Delaunay graph

Lee and Drysdale [5] studied a variant of the Voronoi diagram called the Additively Weighted Voronoi diagram, which is defined as follows: Let P be a weighted point set. The *Additively Weighted Voronoi diagram* of P is a partition of the plane into $|P|$ regions such that each region contains exactly the points in the plane having the same closest neighbor in

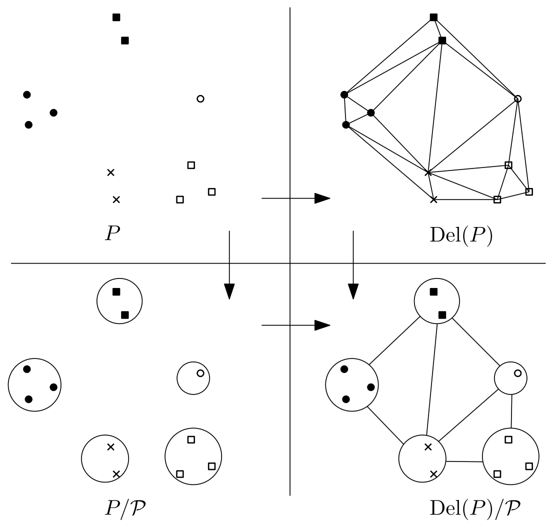

Fig. 9. Illustration of Lemma 2 on the Delaunay triangulation of P.

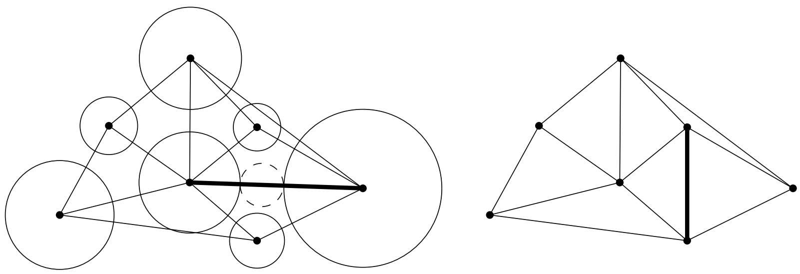

Fig. 10. The Additively Weighted Delaunay graph compared with the Delaunay graph of the disks centers.

P according to the additive distance. In other words, the Voronoi cell of a pair (p_i, r_i) contains the points p such that $d(p, p_i)$ is minimum over all other pairs in P. The *Additively Weighted Delaunay graph* (AW-Delaunay graph) is defined as the face-dual of the Additively Weighted Voronoi diagram.

Alternatively, if all r_i are positive and for all i, j, we have $|p_i p_j| \geqslant r_i + r_j$, then the pairs (p_i, r_i) can be seen as disks D_i of radius r_i centered at p_i and $d(p, D_i)$ is the minimum $|pq|$ over all $q \in D_i$. For a set $\mathcal{D}$ of disks in the plane, we denote the AW-Delaunay graph computed from $\mathcal{D}$ as $\mathrm{Del}(\mathcal{D})$. When no two disks intersect, the AW-Delaunay graph is a natural generalization of the Delaunay graph of a set of points. We say that two disks A and B *properly* intersect if $|A \cap B| > 1$.

Proposition 1. *Let $\mathcal{D}$ be a set of disjoint disks in the plane, and $A, B \in \mathcal{D}$. The edge (A, B) is in $\mathrm{Del}(\mathcal{D})$ if and only if there is a disk C that is tangent to both A and B and does not properly intersect any other disk in $\mathcal{D}$.*

Proof. Suppose (A, B) is in $\mathrm{Del}(\mathcal{D})$, and let c be a point on the boundary of the Voronoi cells of A and B and r be the distance from c to A. Since c is equidistant from A and B, it is also at distance r from B. This means that the disk C centered at c is tangent to both A and B. This disk cannot properly intersect any other disk of $\mathcal{D}$, since this would contradict the fact that c is in the Voronoi cells of A and B. Similarly, if there is a disk that is tangent to both A and B but does not properly intersect any other disk of $\mathcal{D}$, then A and B are Voronoi neighbors. $\square$

Note that the Additively Weighted Delaunay graph is not necessarily isomorphic to the Delaunay graph of the centers of the disks (see Fig. 10). When all radii are equal, however, the two graphs coincide. We now show that if $\mathcal{D}$ is a set of disks in the plane, then $\mathrm{Del}(\mathcal{D})$ is a spanner of $\mathcal{D}$. The intuition behind the proof is the following: we show the existence of a finite set of points P such that $K(P)/\mathcal{D}$ (where $K(P)$ is the complete graph with vertex set P) is isomorphic to the complete graph on $\mathcal{D}$ and $\mathrm{Del}(P)/\mathcal{D}$ is a subgraph of $\mathrm{Del}(\mathcal{D})$. Then, we use Lemma 2 to prove that $\mathrm{Del}(P)/\mathcal{D}$ is a spanner of $\mathcal{D}$, which implies that $\mathrm{Del}(\mathcal{D})$ is a spanner of $\mathcal{D}$.

Definition 5. Let A, B be disjoint disks and S a set of points such that $A \cap S = \emptyset$ and $B \cap S = \emptyset$. A set of points R *represents* S with respect to A and B if for every disk F that is tangent to both A and B, we have $F \cap S \neq \emptyset \Rightarrow F \cap R \neq \emptyset$. If $\mathcal{D}$ is a

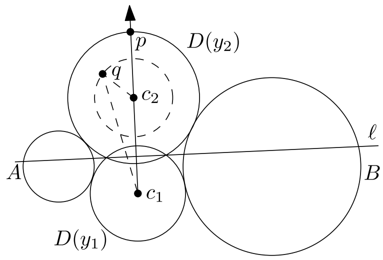

Fig. 11. Illustration of the proof of Lemma 3.

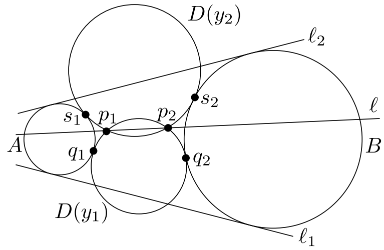

Fig. 12. Illustration of the proof of Lemma 4.

set of disjoint disks, then a set of points $\mathcal{R}$ *represents* $\mathcal{D}$ if for all $A, B, C \in \mathcal{D}$, there is a subset of $\mathcal{R}$ that represents C with respect to A and B.

From here to the end of the proof of Lemma 6, unless stated otherwise, let

1. A, B be two disjoint disks in the plane having their center on the x-axis;
2. $D(y)$ be the disk that is tangent to both A and B and whose center has the y-coordinate equal to y;
3. $y(D)$ be the y-coordinate of the center of a disk D;
4. ℓ_1, ℓ_2 be the two lines that are outer-tangent to both A and B (respectively, from below and above);
5. y_1, y_2 be such that $y_1 < y_2$ and $D(y_1) \cap D(y_2) \neq \emptyset$;
6. ℓ be the line through the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ (if $D(y_1)$ and $D(y_2)$ are tangent, then ℓ is the unique line that is tangent to both $D(y_1)$ and $D(y_2)$);
7. $T(A, B)$ denote the region below ℓ_2, above ℓ_1 and between A and B; and
8. l^+ (l^-) be the closed half-plane above (below) a non-vertical line l.

Throughout this section, it is implicitly assumed that $D(\infty) = \ell_2^+$ and $D(-\infty) = \ell_1^-$.

Lemma 3. *Given $y_1 < y_2$ and $D(y_1) \cap D(y_2) \neq \emptyset$, we have $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$ and $D(y_2) \cap \ell^- \subset D(y_1) \cap \ell^-$ (see Fig. 11).*

Proof. Notice that either $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$ or $D(y_2) \cap \ell^+ \subset D(y_1) \cap \ell^+$. Therefore, all we need to show is that $(D(y_2) \cap \ell^+) \setminus (D(y_1) \cap \ell^+)$ is not empty. Let c_1, c_2 be the respective centers of $D(y_1)$ and $D(y_2)$, and p be the intersection point of the infinite ray from c_1 through c_2 with the boundary of $D(y_1) \cup D(y_2)$.

We show by contradiction that p is not in $D(y_1)$. If that was the case, then $D(y_2)$ would be completely contained in $D(y_1)$. The reason for this is that there is no point of $D(y_2)$ that is farther from c_1 than p. Let q be a point of $D(y_2)$. Then $|qc_1| \leqslant |qc_2| + |c_2c_1| \leqslant |pc_2| + |c_2c_1| = |pc_1|$. But the fact that $D(y_2)$ is completely contained in $D(y_1)$ contradicts the fact that they are both tangent to A and B.

Therefore, since $p \in \ell^+$, we have $p \in (D(y_2) \cap \ell^+) \setminus (D(y_1) \cap \ell^+)$, which implies that $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$. Similarly, $D(y_2) \cap \ell^- \subset D(y_1) \cap \ell^-$. $\square$

Lemma 4. *Let p_1, p_2 be the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ (if $D(y_1)$ and $D(y_2)$ are tangent, then $p_1 = p_2$). Then p_1 and p_2 are in ℓ_2^- and in ℓ_1^+ (see Fig. 12).*

Proof. Let q_1, q_2 be the tangency points of $D(y_1)$ with A and B and s_1, s_2 be the tangency points of $D(y_2)$ with A and B. By Lemma 3, q_1, q_2 are below ℓ and s_1, s_2 are above ℓ. Since ℓ is above q_1 and q_2, which are in turn above ℓ_1, it follows that p_1 and p_2 are above ℓ_1. By a symmetric argument, p_1 and p_2 are below ℓ_2. $\square$

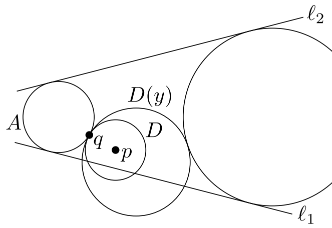

Fig. 13. Illustration of the proof of Lemma 5(3) (first part).

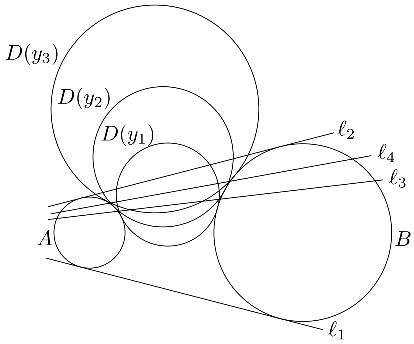

Fig. 14. Illustration of the proof of Lemma 5(3) (second part).

Lemma 5. *The following are true*:

1. *For all $p \in \ell_2^+$, there exists a line $y = y_0$ such that for every disk E that is tangent to both A and B, if the center of E is above y_0 then $p \in E$.*
2. *For all $p \in \ell_1^-$, there exists a line $y = y_1$ such that for every disk E that is tangent to both A and B, if the center of E is below y_1 then $p \in E$.*
3. *For all p in $T(A, B)$, there exist two lines $y = y_0$ and $y = y_1$ such that for every disk E that is tangent to both A and B, $p \in E$ if and only if the center of E is between y_0 and y_1.*

Proof. For (1), the existence of y_0 is guaranteed by the fact that $\lim_{y\to\infty} D(y) = \ell_2^+$. Now, let y_0 be such that $p \in D(y_0)$ and $y' > y_0$. Let $L(y_0)$ and $L(y')$ be the lines respectively defined by the intersection of $D(y_0)$ and $D(y')$ with the half-plane above ℓ_2. By Lemma 4, the two points where the boundaries of $D(y_0)$ and $D(y')$ intersect are below ℓ_2. Therefore, we have either $L(y_0) \subset L(y')$ or $L(y') \subset L(y_0)$. But since $y' > y_0$, by Lemma 3 we have $L(y_0) \subset L(y')$ and therefore $p \in L(y')$. The proof of (2) is symmetric.

For (3), the existence is easy to show. Without loss of generality, assume $d(p, A) \leqslant d(p, B)$. Let D be the disk centered at p that is tangent to A and let q be the tangency point of A and D see Fig. 13. Since $q \in T(A, B)$, there exists y such that $D(y) \cap A = q$. Since $D \subseteq D(y)$, there exists a disk that is tangent to both A and B and contains p.

We now show that $y_1 < y_2 < y_3$ implies $D(y_1) \cap D(y_3) \subseteq D(y_2)$ (see Fig. 14). Let ℓ_3 be the line through the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ and let ℓ_4 be the line through the intersection points of the boundaries of $D(y_2)$ and $D(y_3)$. Let $p \in D(y_1) \cap D(y_3)$. Since ℓ_4 is above ℓ_3 in $D(y_1) \cap D(y_3)$, p is either above ℓ_3, below ℓ_4 or both. If $p \in \ell_3^+$, then since $y_1 < y_2$, by Lemma 3 we have that $D(y_1) \cap \ell_3^+ \subseteq D(y_2) \cap \ell_3^+$ and $p \in D(y_1) \cap D(y_2)$. Similarly, if $p \in \ell_4^-$, then since $y_2 < y_3$, by Lemma 3 we have that $D(y_3) \cap \ell_4^- \subseteq D(y_2) \cap \ell_4^-$ and $p \in D(y_3) \cap D(y_2)$. In either case, $p \in D(y_2)$, which completes the proof. □

Lemma 6. *Let C be a disk that is disjoint of both A and B. There exists a set of at most six points that represents C with respect to A and B.*

Proof. Let

$$C_1 := \left(C \cap \ell_1^-\right) \setminus \ell_2^+$$
$$C_2 := \left(C \cap \ell_2^+\right) \setminus \ell_1^-$$
$$C_3 := C \cap \ell_1^- \cap \ell_2^+$$
$$C_4 := C \cap T(A, B)$$
$$C_5 := \left(C \cap \ell_1^+ \cap \ell_2^-\right) \setminus T(A, B)$$

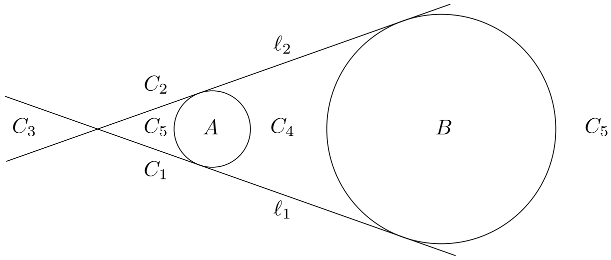

Fig. 15. The five regions for Lemma 6.

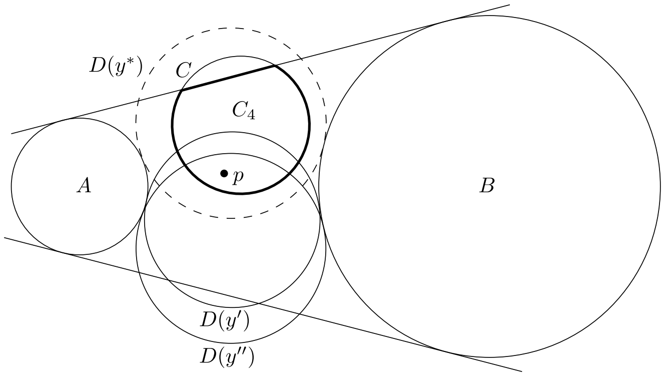

Fig. 16. Case C_4 of the proof of Lemma 6.

These five regions partition the disk C (see Fig. 15). We show that for each region, there is a finite set of points that represents it. The cardinality of the union of the sets is no more than six.

If $C_1 \neq \emptyset$, then let y_0 be the minimum y such that $D(y)$ intersects C_1. Let $p_1 \in C_1 \cap D(y_0)$. By definition of y_0, for any disk E that is tangent to both A and B and intersects C_1, we have $y(E) \geqslant y_0$, and by Lemma 5, we have $p_1 \in E$.

Similarly, if $C_2 \neq \emptyset$, then let y_1 be the maximum y such that $D(y)$ intersects C_2. Let $p_2 \in C_2 \cap D(y_1)$. By definition of y_1, for any disk E that is tangent to both A and B and intersects C_2, we have $y(E) \leqslant y_1$, and by Lemma 5, we have $p_2 \in E$.

If $C_3 \neq \emptyset$, then let y_0 be the minimum $y > 0$ such that $D(y)$ intersects C_3 and y_1 as the maximum $y < 0$ such that $D(y)$ intersects C_3. Let $p_3 \in C_3 \cap D(y_0)$ and $p_4 \in C_3 \cap D(y_1)$. By definition of y_0, for any disk E with $y(E) > 0$ that is tangent to both A and B and intersects C_3, we have $y(E) \geqslant y_0$, and by Lemma 5, we have $p_3 \in E$. The same reasoning applies to p_4 when $y(E) < 0$.

If $C_4 \neq \emptyset$, then let y_0 be the minimum y such that $D(y)$ intersects C_4 and y_1 as the maximum y such that $D(y)$ intersects C_4. Let $p_5 \in C_4 \cap D(y_0)$ and $p_6 \in C_4 \cap D(y_1)$. Let y^* be such that $C \subseteq D(y^*)$ (see Fig. 16). Let E be a disk that is tangent to both A and B and intersects C_4. We show that $y(E) \leqslant y^* \Rightarrow p_5 \in E$ (and similarly, $y(E) \geqslant y^* \Rightarrow p_6 \in E$). It is sufficient to show that $y'' < y' < y^* \Rightarrow C \cap D(y'') \subset C \cap D(y')$. Let $p \in D(y'') \cap C$. By Lemma 5, $\exists y_0(p), y_1(p)$ such that $\forall$ disk E tangent to both A and B, we have $y_0(p) \leqslant y(E) \leqslant y_1(p) \Leftrightarrow p \in E$. Therefore, the following hold:

$$y_0(p) \leqslant y^* \leqslant y_1(p)$$

$$y_0(p) \leqslant y'' \leqslant y_1(p)$$

But since $y'' < y' < y^*$, we have $y'' < y' < y^*$, which implies that $p \in C \cap D(y')$.

Finally, since $C_5 \cap E = \emptyset$ for any disk E that is tangent to both A and B, there is no need to select representative points for C_5. $\square$

Careful analysis of the proof of Lemma 6 allows us to observe that in fact, only two points are necessary to represent a disk C with respect to two other disks A and B. First, note that $C_4 \neq \emptyset \Rightarrow C_3 = \emptyset$ and $C_3 \neq \emptyset \Rightarrow C_4 = \emptyset$. This reduces to four the number of points that are necessary. Also, if $C_1 \neq \emptyset$ and $C_4 \neq \emptyset$, then p_6 is on ℓ_2 and is not required since any disk that contains it also intersects C_1 and therefore contains p_1. Similarly, if $C_2 \neq \emptyset$ and $C_4 \neq \emptyset$, then p_5 is not required since any disk that contains it also intersects C_2 and therefore contains p_2. Therefore, if $C_4 \neq \emptyset$, then the number of points that are necessary is at most two. A similar argument applies to the case where $C_3 \neq \emptyset$. Finally, if both C_3 and C_4 are empty, then only p_1 and p_2 may be required. Therefore, we have the following corollary:

Corollary 2. *Let $\mathcal{D}$ be a set of n disjoint disks. There exists a set of at most $2\binom{n}{3}$ points that represents $\mathcal{D}$.*

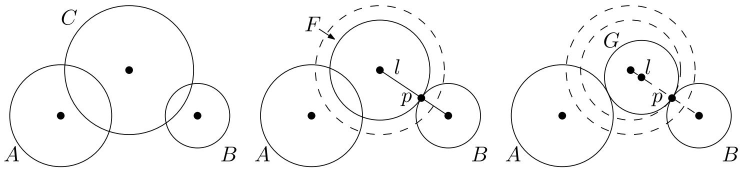

Fig. 17. Proof of Lemma 7.

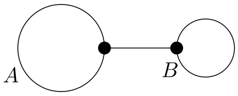

Fig. 18. The distance points of A and B.

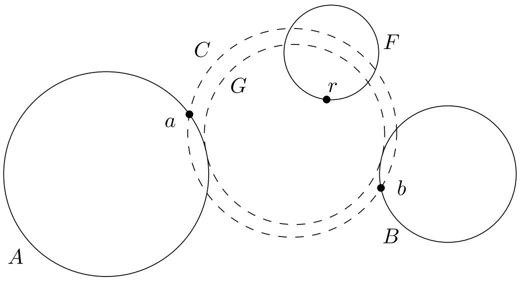

Fig. 19. Illustration of the proof of Theorem 2.

Lemma 7. *Let A and B be two disjoint disks and C be a disk intersecting both of them. Then there exists a disk G inside C that is tangent to both A and B.*

Proof. We show how to construct G. Let a, b, c and r_A, r_B, r_C respectively be the centers and radii of A, B and C. Without loss of generality, assume $|ac| - r_C \leqslant |bc| - r_B$. Let F be the disk centered at c and having radius $r_F = |bc| - r_B$ (see Fig. 17). The disk F is tangent to B. If F is also tangent to A, then let $G = F$ and we are done. Otherwise, F is properly intersecting A. In that case, let p be the tangency point of F and B, l be the line through b and c, and G be the disk through p having its center on l and tangent to A. The result follows from the fact that G is tangent to B and inside C. $\square$

Definition 6. Let A and B be two disks in the plane. The *distance points* of A and B are the two ends of the shortest line segment between A and B (see Fig. 18). If $\mathcal{D}$ is a set of disjoint disks, then the set of *distance points* of $\mathcal{D}$ is the set containing the distance points of every pair of disks in $\mathcal{D}$.

Theorem 2. *Let $\mathcal{D}$ be a set of n disjoint disks. Then* $\mathrm{Del}(\mathcal{D})$ *is a t-spanner of $\mathcal{D}$, where t is the spanning ratio of the Delaunay triangulation of a set of points.*

Proof. By Corollary 2, let R be a set of size at most $2\binom{n}{3}$ that represents $\mathcal{D}$, let S be the set of distance points of $\mathcal{D}$, and let $P = R \cup S$. Since $\mathrm{Del}(P)$ is a t-spanner of P (where t is the spanning ratio of the Delaunay graph of a set of points), by Lemma 2, we have $\mathrm{Del}(P)/\mathcal{D}$ is a t-spanner of $K(P)/\mathcal{D}$, where $K(P)$ is the complete graph with vertex set P. Since P contains the distance points of $\mathcal{D}$, $K(P)/\mathcal{D}$ is isomorphic to the complete graph defined on $\mathcal{D}$. We show that each edge (A, B) of $\mathrm{Del}(P)/\mathcal{D}$ is in $\mathrm{Del}(\mathcal{D})$ (see Fig. 19). Let (A, B) be an edge of $\mathrm{Del}(P)/\mathcal{D}$. This means that in P, there are two points a and b with $a \in A$, $b \in B$ such that there is an empty circle C through a and b. By Lemma 7, C contains a disk G that is tangent to both A and B. The disk G is a witness of the presence of the edge (A, B) in $\mathrm{Del}(\mathcal{D})$. If that was not the case, this would mean that there exists a disk $F \in \mathcal{D}$ such that $G \cap F \neq \emptyset$. By definition of R, this implies that $G \cap R \neq \emptyset$ and thus $C \cap P \neq \emptyset$, which contradicts the fact that C is an empty circle. Therefore, the edge (A, B) is in $\mathrm{Del}(\mathcal{D})$. Since $\mathrm{Del}(P)/\mathcal{D}$ is a t-spanner of $\mathcal{D}$ and a subgraph of $\mathrm{Del}(\mathcal{D})$, we conclude that $\mathrm{Del}(\mathcal{D})$ is a t-spanner of $\mathcal{D}$. $\square$

7. Computing a plane straight-line embedding

Note that the embedding of the AW-Delaunay graph that consists of straight-line segments between the centers of the disks is not necessarily a plane graph (see Fig. 20). However, the Voronoi diagram of a set of disks $\mathcal{D}$, denoted by $\mathrm{Vor}(\mathcal{D})$, is planar [7]. Since $\mathrm{Del}(\mathcal{D})$ is the face-dual of $\mathrm{Vor}(\mathcal{D})$, it is also planar. An important characteristic of the Delaunay graph of

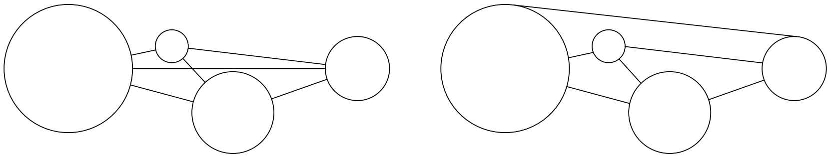

Fig. 20. The AW-Delaunay graph that consists of straight-line segments between the centers of the disks is not necessarily a plane graph, but it is planar. The plane straight-line embedding computed by our algorithm is shown on the right.

a set of points regarded as a spanner is that it is a plane graph. Therefore, a natural question is whether $\mathrm{Del}(\mathcal{D})$ has a plane straight-line embedding that is also a spanner.

The proof of Theorem 2 suggests the existence of an algorithm allowing to compute such an embedding: compute the Delaunay triangulation of the set P that contains the distance points and the representative of $\mathcal{D}$. The graph $\mathrm{Del}(P)$ can be regarded as a multigraph whose vertex set is $\mathcal{D}$. Then, for each pair of disks that share one or more edges, just keep the shortest of those edges. This simple algorithm allows us to compute a plane straight-line embedding of $\mathrm{Del}(\mathcal{D})$ that is also a spanner of $\mathcal{D}$. However, its running time is $O(n^3 \log n)$.

On the other hand, it is also possible to compute in time $O(n \log n)$ a plane spanner of $\mathcal{D}$ whose spanning ratio is t^2, the square of the spanning ratio of the Delaunay graph of a set of points. Here is how to do this: First, compute $\mathrm{Del}(\mathcal{D})$. Then, let P be the set of distance points of all pairs of disks that share an edge in $\mathrm{Del}(\mathcal{D})$. Compute $\mathrm{Del}(P)$. Since P has size $O(n)$, this can be done in time $O(n \log n)$. Also, $\mathrm{Del}(P)$ is a plane graph. As in the above paragraph, the graph $\mathrm{Del}(P)$ can be regarded as a multigraph whose vertex set is $\mathcal{D}$. Again, for each pair of disks that share one or more edges, just keep the shortest of those edges. All that remains to explain is why the resulting graph is a (t^2)-spanner of $\mathcal{D}$. Let $D_1, D_2 \in \mathcal{D}$. The straight-line embedding of $\mathrm{Del}(\mathcal{D})$ contains a t-spanning path between D_1 and D_2. The endpoints of the edges of that path are the distance points between the disks. In $\mathrm{Del}(P)$, each of those edges is approximated within a factor of t, leading to a spanning ratio of t^2. In fact, the vertex set here can be the centers of the disks since we use distance points, which is not the case for the $O(n^3 \log n)$ time algorithm. We summarize our results below:

Theorem 3. *Let $\mathcal{D}$ be a set of n disjoint disks and t be the smallest upper bound on the spanning ratio of the Delaunay triangulation of a set of points. In time $O(n^3 \log n)$, it is possible to compute a plane t-spanner of $\mathcal{D}$, and in time $O(n \log n)$, it is possible to compute a plane t^2-spanner of $\mathcal{D}$.*

Whether or not it is possible to compute a plane straight-line embedding of $\mathrm{Del}(\mathcal{D})$ that is also a t-spanner of $\mathcal{D}$ in time $O(n \log n)$ remains an open question.

8. Conclusion

In this paper, we showed how, given a weighted point set where weights are positive and $|p_i p_j| \geqslant r_i + r_j$ for all $i \neq j$, it is possible to compute a $(1+\epsilon)$-spanner of that point set that has a linear number of edges. We also showed that the Additively Weighted Delaunay graph is a t-spanner of an additively weighted point set in the same case. The constant t is the same as for the Delaunay triangulation of a point set (the best current value is 2.42 [4]). We could not see how the Well-Separated Pair Decomposition (WSPD) can be adapted to solve this problem. The first difficulty resides in the fact that it is not even clear that, given a weighted point set, a WSPD of that point set always exists. Other obvious open questions are whether our results still hold when some weights are negative or $|p_i p_j| < r_i + r_j$ for some $i \neq j$. Also, we did not verify whether our variant of the Yao graph can be computed in time $O(n \log n)$. Finally, another problem that could be explored is whether it is possible to compute t-spanners for multiplicatively weighted point sets.

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