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Integral Formulas with Distribution Kernels for Irreducible Projections in L^2 of a Nilmanifold*

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Let N be a simply connected nilpotent Lie group and Γ a discrete uniform subgroup. The authors consider irreducible representations σ in the spectrum of the quasi-regular representation $N \times L^2(\Gamma\backslash N) \to L^2(\Gamma\backslash N)$ which are induced from *normal* maximal subordinate subgroups $M \subseteq N$. The primary projection P_σ and all irreducible projections $P \leqslant P_\sigma$ are given by convolutions involving right Γ-invariant distributions D on $\Gamma\backslash N$,

$$Pf(\Gamma n) = D * f(\Gamma n) = \langle D, n \cdot f\rangle \qquad \text{all } f \in C^\infty(\Gamma/N),$$

where $n \cdot f(\zeta) = f(\zeta \cdot n)$. Extending earlier work of Auslander and Brezin, and L. Richardson, the authors give explicit character formulas for the distributions, interpreting them as sums of characters on the torus $T^k = (\Gamma \cap M)$ $[M, M]\backslash M$. By examining these structural formulas, they obtain fairly sharp estimates on the order of the distributions: if σ is associated with an orbit $\mathcal{O} \subseteq \mathfrak{n}^*$, and if $\mathscr{V} \subseteq \mathfrak{n}^*$ is the largest subspace which saturates $\mathcal{O}$ in the sense that $\mathcal{O} + \mathscr{V} = \mathcal{O}$, order$(D) \leqslant d = 1 + [\frac{1}{2}(\dim \mathcal{O} - \dim \mathscr{V})]$. As a corollary they obtain Richardson's criterion for a projection to map $C^0(\Gamma\backslash N)$ into itself. The authors also resolve a conjecture of Brezin, proving a Zero–One law which says, among other things, that if the primary projection P_σ maps $C^r(\Gamma\backslash N)$ into $C^0(\Gamma\backslash N)$, so do *all* irreducible projections $P \leqslant P_\sigma$. This proof is based on a classical lemma on the extent to which integral points on a polynomial graph in $\mathbf{R}^n$ lie in the coset ring of $\mathbf{Z}^n$ (the finitely additive Boolean algebra generated by cosets of subgroups in $\mathbf{Z}^n$). This lemma may be useful in other investigations of nilmanifolds.

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1. Introduction

Throughout this paper N will be a simply connected nilpotent Lie group, Γ a uniform discrete subgroup ($\Gamma\backslash N$ compact); we consider the induced *quasi-regular representation* $U^1 = \mathrm{Ind}(\Gamma \uparrow N, 1)$ on $L^2(\Gamma\backslash N)$. Then U^1 splits into a direct sum of certain irreducibles, the *spectrum* of $\Gamma\backslash N$, which we denote by $(N : \Gamma)^\wedge$,

$$U^1 = \bigoplus_{\sigma\in(N:\Gamma)^\wedge} m(\sigma)\sigma,$$

each with finite multiplicity. For each $\sigma \in (N : \Gamma)^\wedge$ let $\mathscr{H}_\sigma \subseteq L^2(\Gamma\backslash N)$ be the σ-primary subspace. We will study the projection P_σ onto a primary subspace, and also projections to irreducible invariant subspaces contained in $\mathscr{H}_\sigma$.

Auslander and Brezin [1] have shown that all bounded projections $E : L^2(\Gamma\backslash N) \to L^2(\Gamma\backslash N)$ commuting with the action of N map $C^\infty(\Gamma\backslash N)$ into itself, and are determined by convolution-type formulas from right Γ-invariant Schwartz distributions D_E ($\langle D_E , \gamma \cdot f\rangle = \langle D_E ,f\rangle$, all $\gamma \in \Gamma$),

$$\begin{aligned}&\langle D_E , \phi\rangle = E\phi(\Gamma e)\\ &E\phi(\Gamma n) = D_E * \phi(\Gamma n) = \langle D_E , n \cdot \phi\rangle\end{aligned} \qquad \text{all } \phi \in C^\infty(\Gamma\backslash N), \tag{1}$$

where $n \cdot \phi(x) = \phi(x \cdot n)$. They show that these distributions have order $\leqslant n + 1$ ($n = \dim N$), so that E actually maps $C^{n+1}(\Gamma\backslash N)$ into $C^0(\Gamma\backslash N)$. One of our main goals is to extend these existence theorems to give a constructive description of the distributions associated with a σ-primary projection P_σ and the irreducible projections $P \leqslant P_\sigma$.

In recent work [12] L. Richardson has studied the action on continuous functions of σ-primary projections P_σ and of the irreducible projections $P \leqslant P_\sigma$. He asks:

Question 1. Does P_σ always map $C^0(\Gamma\backslash N)$ into $C^0(\Gamma\backslash N)$?

Question 2. If the σ-primary projection P_σ maps C^0 into C^0, do some (all) of the irreducible projections $P \leqslant P_\sigma$ do so too?

The first question amounts to asking whether the associated distributions D_σ have order zero (are finite measures on $\Gamma\backslash N$). He showed that the answer to Q. 1 is sometimes negative, and for a large class of $\sigma \in (N : \Gamma)^\wedge$ gave necessary and sufficient conditions that $P_\sigma : C^0 \to C^0$. This "smoothness" of P_σ is tested by associating σ with a set of characters on a certain torus T^k; the distribution D_σ is then of order zero $\Leftrightarrow$ this set lies in the coset ring $\mathrm{COS}(\mathbf{Z}^k)$, the finitely additive Boolean algebra of sets generated by the cosets of subgroups in $\mathbf{Z}^k = \hat{T}^k$. These results applied essentially to those $\sigma \in (N : \Gamma)^\wedge$ induced from *normal* maximal subordinate subgroups M such that $N = M \cdot X$ (semidirect product of M with a closed abelian subgroup X).

We simplify and extend this work by taking a different point of view. Our approach handles all σ induced from normal maximal subordinate subgroups M; no additional structure $N = M \cdot X$ is required. Richardson worked with irreducible projections into $\mathscr{H}_\sigma$ by determining which functions in $L^2(\Gamma \backslash N)$ can appear in their range, using earlier results of his [11] to construct and describe these range spaces. We use the results of [11], and some technical refinements presented in [4, Sect. 5], to describe the projection operators themselves via integration formulas with distribution kernels, as in (1). These formulas directly yield character formulas for the associated distributions. In addition to extending Richardson's results, we are able to give fairly sharp answers to a more general question:

Question 3. When does the primary projection P_σ, or an irreducible projection $P \leqslant P_\sigma$, map $C^r(\Gamma \backslash N)$ into $C^0(\Gamma \backslash N)$?
This amounts to giving estimates for the orders of the associated distributions.

1.1. THEOREM. *Suppose $\sigma \in (N : \Gamma)^\wedge$ is induced from a normal maximal subordinate subgroup M such that $\Gamma \cap M \backslash M$ is compact. Let $\mathcal{O} \subseteq \mathfrak{n}^*$ be the orbit associated with σ and let $\mathscr{V}$ be the largest subspace in $\mathfrak{n}^*$ which saturates $\mathcal{O}$ in the sense that $f \in \mathcal{O} \Rightarrow f + \mathscr{V} \subseteq \mathcal{O}$.* If $d = \dim(\mathcal{O}) - \dim(\mathscr{V})$, *then* $\operatorname{order}(D) \leqslant s$, *where s = smallest integer greater than $d/2$, if D is associated with either P_σ or an irreducible projection $P \leqslant P_\sigma$.*

We also resolve Question 2 in the form of a Zero–One law which shows, among other things, that the irreducible projections $P \leqslant P_\sigma$ are all as well behaved as the primary projection P_σ.

1.2. THEOREM. *If $\sigma \in (N : \Gamma)^\wedge$ is induced from a normal maximal subordinate subgroup M such that $\Gamma \cap M \backslash M$ is compact, and if the primary projection P_σ maps $C^r(\Gamma \backslash N)$ into $C^0(\Gamma \backslash N)$ for some $r \geqslant 0$, then the same is true for all irreducible projections $P \leqslant P_\sigma$.*

Thus the primary projection is already responsible for all loss of smoothness. (Our actual Zero–One law is somewhat more general, and technical, than the result just cited.)

All of these results apply to σ induced from normal M. We offer some speculations on the nonnormal case. Our basic result, set forth in detail in the next section, constructs the distributions associated with the primary projection P_σ and certain "constructible" irreducible projections $P \leqslant P_\sigma$. They are described as the (distribution) sums of certain sets of characters on the torus $T^k = (\Gamma \cap M)[M, M]\backslash M$. These formulas overlap some results presented by Jon Brezin in a note [2] which came to our attention after a preliminary version of the present manuscript had been prepared. Brezin's methods are based on the Mackey Machine, and are very different from ours. They seem to give information only about the primary projection P_σ, but on the other hand they work for $L^2(\Gamma\backslash G)$ in cases where G is not nilpotent. Information about irreducible projections $P \leqslant P_\sigma$ is necessary in resolving the Zero–One law. We wish to thank Jon Brezin for communicating his results to us, and for a continuing correspondence on harmonic analysis on nilmanifolds. His comments on the proper form of the basic Zero–One law (Theorem 6.1) were particularly helpful. We are also indebted to L. Richardson for an extensive exchange of correspondence concerning [12] and more recent developments, and for his hospitality in arranging a short conference on nilmanifolds at Louisiana State University, which led to a valuable exchange of ideas.

2. The Character Formulas

We presume familiarity with the work of Kirillov [9] and Richardson [11] (see also Howe [7]). We refer to the one-dimensional unitary representations of a group G as its *characters*. Following [11], a *maximal character* for N is any pair (χ, M) such that

(i) M is a closed connected subgroup and χ is a character on it;

(ii) there is an $f \in \mathfrak{n}^*$ such that $\chi = e^{2\pi i f \circ \log} \mid M$, and the Lie algebra $\mathfrak{m}$ is maximal subordinate to f.

Then (χ, M) is an *integral maximal character* if

(iii) M is rational (i.e., $\Gamma \cap M\backslash M$ is compact);

(iv) $\chi \equiv 1$ on $\Gamma \cap M$.

This is not quite Richardson's definition in [11]. He imposed additional "speciality" requirements on M; these can be eliminated using refinements of his paper given in [4, Sect. 5]. For a while we will allow M to be nonnormal. Beginning with Section 3 we confine our attention to *normal* M.

As explained in [11] or [4, Sect. 5], if (χ, M) is maximal integral the induced representation $\sigma = \mathrm{Ind}(M \uparrow N, \chi)$ lies in the spectrum, $\sigma \in (N : \Gamma)^\wedge$. Furthermore, every $\sigma \in (N : \Gamma)^\wedge$ arises in this way. For each maximal character we get an explicit intertwining isometry $B : \mathscr{H}(U^\chi) \to \mathscr{H}_{(\chi, M)} \subseteq \mathscr{H}_\sigma$ via the following averaging process. Regarding $\mathscr{H}(U^\chi)$ as a space of functions on N, we define

$$BF(\Gamma n) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} F(\gamma n) \qquad \text{all} \quad n \in N. \tag{2}$$

Here $\Gamma_0 = \Gamma \cap M$; the sum is taken over any set of coset representatives for $\Gamma_0 \backslash \Gamma$ and is independent of this arbitrary choice. In [4] we showed that this pointwise formula (2) is valid for $F \in \mathscr{H}(U^\chi)_{00} =$ functions which are bounded and measurable, and have compact support modulo M; then the sum has only finitely many nonzero terms and determines an isometry B on $\mathscr{H}(U^\chi)$. Actually, the arguments in [4] work just as well (the sum is now absolutely convergent pointwise) if $F \in \mathscr{H}(U^\chi)_0 =$ functions which are bounded, measurable, and rapidly vanishing transverse to M cosets in the sense that

$$\int_{\Gamma \backslash N} \left(\sum_{\gamma \in \Gamma_0 \backslash \Gamma} |F(\gamma n)| \right)^2 d\dot{n} < +\infty. \tag{3}$$

Formula (2) is precisely what Richardson used [11] to determine which functions in $L^2(\Gamma \backslash N)$ belong to the "lift space" $\mathscr{H}_{(\chi, M)} =$ range(B), and thus to study what he called the "constructible projections" $P_{(\chi, M)} : L^2(\Gamma \backslash N) \to \mathscr{H}_{(\chi, M)}$, those corresponding to integral maximal characters which induce σ. We shall study the constructible projections $P_{(\chi, M)}$ by directly computing the adjoint $B^* : L^2(\Gamma \backslash N) \to \mathscr{H}(U^\chi)$ and $P_{(\chi, M)} = BB^*$.

2.1. Lemma. *If $f \in C^0(\Gamma \backslash N)$ then*

$$B^*f(n) = \int_{\Gamma_0 \backslash M} \overline{\chi(m)} f(mn)\, d\dot{m} = \int_{\Gamma_0 \backslash M} \overline{\chi(m)}\, n \cdot f(m)\, d\dot{m}, \tag{4}$$

where $n \cdot f(x) = f(x \cdot n)$ and $d\dot{m}$ is normalized so that $\mathrm{Vol}(\Gamma_0 \backslash M) = 1$.

Proof. The right-hand side gives a function $Af(n)$ which is well defined for all $n \in N$, bounded, continuous; Af also has the desired covariance, $Af(mn) = \chi(m)\, Af(n)$ for $m \in M$. If $h \in \mathscr{H}(U^\chi)_{00}$, then (2) insures that if C is a bounded measurable fundamental domain for

$\Gamma\backslash N$, and dx = Haar measure on N normalized so that C has mass 1, then

$$(h \mid B^*f) = (Bh \mid f) = \int_{\Gamma\backslash N} \left(\sum_{\gamma\in\Gamma_0\backslash\Gamma} h(\gamma x) \mid f(x) \right) d\dot{x}$$

$$= \int_C \sum_{\gamma\in\Gamma_0\backslash\Gamma} (h(\gamma x) \mid f(\gamma x))\, dx$$

(since $f(x) = f(\gamma x)$ for all $\gamma \in \Gamma$)

$$= \int_{M\backslash N} \int_{\Gamma_0\backslash M} (h(m \cdot n) \mid f(m \cdot n))\, d\dot{m}\, d\dot{n}$$

$$= \int_{M\backslash N} \int_{\Gamma_0\backslash M} (h(n) \mid \overline{\chi(m)}\, f(mn))\, d\dot{m}\, d\dot{n}$$

$$= \int_{M\backslash N} \left(h(n) \,\middle|\, \int_{\Gamma_0\backslash M} \overline{\chi(m)}\, f(mn)\, d\dot{m}\right) d\dot{n}$$

$$= (h \mid Af).$$

It follows that Af is square integrable. Thus $Af \in \mathscr{H}(U^\chi)$ and $B^*f = Af$. Q.E.D.

Now, formally at least, we may compute $BB^*f(\Gamma n)$ from (2). If $f \in C^0(\Gamma\backslash N)$,

$$BB^*f(\Gamma n) = \sum_{\gamma\in\Gamma_0\backslash\Gamma} B^*f(\gamma n)$$

$$= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{\Gamma_0\backslash M} \overline{\chi(m)}\, f(\Gamma m\gamma n)\, d\dot{m}$$

$$= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{\Gamma_0\backslash M} \overline{\chi(m)}\, f(\Gamma\gamma^{-1}m\gamma \cdot n)\, d\dot{m}. \tag{5}$$

Why doesn't this work? The difficulty is in the first line. The pointwise formula (2) is only known to be valid if $B^*f = F$ vanishes rapidly transverse to M cosets as in (3). In the next section we will show that B^*f satisfies (3) if M is normal and if $f \in C^{k+1}(\Gamma\backslash N)$, $k = \dim \mathfrak{m} - \dim[\mathfrak{m}, \mathfrak{m}]$. Meanwhile, we give a geometric interpretation for this formula, when it is valid. For $n \in N$ let $(\chi, M) \cdot n = (\chi \cdot n, M \cdot n) = (\chi^n, M^n)$ be the maximal character

$$\chi^n(m') = \chi(nm'n^{-1}) \qquad \text{for} \quad m' \in M^n = n^{-1}Mn.$$

We write χ^n or $\chi \cdot n$ as convenience dictates. If M is normal, only χ moves. Let $((\chi, M) \cdot N)_{\#}$ be the integral maximal characters in this N-orbit. For integral points $(\chi', M') = (\chi, M) \cdot n \in ((\chi, M) \cdot N)_{\#}$ let $\mu' = \mu^n$ be a normalized M'-invariant measure on $\Gamma \cap M' \backslash M'$. By integrality, χ' may be regarded as a continuous function on $\Gamma \cap M'\backslash M'$. Define $\nu_{(\chi', M')}(d\dot{m}) = \overline{\chi'(m)}\, \mu'(d\dot{m})$ on $\Gamma \cap M'\backslash M'$. This may be regarded as a signed measure on $\Gamma\backslash N$ of total variation 1, if we identify $\Gamma \cap M'\backslash M' \approx \Gamma\backslash\Gamma M' \subseteq \Gamma\backslash N$.

Elements $n = \gamma \in \Gamma$ preserve integrality. Thus Formula (5) can be written

$$\begin{aligned} P_{(\chi,M)} f(\Gamma n) &= BB^* f(\Gamma n) \\ &= \sum_{\gamma \in \Gamma_0\backslash\Gamma} \int_{\Gamma \cap M^\gamma \backslash M^\gamma} \overline{\chi^\gamma(m)}\, f(\Gamma m n)\, \mu^\gamma(d\dot{m}) \\ &= \sum_{\gamma \in \Gamma_0\backslash\Gamma} \langle \nu_{(\chi,M)\cdot\gamma}, n \cdot f \rangle. \end{aligned} \tag{6}$$

Formulas (1) and (6) strongly suggest that the sum $\sum_\gamma \nu_{(\chi,M)\cdot\gamma}$ should converge as a distribution, to $D_{(\chi,M)}$. This would yield a satisfying interpretation for $D_{(\chi,M)}$. It would be precisely the sum of complex conjugates of all characters in the Γ-orbit $\{(\chi, M) \cdot \gamma : \gamma \in \Gamma_0\backslash\Gamma\}$ which determined the projection $P_{(\chi,M)}$, if we identify characters $(\chi, M) \cdot \gamma$ with the measures $\nu_{(\chi,M)\cdot\gamma}$ on $\Gamma\backslash N$; i.e.,

$$D_{(\chi,M)} = \sum \{\nu_{(\chi',M')} : (\chi', M') \in (\chi, M) \cdot \Gamma\}. \tag{7}$$

By the same token, if $(\chi_1, M_1), \ldots, (\chi_q, M_q)$ are representatives for the various Γ-orbits in $((\chi, M) \cdot N)_{\#}$, then as in [11] the projections $P_i = B_i B_i^* = P_{(\chi_i, M_i)}$ are orthogonal and $P_\sigma = P_1 + \cdots + P_q$. Thus P_σ is associated with the distribution obtained by adding up *all* the integral characters in the orbit $(\chi, M) \cdot N$ instead of those in a single Γ-orbit,

$$D_\sigma = \sum \{\nu_{(\chi',M')} : (\chi', M') \in ((\chi, M) \cdot N)_{\#}\}. \tag{8}$$

In the next section, where M is *normal*, we will prove the validity of Formulas (7) and (8). In the normal case all characters live on M, are integral, and annihilate $[M, M]$, so we may simplify things by regarding them as characters on the torus $T^k = \Gamma_0[M, M]\backslash M$, and the sums (7) and (8) as sums over certain sets in $\hat{T}^k = \mathbf{Z}^k$.

In the nonnormal case M moves and the picture is more complicated. We conjecture that Formulas (7) and (8) remain valid. We have actually proved (7) and (8) valid for certain classes of nonnormal subgroups, but the proofs are complicated and we will not go into them here.

3. Proof of Formula (7)

Of course (7) $\Rightarrow$ (8). Let (χ, M) be an integral maximal character with M normal. Let $\sigma = \mathrm{Ind}(M \uparrow N, \chi)$. We start by elaborating the connection between the sum (7) on $\Gamma \cap M\backslash M$ and the associated sum of characters on the torus $T^k = \Gamma_0[M, M]\backslash M$. Let $M_1 = [M, M]$, $M_2 = \Gamma_0[M, M]$, and let $p : \Gamma_0\backslash M \to T^k = M_2\backslash M$ be the canonical map. We define a "global" averaging map $p^* : C^0(\Gamma\backslash N) \to C^0(N)$,

$$p^*f(n) = \int_H f(\Gamma hn)\, dh \qquad \text{all} \quad n \in N, \tag{9}$$

where H is a bounded measurable fundamental domain for $\Gamma \cap M_2\backslash M_2 = \Gamma_0\backslash M_2$ and dh is Haar measure on M_2 normalized so H has mass one. Now p^*f is constant on cosets of the closed subgroup $\Gamma \cdot M_2$; if $\gamma_0 \in \Gamma$, $m_1 \in [M, M]$, $n \in N$ we have

$$\begin{aligned} p^*f(\gamma_0 m_1 n) &= \int_H f(\Gamma h \gamma_0 m_1 n)\, dh \\ &= \int_{\gamma_0 H \gamma_0^{-1}} f(\Gamma h m_1 n)\, dh \\ &= \int_{H m_1^{-1}} f(\Gamma h n)\, dh = p^*f(n), \end{aligned}$$

because the integrands (left Γ_0-periodic functions of $h \in M_2$) have the same integral over the various fundamental domains H, Hm ($m \in M_1$), $\gamma H \gamma^{-1}$ ($\gamma \in \Gamma$).

If $\pi: N \to \Gamma\backslash N$ is the canonical map, $\pi(M) = \Gamma\backslash\Gamma M \approx \Gamma_0\backslash M$ is a closed submanifold. The values of p^*f on M are determined by the values $f \mid \pi(M)$ of f on $\pi(M)$, so Formula (9) induces a map p_1^*: $C^0(\Gamma_0\backslash M) = C^0(\Gamma\backslash\Gamma M) \to C^0(T^k)$ such that $p_1^*(f \mid \pi(M)) = p^*f \mid M$ when we identify the latter with a function on $T^k = M_2\backslash M$; just set $n = m \in M$ in (9). This restricted operation suffices for our immediate purpose and has the following easily verified properties:

(i) p_1^* is $\|\cdot\|_\infty$-norm continuous;

(ii) $p_1^*(\phi \circ p) = \phi$ for all $\phi \in C^0(T^k)$;

(iii) if dt is normalized measure on T^k,

$$\int_{\Gamma_0\backslash M} f(\Gamma_0 m)\, d\dot{m} = \int_{T^k} p_1^*f(t)\, dt \tag{10}$$

for all $f \in C^0(\Gamma_0\backslash M)$;

(iv) for all $0 \leqslant r < \infty$, p_1^* maps $C^r(\Gamma_0\backslash M)$ *onto* $C^r(T^k)$ and is continuous with respect to the $\|\cdot\|_r$-norms. If $X_1, \ldots, X_r \in \mathfrak{m}$, if

$\psi : M \to M_2\backslash M = T^k$ and $d\psi : \mathfrak{m} \to \mathfrak{t}^k$ are the canonical homomorphisms, then

$$p_1{}^*(X_1 \cdots X_r f) = d\psi(X_1) \cdots d\psi(X_r)\, p_1{}^* f$$

for all $f \in C^r(\Gamma_0\backslash M)$.

Clearly $p_1{}^* : C^\infty(\Gamma_0\backslash M) \to C^\infty(T^k)$ is continuous with respect to the Schwartz topology, giving us an injective liftback of distributions $p_1^{**} : \mathscr{D}'(T^k) \to \mathscr{D}'(\Gamma_0\backslash M)$. Furthermore, if $\pi : N \to \Gamma\backslash N$ is the quotient map, there is the obvious diffeomorphism $i : \Gamma_0\backslash M \to \pi(M) \subseteq \Gamma\backslash N$, $\pi(M)$ a compact submanifold in $\Gamma\backslash N$. This gives the obvious injective map of distributions i^{**}: $\mathscr{D}'(\Gamma_0\backslash M) \to \mathscr{D}'(\Gamma\backslash N)$. Obviously, $\text{order}(S) = \text{order}(p_1^{**}S) = \text{order}(i^{**}(p_1^{**}S))$ under these liftings.

Note. C^r norms $\| \phi \|_r$ are defined in $\Gamma_0\backslash M$ and $\Gamma\backslash N$ by fixing a weak Malcev basis $X_1, \dots, X_m, \dots, X_n$ which runs through M and N. Then regard every $X \in \mathfrak{n}$ (resp. $X \in \mathfrak{m}$) as a differential operator on $\Gamma\backslash N$ (resp. $\Gamma_0\backslash M$),

$$X\phi(\zeta) = \lim_{t\to 0}(1/t)[\phi(\zeta \cdot \exp(tX)) - \phi(\zeta)]$$

(uniformly convergent if $\phi \in C^1$). On $C^r(\Gamma_0\backslash M)$ let

$$\| \phi \|_r = \max\{\| X_1^{i_1} \cdots X_m^{i_m}\phi \|_\infty : 0 \leqslant i_1 + \cdots + i_m \leqslant r\},$$

and similarly on $C^r(\Gamma\backslash N)$.

If χ is an integral maximal character on M, write $\tilde{\chi}$ for the associated character on T^k, $\chi = \tilde{\chi} \circ p$. By (10ii), $\chi = p_1{}^*(\tilde{\chi} \cdot p)$. Thus (10) insures that

$$\int_{T^k} \overline{\tilde{\chi}^\gamma(t)}\, p_1{}^*\phi(t)\, dt = \int_{\Gamma_0\backslash M} \overline{\chi^\gamma(m)}\, \phi(m)\, d\dot{m}$$

for all $\phi \in C^0(\Gamma_0\backslash M)$. On T^k *any* set of characters determines a distribution of order $\leqslant k + 1$. Write $\tilde{S}$ for the distribution on T^k determined by the complex conjugates of the characters in the Γ-orbit $\{\chi^\gamma : \gamma \in \Gamma_0\backslash\Gamma\}$,

$$\tilde{S} = \overline{\sum \{\tilde{\chi}^\gamma : \gamma \in \Gamma_0\backslash\Gamma\}}. \tag{11}$$

By (10iii) its liftback S to $\Gamma_0\backslash M$ or $\Gamma\backslash N$ is given by an absolutely convergent sum,

$$\begin{aligned}\langle S, \phi\rangle = \langle \tilde{S}, p_1{}^*\phi\rangle &= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{T^k} \overline{\tilde{\chi}^\gamma(t)}\, p_1{}^*\phi(t)\, dt \\ &= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \int_{\Gamma_0\backslash M} \overline{\chi^\gamma(m)}\, \phi(m)\, d\dot{m} \\ &= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \langle \nu_{(\chi,M)\cdot\gamma}, \phi\rangle\end{aligned}$$

for $\phi \in C^{k+1}(\Gamma_0\backslash M)$. Thus $S = \sum \{\nu_{(x,M)\cdot\gamma} : \gamma \in \Gamma_0\backslash\Gamma\}$ converges to a Schwartz distribution of order $\leqslant k+1$ on $\Gamma_0\backslash M$ or $\Gamma\backslash N$.

It remains to show that the left and right sides of (6) agree for $f \in C^{k+1}$ (or $f \in C^\infty$), and thus that $D_{(x,M)} = S =$ liftback from T^k of (6). Let K be a bounded, measurable fundamental domain for $\Gamma\backslash N$. If $f \in C^{k+1}(\Gamma\backslash N)$, set

$$F(n) = B^*f(n) = \int_{\Gamma_0\backslash M} \overline{\chi(m)}\, n \cdot f(m)\, d\dot{m}, \qquad \text{all} \quad n \in N.$$

If we decompose $n = \gamma \cdot k$ $(k \in K)$ we get

$$F(n) = \int_{\Gamma_0\backslash M} \overline{\chi^\gamma(m)}\, k \cdot f(m)\, d\dot{m}.$$

For $0 \leqslant r \leqslant +\infty$ the following maps are continuous,

$$N \times C^r(\Gamma\backslash N) \to C^r(\Gamma\backslash N),$$

$$C^r(\Gamma\backslash N) \xrightarrow{\text{restrict}} C^r(\Gamma_0\backslash M) \xrightarrow{p^*} C^r(T^k),$$

where we identify $\Gamma_0\backslash M \approx \pi(M) \subseteq \Gamma\backslash N$ as above. Compactness of $\bar{K}$ insures precompactness of the set of functions $\{p_1^*(k \cdot f) : k \in K\} \subseteq C^r(T^k)$. If $r = k+1$ this means there is a bound A such that the Fourier transforms satisfy

$$|p_1^*(k \cdot f)^\wedge(\mathbf{n})| \leqslant \frac{A}{(1 + \|\mathbf{n}\|)^{k+1}}, \qquad \text{all} \quad \mathbf{n} \in \mathbf{Z}^k = \hat{T}^k, \text{ all } k \in K.$$

Thus

$$\sum_{\mathbf{n}\in\mathbf{Z}^k} |p_1^*(k \cdot f)^\wedge(\mathbf{n})| \leqslant A' < +\infty, \qquad \text{all} \quad k \in K.$$

This insures that

$$\begin{aligned}
&\int_{K=\Gamma\backslash N} \Big(\sum_{\gamma\in\Gamma_0\backslash\Gamma} |F(\gamma k)|\Big)^2 dk \\
&\quad= \int_K \Big(\sum_{\gamma\in\Gamma_0\backslash\Gamma} \Big|\int_{\Gamma_0\backslash M} \overline{\chi^\gamma(m)}\, k \cdot f(m)\, d\dot{m}\Big|\Big)^2 dk \\
&\quad= \int_K \Big(\sum_{\gamma\in\Gamma_0\backslash\Gamma} \Big|\int_{T^k} \overline{\tilde{\chi}^\gamma(t)}\, p_1^*(k \cdot f)(t)\, dt\Big|\Big)^2 dk \\
&\quad\leqslant \int_K \Big(\sum_{\mathbf{n}\in\mathbf{Z}^k} |p_1^*(k \cdot f)^\wedge(\mathbf{n})|\Big)^2 dk \\
&\quad\leqslant (A')^2 \operatorname{Vol}(\Gamma\backslash N) < +\infty.
\end{aligned}$$

This is just the "rapid vanishing" estimate (3) needed to justify the pointwise formula (2) for $BF(n), F \in \mathscr{H}(U^{\chi})$. Thus,

$$\langle D_{(\chi,M)}, n \cdot f \rangle = BB^*f(n) = \langle S, n \cdot f \rangle$$

for all $n \in N$, and Formula (7) is fully justified.

The character formulas (7) and (8) for $D_{(\chi,M)}$ and $D_\sigma = \sum_{j=1}^{q} D_{(\chi_j,M)}$ show that these distributions have order $\leqslant k + 1$ where $k = \dim T^k = \dim \mathfrak{m} - \dim[\mathfrak{m}, \mathfrak{m}]$. We will get sharper estimates below by examining these formulas more carefully. Meanwhile, the theorems of Helson [6] and Rudin [13] on idempotent measures on tori immediately yield Richardson's necessary and sufficient condition for $D_{(\chi,M)}$ or D_σ to have order zero (be a finite measure on $\Gamma\backslash N$).

3.2. COROLLARY. *If (χ, M) is an integral maximal character with M normal, then $D_{(\chi,M)}$ has order zero $\Leftrightarrow$ the set of characters $\{\tilde{\chi}^{\gamma} : \gamma \in \Gamma_0\backslash\Gamma\} \subseteq \hat{T}^k = \mathbf{Z}^k$ lies in the coset ring* COS($\mathbf{Z}^k$). *Similarly, D_σ is a measure $\Leftrightarrow$ the full set of integral characters $\{\chi' : (\chi', M) \in ((\chi, M) \cdot N)_{\#}\}$* lies in COS($\mathbf{Z}^k$).

Brezin has given an interesting variant of this criterion for the primary distribution in [2].

If we regard $D_{(\chi,M)}$ and D_σ as distributions on $\Gamma\backslash N$, their values when bracketed with $\phi \in C^{k+1}(\Gamma\backslash N)$ are determined solely by $\phi \mid \pi(M)$. Thus they are supported on $\pi(M) \approx \Gamma_0\backslash M$ and involve no differentiations transverse to the compact submanifold $\pi(M)$. Finally, D_σ cannot depend on the particular choice of integral character (χ, M) from which σ was induced. If we have other maximal subordinate subgroups M' such that $\sigma = \mathrm{Ind}(M' \uparrow N, \chi')$, the remarks about supports still apply, so we conclude that $\mathrm{supp}(D_\sigma) \subseteq \bigcap\{\pi(M') : M'$ normal, and M' is associated with an integral maximal character (χ', M') which induces $\sigma\}$.

4. ORDER ESTIMATES FOR THE PRIMARY DISTRIBUTION

Fix $\sigma \in (N : \Gamma)^\wedge$ and assume σ is associated with an integral maximal character (χ_0, M) such that M is normal. Action of Γ preserves integrality, so the set $((\chi_0, M) \cdot N)_{\#}$ of integral characters in the N-orbit splits into Γ-orbits. Let $\{\chi_1, \dots, \chi_q\}$ be orbit representatives and let $D_i = D_{(\chi_i,M)}$ and D_σ be the distributions associated with $P_i = P_{(\chi_i,M)} = B_i B_i^*$ and P_σ. We now give an estimate for order(D_σ); later we will get the same estimate on order(D_i) indirectly. These arguments are based on the following lemmas together with the character formulas (7) and (8).

4.1. LEMMA.[1] *Let* $d = 0, 1, 2, \ldots$, *let* $P_1, \ldots, P_l$ *be polynomials on* $\mathbf{R}^d$, *and consider their "integral graph" in* $\mathbf{Z}^{l+d} = \mathbf{Z}^d \times \mathbf{Z}^l$,

$$G = \mathbf{Z}^{l+d} \cap \{(a_1, \ldots, a_d, P_1(\mathbf{a}), \ldots, P_l(\mathbf{a})) : \mathbf{a} = (a_1, \ldots, a_d) \in \mathbf{Z}^d\}.$$

(If $d = 0$, *interpret* G *as a single point in* $\mathbf{Z}^{l+d}$.*) Let* E *be any subset of this graph. Identifying the dual of the torus* T^{l+d} *as* $\hat{T}^{l+d} = \mathbf{Z}^{l+d}$ *in the usual way, define a distribution on* T^{l+d}

$$D = \sum \{\chi_{\mathbf{v}} : \mathbf{v} \in E\} \qquad \textit{where} \quad \chi_{\mathbf{v}}(\mathbf{t}) = e^{2\pi i(\mathbf{v}|\mathbf{t})}$$

(Take $D = 0$ *if* $E = \varnothing$.*) Then* order$(D) \leqslant s$, *where* s = *smallest integer greater than* $d/2$.

Proof. Let π be the projection of $\mathbf{Z}^d \times \mathbf{Z}^l \to \mathbf{Z}^d \times (\mathbf{0})$ and let $I = \pi(E)$. For $1 \leqslant i \leqslant d$ let $I_i = \{\mathbf{n} \in I : \mathbf{n} \neq 0, \mid n_i \mid \geqslant \mid n_k \mid$, all k, and $\mid n_i \mid > \mid n_k \mid$ for $1 \leqslant k < i\}$; then the I_i are disjoint and $I \subseteq \{\mathbf{0}\} \cup I_1 \cup \cdots \cup I_d$. Let $E_i = E \cap \pi^{-1}(I_i)$, so that except possibly for a single point in E sitting over $\mathbf{n} = \{\mathbf{0}\} \in \mathbf{Z}^d$ we have $E = \bigcup\{E_i : 1 \leqslant i \leqslant d\}$ (disjoint union). Now set

$$f_i = \sum \{\mid n_i \mid^{-s} \chi_{\mathbf{n}} : \mathbf{n} \in E_i\} \qquad \text{(taking } s \text{ as above).}$$

Identifying $\hat{T}^{d+l} = \mathbf{Z}^{d+l}$, f_i is square summable on T^{d+l}; in fact, for fixed i, look at the possibilities $\mid n_i \mid = p$, $p = 1, 2, \ldots$. Then crd$\{\mathbf{m} \in \mathbf{Z}^d : \mathbf{m} \in I_i, \mid m_i \mid = p\} \leqslant$ crd$\{\mathbf{m} \in \mathbf{Z}^d : \mid m_k \mid \leqslant p$, all k, and $\mid m_i \mid = p\} = 2(2p + 1)^{d-1}$. Hence

$$\|f_i\|^2 = \sum \{\mid n_i \mid^{-2s} : \mathbf{n} \in E_i\}$$

$$\leqslant \sum_{p=1}^{\infty} 2(2p+1)^{d-1} \cdot p^{-2s}$$

$$\sim \sum_{p=1}^{\infty} p^{d-1-2s} < +\infty$$

by definition of s. Obviously $\partial^s f_i / \partial x_i^s$ is a scalar multiple of the distribution $F_i = \sum \{\chi_{\mathbf{n}} : \mathbf{n} \in E_i\}$ on the torus, so order$(F_i) \leqslant s$. Finally, D = (single character) $+ F_1 + \cdots + F_d$, so order$(D) \leqslant s$. Q.E.D.

[1] See "Note added in proof" concerning this lemma.

4.2. LEMMA. *Let N be a simply connected nilpotent Lie group with discrete uniform subgroup Γ. Let N act as unipotent linear operators on a real finite dimensional vector space V in which there is a Jordan–Holder* $\mathbf{R}$*-basis* $\{e_1, \ldots, e_n\}$ *for the action of N, such that Γ maps the integral points* $V_{\mathbf{Z}} = \mathbf{Z}$*-span*$\{e_1, \ldots, e_n\}$ *into themselves. Identify each $v \in V$ with its coordinates* $\mathbf{a} = (a_1, \ldots, a_n)$, $v = \sum_j a_j e_j$. *Fix an integral point $v \in V_{\mathbf{Z}}$ and let d = dimension of its orbit $\mathcal{O} = v \cdot N$. Then there exist indices $i_1 < \cdots < i_d$ such that*

(i) *the forgetful projection $p\colon (a_1, \ldots, a_n) \to (a_{i_1}, \ldots, a_{i_d}) \in \mathbf{R}^d$ is a homeomorphism of $\mathcal{O}$ onto $\mathbf{R}^d$;*

(ii) *the inverse $p^{-1} : \mathbf{R}^d \to \mathcal{O}$ is given by polynomials with rational coefficients, $p^{-1}(\mathbf{a}) = (T_1(\mathbf{a}), \ldots, T_n(\mathbf{a}))$ such that $T_{i_j}(\mathbf{a}) = a_{i_j}$ for $1 \leqslant j \leqslant d$.*

For any such choice,

(iii) *the image $p(v \cdot \Gamma)$ lies in the coset ring* $\mathrm{COS}(\mathbf{Z}^d)$, *where* $\mathrm{COS}(\mathbf{Z}^d)$ *is the finitely additive Boolean algebra of sets in $\mathbf{Z}^d$ generated by all cosets of subgroups in $\mathbf{Z}^d$.*

Note. The original Γ-orbit $v \cdot \Gamma$ lies in $\mathbf{Z}^n \cong V_{\mathbf{Z}}$ but need not lie in the coset ring of $\mathbf{Z}^n$. In essence, a proper choice of projection irons out the nonlinearities of the orbit. This will help us get sharp order estimates for sums of characters. We believe that this lemma will also be of use in other places.

Proof. Let N_0 = stabilizer of v in N. It is not hard to see that N_0 is rational, $\Gamma \cap N_0 \backslash N_0$ compact, due to integrality of v and our other hypotheses about the action of N. Pick a weak Malcev basis (cf. [5, Sect. 3] for definition and basic facts) for N, $Y_1, \ldots, Y_{n-d}, X_1, \ldots, X_d$ so that $Y_1, \ldots, Y_{n-d}$ spans N_0. Write $\gamma(t_1, \ldots, t_d) = \exp(t_1 X_1) \cdot \cdots \cdot \exp(t_d X_d)$; then we get a homeomorphism $\mathbf{t} = (t_1, \ldots, t_d) \to v \cdot \gamma(\mathbf{t})$ from $\mathbf{R}^d$ to $\mathcal{O}$ which carries $\mathbf{Z}^d$ one-to-one onto the Γ-orbit $v \cdot \Gamma$. By adapting (and simplifying) the well-known arguments of Pukanszky [10, pp. 50–54] we can find indices $i_1 < \cdots < i_d$ such that the projection $p\colon \mathcal{O} \to \mathbf{R}^d$ is a homeomorphism. Write $\mathbf{s} = (s_1, \ldots, s_d)$ for the coordinates $(a_{i_1}, \ldots, a_{i_d})$ in the range of p. Since Γ preserves $V_{\mathbf{Z}}$, $p(v \cdot \Gamma)$ lies within $\mathbf{Z}^d$. We now observe that

The induced homeomorphism $\psi : (s_1, \ldots, s_d) \to \mathcal{O} \to (t_1, \ldots, t_d)$ is given by rational polynomials $t_i = T_i(s_1, \ldots, s_d)$ for $1 \leqslant i \leqslant d$.

This is pretty obvious, so we omit details. Clearly,

$$\mathbf{s} = (s_1, \ldots, s_d) \text{ corresponds to a point } \mathbf{a} = (a_1, \ldots, a_n) \in v \cdot \Gamma$$
$$\text{if and only if } \mathbf{s} \in \mathbf{Z}^d \text{ and } t_i = T_i(s_1, \ldots, s_d) \in \mathbf{Z} \text{ for each } 1 \leqslant i \leqslant d. \quad (12)$$

Since the T_i have rational coefficients there is an integer K such that that $T_i = (1/K) Q_i$ where the Q_i have integer coefficients. Thus,

$$\mathbf{s} = (s_1, \ldots, s_d) \in p(v \cdot \Gamma) \Leftrightarrow \mathbf{s} \in \mathbf{Z}^d \quad \text{and} \quad Q_i(\mathbf{s}) \equiv 0 \pmod{K}$$
$$\text{for each } 1 \leqslant i \leqslant d.$$

Therefore to prove (iii) it suffices to show that $E_i = \{\mathbf{s} : \mathbf{s} \in \mathbf{Z}^d$ and $Q_i(\mathbf{s}) \equiv 0 \pmod{K}\}$ is in $\mathrm{COS}(\mathbf{Z}^d)$ for each $1 \leqslant i \leqslant d$. First note that, if $\mathbf{s} \in \mathbf{Z}^d$ and $Q_i(\mathbf{s}) \equiv 0 \pmod{K}$, then

$$Q_i(\mathbf{s} + K\mathbf{Z}^d) \equiv 0 \pmod{K},$$

so that E_i is a union of cosets of the lattice $K\mathbf{Z}^d$. This follows by binomial expansion: if $Q_i = \sum m_{i_1 \cdots i_d} s_1^{i_1} \cdots s_d^{i_d}$ then, because $s_j \in \mathbf{Z}$ for $1 \leqslant j \leqslant d$, we have

$$\begin{aligned}(s_j + Kn_j)^r &= (s_j)^r + r(Kn_j)(s_j)^{r-1} + \cdots + (Kn_j)^r \\ &= (s_j)^r + (\text{integer multiple of } K)\end{aligned}$$

for $r = 0, 1, 2, \ldots$. Consequently, if $\mathbf{s} \in E_i$ and $\mathbf{n} \in \mathbf{Z}^d$,

$$\begin{aligned}&Q_i(s_1 + Kn_1, \ldots, s_d + Kn_d) \\ &\quad = \sum m_{i_1 \cdots i_d} (s_1)^{i_1} \cdots (s_d)^{i_d} + (\text{integer multiple of } K) \\ &\quad = Q_i(\mathbf{s}) + (\text{integer multiple of } K) \\ &\quad = (\text{integer multiple of } K).\end{aligned}$$

On the other hand E_i can be a union of at most K^d cosets of $K\mathbf{Z}^d$, so that $E_i \in \mathrm{COS}(\mathbf{Z}^d)$. Q.E.D.

By abuse of notation we write D_σ for the distribution on T^k from which the actual primary distribution D_σ is lifted. We shall apply Lemma 4.1 to estimate order(D_σ) after we establish some notation. Let $H = [M, M]\backslash M$, so that its Lie algebra is $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]\backslash\mathfrak{m}$ and $\mathfrak{h}^* = [\mathfrak{m}, \mathfrak{m}]^\perp \subseteq \mathfrak{m}^*$. Since H is abelian we identify $H = (\mathfrak{h}, +) \cong \mathbf{R}^k$ and its dual $H^\wedge = \mathfrak{h}^*$ via $\chi_l = e^{2\pi i l}$, all $l \in [\mathfrak{m}, \mathfrak{m}]^\perp$. Let Λ be the image of $\Gamma \cap M$ in H under the canonical homomorphism; thus $\Lambda\backslash H \approx \Gamma_0[M, M]\backslash M$ is a torus T^k. Identify $\hat{T}^k$ with the integral points $\mathfrak{h}_{\mathbf{Z}}^* = \{l \in \mathfrak{h}^*: l(\log \Lambda) \subseteq \mathbf{Z}\}$ in $\mathfrak{h}^*$. Now the adjoint action of N on $\mathfrak{n}^*$

induces obvious actions on $\mathfrak{m}^*$ and $\mathfrak{h}^* = [\mathfrak{m}, \mathfrak{m}]^\perp \subseteq \mathfrak{m}^*$ since $\mathfrak{m} \lhd \mathfrak{n}$. Let $l_0 \in \mathfrak{m}^*$ correspond to the integral character $\chi_0 = e^{2\pi i l_0}$. Actually, $l_0 \in [\mathfrak{m}, \mathfrak{m}]^\perp = \mathfrak{h}^*$, and in fact $l_0 \in \mathfrak{h}_Z{}^*$. Write $\mathcal{O}' = l_0 \cdot N$ and $\mathcal{O}_{\#}' = \mathcal{O}' \cap \mathfrak{h}_Z{}^*$ for the integral points in this orbit. (These correspond to the points in $(\chi_0 \cdot N)_{\#}$). Now D_σ is identified as the sum

$$D_\sigma = \sum \{e^{2\pi i l} : l \in \mathcal{O}_{\#}'\} \tag{13}$$

on T^k.

If we choose any $f_0 \in \mathfrak{n}^*$ such that $f_0 \mid \mathfrak{m} = l_0$, let $\mathcal{O} = f_0 \cdot N$ and let $\mathscr{V} \subseteq \mathfrak{n}^*$ be the largest subspace which saturates $\mathcal{O}$ in the sense that $f \in \mathcal{O} \Rightarrow f + \mathscr{V} \subseteq \mathcal{O}$. Let $\mathscr{V}'$ be the largest subspace of $\mathfrak{h}^*$ which saturates $\mathcal{O}'$. The elements in $\mathcal{O}$ and $\mathscr{V}$ all annihilate $[\mathfrak{m}, \mathfrak{m}]$ since f_0 does. The natural linear surjection $\Phi: \{f \in \mathfrak{n}^*: f[\mathfrak{m}, \mathfrak{m}] = 0\} \to \mathfrak{h}^*$ (with Ker $\Phi = \mathfrak{m}^\perp$) carries $\mathcal{O}$ onto $\mathcal{O}'$ and $\mathscr{V}$ onto $\mathscr{V}'$, since $\mathscr{V} \supseteq \mathfrak{m}^\perp$ by a result of Pukanszky [10, pp. 158–159]. Our estimate of order(D_σ) says: if $d = \dim(\mathcal{O}) - \dim(\mathscr{V}) = \dim(\mathcal{O}') - \dim(\mathscr{V}')$, then

$$\text{order}(D_\sigma) \leqslant s \qquad (s = \text{smallest integer greater than } d/2). \tag{14}$$

Before proving (14) we give a more algebraic interpretation of d by noting that

$$\dim \mathcal{O} - \dim \mathscr{V} = \dim \mathfrak{i} - \dim \mathfrak{r} \tag{15}$$

where $\mathfrak{r}$ is the radical of f_0, and $\mathfrak{i}$ is the smallest ideal in $\mathfrak{n}$ which contains $\mathfrak{r}$. From (15) we see that the order estimate (14) refines the one (order $\leqslant k + 1$) in Section 3; since $\mathfrak{m} \lhd \mathfrak{n}$ we have $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{r}$ and $\mathfrak{i} \subseteq \mathfrak{m}$, so $\dim \mathfrak{i} - \dim \mathfrak{r} \leqslant \dim \mathfrak{m} - \dim[\mathfrak{m}, \mathfrak{m}] = k$.

4.3. LEMMA. *Let $f_0 \in \mathfrak{n}^*$, $\mathcal{O} = f_0 \cdot N$, $\mathfrak{r} =$ radical of f_0*, and $\mathfrak{i} =$ *the smallest ideal in $\mathfrak{n}$ such that $\mathfrak{i} \supseteq \mathfrak{r}$. Then*

(i) *$\mathcal{O}$ is saturated with respect to $\mathfrak{i}^\perp$;*

(ii) *$\mathfrak{i}^\perp$ is the largest subspace in $\mathfrak{n}^*$ which saturates $\mathcal{O}$;*

(iii) *if $f \in \mathcal{O}$, then $f + \mathfrak{i}^\perp = f \cdot I_f{}^0$ for a certain connected subgroup of N.*

Proof. If f is a typical point in $\mathcal{O}$, $f = f_0 \cdot x$, then $\mathfrak{r}_f =$ radical of $f = \text{Ad}(x^{-1})\mathfrak{r}$ lies in $\mathfrak{i}$. Let $\mathfrak{i}_f{}^0 = \{X \in \mathfrak{n}: f[X, \mathfrak{i}] = 0\}$. This is an *algebra* which corresponds to a Lie subgroup $I_f{}^0$ containing the stabilizer $R_f = \{y \in N : f \cdot y = f\}$. Now $f \cdot I_f{}^0 \subseteq f + \mathfrak{i}^\perp$ because $y = \exp(Y) \in I_f{}^0 \Rightarrow$

$$\langle Ad'(y)f, X\rangle = \langle f, X - [Y, X] + \tfrac{1}{2}[Y, [Y, X]] - \cdots\rangle = \langle f, X\rangle$$

for all $X \in \mathfrak{i}$. On the other hand, the I_f^0-orbit must fill $f + \mathfrak{i}^\perp$ because its dimension is $\dim(\mathfrak{i}_f^0/\mathfrak{r}_f) = \dim(\mathfrak{n}/\mathfrak{r}_f) - \dim(\mathfrak{i}/\mathfrak{r}_f) = \dim \mathfrak{n} - \dim \mathfrak{i} = \dim \mathfrak{i}^\perp$. Thus $f \cdot I_f^0 = f + \mathfrak{i}^\perp$ for any $f \in \mathcal{O}$, so that $\mathfrak{i}^\perp$ saturates $\mathcal{O}$. To see that $\mathfrak{i}^\perp$ is as large as possible, note that the tangent plane to $f \in \mathcal{O}$ is $ad'(\mathfrak{n})f = f + (\mathfrak{r}_f)^\perp$, where $\langle ad'(X)f, Y\rangle = -\langle f, [X, Y]\rangle$, and has dimension $= \dim \mathfrak{r}_f^\perp$. If $\mathcal{O}$ is $\mathscr{V}$-saturated we must have $f + \mathscr{V} \subseteq$ (tangent plane); thus,

$$\mathscr{V} \subseteq \bigcap \{\mathfrak{r}_f^\perp : f \in \mathcal{O}\} = \bigcap \{(\mathfrak{r}_{Ad'(x)f_0})^\perp : x \in N\}$$

$$= \bigcap \{(Ad(x)\, \mathfrak{r}_{f_0})^\perp : x \in N\}$$

$$= \Big(\sum \{Ad(x)(\mathfrak{r}_{f_0}) : x \in N\}\Big)^\perp = \mathfrak{i}_1^\perp .$$

Obviously $\mathfrak{i}_1 = \sum \{\mathfrak{r}_{f_0} \cdot x : x \in N\}$ is the smallest ideal in $\mathfrak{n}$ generated by $\mathfrak{r} = \mathfrak{r}_{f_0}$, so $\mathfrak{i}_1 = \mathfrak{i}$. Q.E.D.

Since l_0 is rational on $\mathfrak{m}$, we may take $f_0 \in \mathfrak{n}^*$ to be a rational extension, $f_0(\log \Gamma) \subseteq \mathbf{Q}$. Then $R = \{y \in N : f_0 \cdot y = f_0\}$ is rational and so are $I = \exp(\mathfrak{i})$ and $[M, M]$. Take a Malcev basis $X_1, \ldots, X_n$ which spans successively $[M, M] \subseteq I \subseteq M \subseteq N$ (possible since $[M, M]$, I, M are normal in N). Let $\lambda : M \to H$ be the canonical map. Then $Y_1, \ldots, Y_k$ with $Y_i = d\lambda(X_{m-k+i})$ form a Jordan–Holder basis for $\mathfrak{h}$ with respect to the action of N, and a $\mathbf{Z}$-basis for the additive lattice of integral points $\mathfrak{h}_\mathbf{Z} = \log \Lambda = \log \lambda(\Gamma \cap M)$. The dual basis $Y_k^*, \ldots, Y_1^*$ is a Jordan–Holder basis for the contragredient action of N on $\mathfrak{h}^*$, and is a $\mathbf{Z}$-basis for $\mathfrak{h}_\mathbf{Z}^*$. Since the X_i ran through $\mathfrak{i}$, and since $\mathscr{V} = \mathfrak{i}^\perp$, it follows that $Y_k^*, \ldots, Y_{k-r+1}^*$ span $\mathscr{V}'$, forming a Jordan–Holder basis for this N-invariant subspace and a $\mathbf{Z}$-basis for the integral points $\mathscr{V}_\mathbf{Z}' = \mathscr{V}' \cap \mathfrak{h}_\mathbf{Z}^*$ ($r = \dim \mathscr{V}' = \dim \mathscr{V} - \dim \mathfrak{m}^\perp = \dim \mathfrak{m} - \dim \mathfrak{i}$; $\mathscr{V}_\mathbf{Z}' \cong \mathbf{Z}^r$). Now let $\mathscr{W}' = \mathbf{R}\text{-span}\{Y_{k-r}^*, \ldots, Y_1^*\}$. Obviously $\mathfrak{h}_\mathbf{Z}^*$ decomposes into an internal direct sum

$$\mathfrak{h}_\mathbf{Z}^* = \mathbf{Z}\text{-span}\{Y_k^*, \ldots, Y_{k-r+1}^*\} \oplus \mathbf{Z}\text{-span}\{Y_{k-r}^*, \ldots, Y_1^*\}. \tag{16}$$

Clearly $Y_{k-r}^*, \ldots, Y_1^*$ form a $\mathbf{Z}$-basis for the set of integral points $\mathscr{W}_\mathbf{Z}' = \mathscr{W}' \cap \mathfrak{h}_\mathbf{Z}^*$.

Returning to our orbit $\mathcal{O}' = l_0 \cdot N$, $l_0 \in \mathfrak{h}_\mathbf{Z}^*$, let $E = \mathscr{W}_\mathbf{Z}' \cap \mathcal{O}'$. A fairly straightforward calculation shows that

$$\mathcal{O}_{\#}' = \mathscr{V}_\mathbf{Z}' \oplus E. \tag{17}$$

Letting $T^k = \Lambda \backslash H$ and identifying $\hat{T}^k = \mathbf{Z}^k = \mathfrak{h}_{\mathbf{Z}}{}^*$ as above, we want to determine the order of D_σ. Since D_σ is the sum of all characters in $\mathcal{O}_{\#}' = \mathcal{O}' \cap \mathfrak{h}_{\mathbf{Z}}{}^*$ as in (13), its Fourier transform is the characteristic function of $\mathcal{O}_{\#}'$. We have decomposed $\mathbf{Z}^k = \mathbf{Z}^r \oplus \mathbf{Z}^{k-r} = \mathscr{V}_{\mathbf{Z}}' \oplus \mathscr{W}_{\mathbf{Z}}'$ and found that $\mathcal{O}_{\#}'$ is a union of cosets of $\mathbf{Z}^r + (0) = \mathscr{V}_{\mathbf{Z}}'$. Take A = annihilator in T^k of $\mathscr{V}_{\mathbf{Z}}' = \mathbf{Z}^r + (0)$, a torus of dimension $k - r$. Let $\rho : C^0(T^k) \to C^0(A)$ be the restriction map. It is not hard to show that any distribution $D \in \mathscr{D}'(T^k)$, such as $D = D_\sigma$, whose Fourier transform is constant on cosets of $\mathscr{V}_{\mathbf{Z}}' = \mathbf{Z}^r + (0)$ is in the range of the obvious injection $\rho^*\colon \mathscr{D}'(A) \to \mathscr{D}'(T^k)$. That is, $\mathrm{supp}(D) \subseteq A$ and D does not involve any differentiations transverse to A; to put it another way, $\langle D, \phi \rangle$ is determined solely by the restriction $\phi \mid A$ for all $\phi \in C^\infty(T^k)$. Clearly, $\mathrm{order}(\rho^* S) = \mathrm{order}(S)$ for all $S \in \mathscr{D}'(A)$.

Let $\bar{D}_\sigma \in \mathscr{D}'(A)$ be the distribution such that $\rho^*(\bar{D}_\sigma) = D_\sigma$. We study $\bar{D}_\sigma$ by passing to the quotient space $(\mathscr{V}' \backslash \mathfrak{h}^*)$. With respect to the induced actions of N and Γ in $\mathscr{V}' \backslash \mathfrak{h}^*$, the vectors $Z_{k-r}^* = Y_{k-r}^* + \mathscr{V}', \ldots, Z_1{}^* = Y_1{}^* + \mathscr{V}'$ form a Jordan–Holder basis. Furthermore, if $\theta\colon \mathfrak{h}^* \to \mathscr{V}' \backslash \mathfrak{h}^*$ is the quotient map and we define integral points in this quotient space to be $(\mathscr{V}' \backslash \mathfrak{h}^*)_{\mathbf{Z}} = \theta(\mathfrak{h}_{\mathbf{Z}}{}^*) = \theta(\mathscr{W}_{\mathbf{Z}}')$, these integral points are Γ-invariant and the $\{Z_{k-r}^*, \ldots, Z_1{}^*\}$ form a $\mathbf{Z}$-basis for them, in view of (16). Now θ is equivariant with respect to the actions of N, so $l_0'' = \theta(l_0)$ is an integral point in the N-orbit $\mathcal{O}'' = l_0'' \cdot N = \theta(\mathcal{O}')$, which has dimension $d = \dim \mathcal{O}' - \dim \mathscr{V}'$. The integral points $\mathcal{O}_{\#}'' = \mathcal{O}'' \cap (\mathscr{V}' \backslash \mathfrak{h}^*)_{\mathbf{Z}}$ in this orbit are precisely the points $\theta(\mathcal{O}_{\#}') = \theta(E) = \theta(\mathcal{O}' \cap \mathscr{W}_{\mathbf{Z}}')$. Now characters on A are identified in an obvious way with the integral points $(\mathscr{V}' \backslash \mathfrak{h}^*)_{\mathbf{Z}}$: identify $l + \mathscr{V}'$ with $e^{2\pi i l} \mid A$. Since D_σ is the sum of all characters on T^k in $\mathcal{O}_{\#}'$, $\bar{D}_\sigma$ is the sum of all characters on $A = T^{k-r}$ corresponding to the integral points $\mathcal{O}_{\#}''$.

Apply parts (i) and (ii) of Lemma 4.2, identifying $V = (\mathscr{V}' \backslash \mathfrak{h}^*)$, $V_{\mathbf{Z}} = (\mathscr{V}' \backslash \mathfrak{h}^*)_{\mathbf{Z}}$, and $\{e_1, \ldots, e_n\} = \{Z_{k-r}^*, \ldots, Z_1{}^*\}$. After permuting the labels on the basis $Z_{k-r}^*, \ldots, Z_1{}^*$ (which is all right since we will make no further use of the Jordan–Holder property), we see that $\mathcal{O}_{\#}''$ is a subset of the integral points on a polynomial graph determined by d free variables,

$$\mathcal{O}_{\#}'' \subseteq \mathbf{Z}^{k-r} \cap \{(a_1, \ldots, a_d, P_1(\mathbf{a}), \ldots, P_{k-r-d}(\mathbf{a})) : \mathbf{a} \in \mathbf{Z}^d\}.$$

Now apply Lemma 4.1 to conclude that $\mathrm{order}(\bar{D}_\sigma) \leqslant s$. This completes the proof of Theorem 1.1 for the distribution associated with the primary projection P_σ. Q.E.D.

5. Irreducible Projections as Convolutions with Measures

In this section we prove that the constructible projections $P_{(\chi, M)} \leqslant P_\sigma$ have associated distributions such that order$(D_{(\chi, M)}) \leqslant$ order(D_σ). In the next section we will prove a Zero–One law which implies, among other things, that *all* irreducible projections $P \leqslant P_\sigma$ satisfy this estimate.

As usual, we assume that σ is induced from an integral maximal character (χ, M) such that M is normal. Form the N-orbit $\chi \cdot N$, pick Γ-orbit representatives $\chi_1, ..., \chi_q$ in $(\chi \cdot N)_\#$, and take $n_i \in N$ such that $\chi_i = \chi \cdot n_i$. (Take $n_1 = e$; the n_i can be chosen to be *rational* elements in N.) Let $P_i = B_i B_i^* = P_{(\chi_i, M)}$.

5.1. Proposition. *Let (χ, M) be an integral maximal character such that M is normal and let $\sigma = \text{Ind}(M \uparrow N, \chi)$. Then the constructible irreducible projection $P = P_{(\chi, M)}$ maps $C^0(\Gamma\backslash N) \cap \mathscr{H}_\sigma$ into itself. Thus,*

$$f \in C^0(\Gamma\backslash N) \quad \textit{and} \quad P_\sigma f = f \Rightarrow Pf \in C^0(\Gamma\backslash N).$$

In fact, there exists a finite measure μ on $\Gamma\backslash N$ (not necessarily right Γ-invariant) such that $P_{(\chi, M)} f = \mu * f$ for all $f \in C^0 \cap \mathscr{H}_\sigma$.

Note. If $f \in C^0(\Gamma\backslash N)$ we define $\mu * f(n) = \langle \mu, n \cdot f \rangle$ as a function on N. It does not follow automatically that $\mu * f$ is left Γ-invariant; this might not be true for arbitrary f. It does turn out to be true for $f \in C^0 \cap \mathscr{H}_\sigma$.

Since $f \in C^r \Rightarrow P_i f = P_i(P_\sigma f) = \mu_i * (P_\sigma f) \in C^0$, it follows that

5.2. Corollary. *The primary projection P_σ maps $C^r(\Gamma\backslash N)$ into $C^0(\Gamma\backslash N)$ for some $r \geqslant 0 \Leftrightarrow$ the same is true of every constructible irreducible projection $P_{(\chi, M)}$.*

By the Sobolev arguments of Auslander–Brezin [1] and the closed graph theorem, one can show that order$(D_P) \leqslant r \Leftrightarrow P$ maps C^r into C^0.

Proof of 5.1. Let $l_0 \in \mathfrak{m}^*$ be chosen such that $\chi = e^{2\pi i l_0}$.

We adopt the notation of Section 4, defining $H = [M, M]\backslash M$, Λ, $T^k = \Lambda\backslash H$, and then identifying $\hat{T}^k = \mathfrak{h}_{\mathbf{Z}}^*$. Thus D_σ is the liftback to $\Gamma \cap M\backslash M \approx \Gamma\backslash\Gamma M$ of the distribution $\bar{D}_\sigma$ on T^k obtained by summing the characters in $\mathcal{O}_\#' = \mathcal{O}' \cap \mathfrak{h}_{\mathbf{Z}}^*$, where $\mathcal{O}' = l_0 \cdot N$. Similarly, the irreducible projection $P = P_{(\chi, M)}$ associated with the Γ-orbit $\chi \cdot \Gamma$ is determined by the distribution $\bar{D}$ on T^k,

$$\bar{D} = \sum \{e^{2\pi i l} : l \in l_0 \cdot \Gamma\}. \tag{18}$$

In $\mathfrak{h}^*$ introduce the basis $e_1 = Y_k^*, \ldots, e_k = Y_1^*$ used in the order estimate of Section 4. Since $l_0 \in \mathfrak{h}_{\mathbf{Z}}^*$, and since the $\{Y_i^*\}$ are a Jordan–Holder basis for the action of N on $\mathfrak{h}^*$ and a $\mathbf{Z}$-basis for the Γ-invariant set of integral points $\mathfrak{h}_{\mathbf{Z}}^*$, we may apply the full strength of Lemma 4.2. Taking coordinates $(a_1, \ldots, a_k) \to \sum_i a_i e_i$ in $\mathfrak{h}^*$, and choosing $i_1 < \cdots < i_d$ ($d = \dim l_0 \cdot N$) as in parts (i) and (ii) of the lemma, we are assured that the coordinate projection p: $(a_1, \ldots, a_k) \to (a_{i_1}, \ldots, a_{i_d}) \in \mathbf{R}^d$ maps the Γ-orbit to $p(l_0 \cdot \Gamma) \in \mathrm{COS}(\mathbf{Z}^d)$. Then $F = p^{-1}(p(l_0 \cdot \Gamma)) \cap \mathbf{Z}^k$ is in $\mathrm{COS}(\mathbf{Z}^k)$, if we identify $\mathfrak{h}_{\mathbf{Z}}^* = \mathbf{Z}^k$ in these coodinates. Thus there is a finite measure $\bar{\mu}$ on T^k whose Fourier transform is $\bar{\mu}\hat{} =$ characteristic function of F. Obviously the primary distribution given by (13), and the irreducible distribution given by (18) on T^k have Fourier transforms

$$\bar{D}_\sigma\hat{} = \text{characteristic function of } \mathcal{O}_{\#}' = \mathcal{O}' \cap \mathfrak{h}_{\mathbf{Z}}^*,$$

$$\bar{D}\hat{} = \text{characteristic function of } l_0 \cdot \Gamma = \mathcal{O}_{\#}' \cap F.$$

Thus

$\bar{D} = \bar{\mu} \star \bar{D}_\sigma$ where $(\star)$ stands for convolution in the abelian group T^k,

$$\langle \bar{\mu} \star \bar{D}_\sigma, \phi \rangle = \int_{T^k} \langle \bar{D}_\sigma, s \cdot \phi \rangle \, d\bar{\mu}(s). \tag{19}$$

Let μ be the liftback $p_1^{**}(\bar{\mu})$. Formula (19) suggests that we have $D = \mu * D_\sigma$ under a suitably defined convolution on $\Gamma \backslash N$. If so, the proof is finished: $f \to \mu * f$ carries C^0 into C^0 so that $f \in C^0 \cap \mathscr{H}_\sigma \Rightarrow Pf = D * f = \mu * (D_\sigma * f) = \mu * (P_\sigma f) = \mu * f \in C^0 \cap \mathscr{H}_\sigma$. The main obstacle is that μ (unlike D and D_σ) need not be right Γ-invariant, which makes for trouble in defining convolution on $\Gamma \backslash N$.

This idea can be made into a valid argument as follows. If D is any distribution on Γ / N we define a map $f \to D * f$ from $C^\infty(\Gamma \backslash N)$ into $C^\infty(N)$ via $D * f(n) = \langle D, n \cdot f \rangle$. There is no guarantee that $D * f$ is left Γ-invariant. Let $D = \mu$ be a finite Borel measure on $\Gamma \backslash N$; then $\mu * f$ is bounded on N. Let F be any bounded (compact closure) measurable fundamental domain for $\Gamma \backslash N$. We assert that there is a constant C_F such that

$$\int_F | \mu * f(n) |^2 \, dn \leqslant C_F \| \mu \|^2 \left(\int_{\Gamma \backslash N} | f |^2 \, d\dot{n} \right) \tag{20}$$

for all $f \in C^0(\Gamma \backslash N)$. The proof follows a suggestion of L. Richardson. The canonical map π: $N \to \Gamma \backslash N$ is a Borel isomorphism between F and $\Gamma \backslash N$. Define $\tilde{\mu}$ on N via $\tilde{\mu}(A) = \tilde{\mu}(A \cap F) = \mu(\pi(A \cap F))$. Since

$\overline{FF}$ is compact and F bounded, only a finite number of distinct translates of F can meet $\overline{FF}$, so we get a cover $U = \gamma_1 F \cup \cdots \cup \gamma_q F \supseteq \overline{FF}$. Letting χ_A = characteristic function of a set A, we get

$$\begin{aligned}
\int_F | \mu * f(n)|^2 \, dn &= \int_F \left| \int_N f(\Gamma xn) \, d\tilde{\mu}(x) \right|^2 dn \\
&\leqslant \int_F \left[\int_N | f(\Gamma xn)| \, d \, | \tilde{\mu} | \, (x) \right]^2 dn \\
&\leqslant \| \tilde{\mu} \| \int_F \int_N | f(\Gamma xn)|^2 \, d \, | \tilde{\mu} | \, (x) \, dn \\
&= \| \mu \| \int_N \int_N \chi_F(n) \, | f(\Gamma xn)|^2 \, d \, | \tilde{\mu} | \, (x) \, dn \\
&= \| \mu \| \int_N | f(\Gamma n)|^2 \left[\int \chi_F(x^{-1}n) \cdot \chi_F(x) \, d \, | \tilde{\mu} | (x) \right] dn \\
&\leqslant \| \mu \| \int_N | f(\Gamma n)|^2 \int \chi_U(n) \, d \, | \tilde{\mu} | \, (x) \, dn \\
&= \| \mu \|^2 \sum_{i=1}^{q} \int_{\gamma_i F} | f(\Gamma n)|^2 \, dn \\
&= q \, \| \mu \|^2 \int_F | f(\Gamma n)|^2 \, dn.
\end{aligned}$$

When $D \in \mathscr{D}'(\Gamma \backslash N)$ is right Γ-invariant, $\langle D, \gamma \cdot f \rangle = \langle D, f \rangle$, then $D * f$ is left Γ-invariant on N, hence is well defined as a function back in $C^\infty(\Gamma \backslash N)$. (If μ is a right invariant *measure*, the estimate (20) shows that $T_\mu f = \mu * f$ is a bounded operator on $L^2(\Gamma \backslash N)$.) Now the distributions D, D_σ associated with P, P_σ are automatically right Γ-invariant, so we shall regard these convolutions as maps from $C^\infty(\Gamma \backslash N)$ into itself, giving $Pf = D * f$ and $P_\sigma f = D_\sigma * f$. If $\mu = p_1^{**}(\tilde{\mu})$ as in (19), we must regard $f \to \mu * f$ as a map from $C^0(\Gamma \backslash N)$ into $C^0(N)$ since there is no reason to expect μ to be right invariant. We want to show that

$$(Pf) \circ \pi = \mu * P_\sigma f = \mu * f \qquad \text{a.e. on } N, \text{ all } f \in C^0 \cap \mathscr{H}_\sigma. \tag{21}$$

The right side is obviously continuous on N, hence P maps $C^0 \cap \mathscr{H}_\sigma$ into itself, which proves the theorem. But (21) follows if we can prove

$$(D * f) \circ \pi = \mu * (D_\sigma * f) \qquad \text{for all } \ f \in C^\infty \cap \mathscr{H}_\sigma . \tag{22}$$

To see this, assume (22) and let F be any bounded, measurable fundamental domain for $\Gamma \backslash N$. Identify $L^2(\Gamma \backslash N, dn) = L^2(F, dn)$. If

$f \in C^0 \cap \mathscr{H}_\sigma$ there exist $f_n \in C^\infty \cap \mathscr{H}_\sigma$ such that $\| f_n - f \|_\infty \to 0$. Since $Pf_n \circ \pi = \mu * (D_\sigma * f_n) = \mu * f_n$, we may apply the L^2 estimate (20) to get (π being the map of $N \to \Gamma\backslash N$),

$$\begin{aligned} &\|(Pf) \circ \pi - \mu * f \|_{2,F} \\ &\quad \leqslant \|(Pf) \circ \pi - (Pf_n) \circ \pi \|_{2,F} + \| \mu * f_n - \mu * f \|_{2,F} \\ &\quad \leqslant \| Pf - Pf_n \|_{2,\Gamma\backslash N} + C_F \| \mu \| \, \| f_n - f \|_{2,F} \to 0; \end{aligned}$$

i.e., $Pf \circ \pi = \mu * f$ a.e. on F. This is true for any choice of F, proving (21).

We prove (22) from the convolution formula (19) on T^k. Notice that $f \in C^\infty \cap \mathscr{H}_\sigma \Rightarrow f \circ \pi$ is a constant on cosets of $[M, M]$, hence also on $\Gamma \cdot [M, M]$ cosets. In fact, if $f \in C^\infty(\Gamma\backslash N)$ we have already noted that p^*f is constant of cosets of $M_1 = [M, M]$. Thus if $m_1 \in M_1$ we get

$$\begin{aligned} P_\sigma f(\pi(m_1 n)) &= \langle D_\sigma , m_1 n \cdot f \rangle \\ &= \langle \bar{D}_\sigma , p^*(m_1 \cdot n \cdot f) | M \rangle \\ &= \langle D_\sigma , n \cdot f \rangle = P_\sigma f(\pi(n)). \end{aligned}$$

Therefore the averaging process $p^*: C^\infty(\Gamma\backslash N) \to C^0(N)$ is trivial on functions in $C^\infty \cap \mathscr{H}_\sigma$. But the map $p_1^*: C^\infty(\Gamma\backslash \Gamma M) \to C^\infty(T^k)$ used to lift distributions in Section 3 is related to the global map p^* by the formula

$$p_1^*(f \mid \pi(M)) = p^*f \mid M \qquad \text{all} \quad f \in C^\infty(\Gamma\backslash N)$$

where $p^*f \mid M$ is identified with the obvious function on $T^k = M_2\backslash M$. From this it is easy to verify that, for $\dot{x} = M_2 x \in T^k$ and $f \in C^\infty \cap \mathscr{H}_\sigma$,

$$p_1^*(D * f \mid \pi(M))(\dot{x}) = \langle \bar{D}, \dot{x} \cdot (p^*f \mid M) \rangle = \langle \bar{D}, p^*(x \cdot f) | M \rangle$$

and likewise for D_σ. Thus if $n \in N$, $f \in C^\infty \cap \mathscr{H}_\sigma$ we get

$$\begin{aligned} \mu * (D_\sigma * f)(n) &= \langle \mu, n \cdot (D_\sigma * f) \rangle = \langle \mu, D_\sigma * (n \cdot f) \rangle \\ &= \langle \bar{\mu}, p^*(D_\sigma * n \cdot f) | M \rangle \\ &= \int_{T^k} \langle \bar{D}_\sigma , p^*(x \cdot n \cdot f) \mid M \rangle \, d\bar{\mu}(\dot{x}) \\ &= \langle \bar{\mu} \star \bar{D}_\sigma , p^*(n \cdot f) \mid M \rangle \\ &= \langle D, n \cdot f \rangle = D * f(\Gamma n) = D * f(\pi(n)), \end{aligned}$$

which proves (22). Q.E.D.

Note. Another way to view this calculation is to say that since all functions $f \in C^\infty \cap \mathscr{H}_\sigma$ are constant on $[M, M]$-cosets, we may as well assume that M is abelian. Then p^* really is trivial. We actually get a stronger statement than (22); since $D * f = D * (D_\sigma * f)$, we get $(D * f) \circ \pi = \mu * (D_\sigma * f)$ for *all* $f \in C^\infty(\Gamma\backslash N)$.

6. The Zero–One Law

The following form of the Zero–One law was posed as a conjecture by Jon Brezin.

6.1. Theorem. (Zero–One law). *Let $\sigma \in (N : \Gamma)^\wedge$ be induced from an integral maximal character (χ, M) such that M is normal. Then all bounded linear operators $T: (\mathscr{H}_\sigma, \|\cdot\|_2) \to (\mathscr{H}_\sigma, \|\cdot\|_2)$ which commute with the action of N are continuity preserving; they map $C^0(\Gamma\backslash N) \cap \mathscr{H}_\sigma$ into itself.*

In particular, if the σ-primary projection P_σ maps C^r into C^0 for some $r \geqslant 0$, so do all bounded linear operators on L^2 which commute with the action of N and have range in $\mathscr{H}_\sigma$. This follows because P_σ is in the center of the Von Neumann algebra generated by $\{U_n{}^1: n \in N\}$, so $AP_\sigma = P_\sigma A$. Thus all loss of smoothness is already due to the σ-primary projection.

Proof of Theorem 6.1. Let $\mathscr{H}_i$ be the subspaces of $\mathscr{H}_\sigma$ associated with the Γ-orbits $\chi_1 \cdot \Gamma, \ldots, \chi_q \cdot \Gamma$ in $(\chi \cdot N)_\#$. Every operator T is a linear combination $T = \sum_{i,j} c_{ij} T_{ij}$ ($c_{ij} \in \mathbf{C}$), where T_{ji} is an isometry of $\mathscr{H}_j$ onto $\mathscr{H}_i$ and $T_{ji}(\mathscr{H}_r) = 0$ if $r \neq i$. By irreducibility of the $\mathscr{H}_r$, T_{ji} is unique up to a scalar of modulus one. Thus it suffices to prove the theorem for the basic operators $T = T_{ji}$. (Proposition 5.1 assures us it is true for the diagonal terms $T_{ii} = P_i$.)

The action of T_{ji} can be written out explicitly for smooth functions $\phi \in C^\infty(\Gamma\backslash N)$; by a completion argument at the end of the proof, our conclusions will be verified for $\phi \in C^0(\Gamma\backslash N)$, too. If $x \in N$,

$$T_{ji}\phi(\Gamma x) = B_j I_{ji} B_i{}^* \phi(\Gamma n)$$
$$= \sum_{\gamma \in \Gamma_0\backslash\Gamma} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i(m)}\, \phi(\Gamma m y \gamma x)\, d\dot{m} \right). \tag{23}$$

Here $\Gamma_0 = \Gamma \cap M$, $y = n_i^{-1} n_j$, and $I_{ji} = \mathscr{H}(U^{\chi_i}) \to \mathscr{H}(U^{\chi_j})$ is defined by

$$I_{ji}F(n) = F(yn) = F(n_i^{-1} n_j n).$$

Now $B_i: \mathscr{H}(U^{\chi_i}) \to \mathscr{H}_i \subseteq \mathscr{H}_\sigma$; its adjoint takes smooth functions $\phi \in C^\infty(\Gamma\backslash N)$ to continuous functions $B_i{}^*\phi$ vanishing rapidly transverse to M cosets. This rapid decrease (3) is also true of $\psi = I_{ji}B_i{}^*\phi \in \mathscr{H}(U^{\chi_j})$, so the pointwise formula (2) is valid for $B_j\psi$, which is what we have written out in (23). The sum $\sum_\gamma (\cdots)$ is absolutely convergent for $\phi \in C^\infty(\Gamma\backslash N)$.

Let $\Delta = \{y \in \Gamma: y\gamma y^{-1} \in \Gamma\}$. Then $[\Gamma : \Delta] < +\infty$ (look at the rational map $\gamma \to y\gamma y^{-1}$ in Malcev coordinates), and likewise $[\Gamma : y\Delta y^{-1}] < +\infty$. Pick coset representatives S for $\Gamma_0\Delta\backslash\Gamma$ (a finite set) and U for $\Gamma_0\backslash\Gamma_0\Delta$. Under the map $\delta \to y\delta y^{-1} = \delta'$, U is mapped to a set U' of coset representatives for $\Gamma_0\backslash\Gamma'$ where $\Gamma' = y(\Gamma_0\Delta)\, y^{-1} = \Gamma_0(y\Delta y^{-1})$. Now

$$\begin{aligned} T_{ji}\phi(\Gamma x) &= \sum_{s\in S}\sum_{\delta\in U}\int_{\Gamma_0\backslash M} \overline{\chi_i(m)}\, \phi(\Gamma m y\delta s x)\, d\dot{m} \\ &= \sum_{s\in S}\sum_{\delta\in U}\int_{\Gamma_0\backslash M} \overline{\chi_i(m)}\, \phi(\Gamma m y\delta y^{-1}\cdot y s x)\, d\dot{m} \\ &= \sum_{s\in S}\sum_{\delta'\in U'}\int_{\Gamma_0\backslash M} \overline{\chi_i\cdot\delta'(m)}\, \phi(\Gamma m y s x)\, d\dot{m}. \end{aligned} \tag{24}$$

Since $\Gamma' \cap M = \Gamma_0 = \Gamma \cap M$, a character on M is integral with respect to $\Gamma \Leftrightarrow$ it is integral with respect to Γ'. Since (χ_i, M) is an integral maximal character with respect to Γ', $\sigma = \mathrm{Ind}(M \uparrow N, \chi_i)$ belongs to the spectrum $(N : \Gamma')^\wedge$.

Let $A: L^2(\Gamma'\backslash N) \to L^2(\Gamma'\backslash N)$ be the projection $A = B_i'(B_i')^*$ to the irreducible subspace $\mathscr{H}_i' = \mathscr{H}'_{(\chi_i, M)} \subseteq \mathscr{H}_\sigma' = \sigma$-primary subspace in $L^2(\Gamma'\backslash N)$, corresponding to the intertwining isometry $B_i': \mathscr{H}(U^{\chi_i}) \to \mathscr{H}_i'$ constructed as in Section 2. If ψ is smooth, say $\psi \in C^\infty(\Gamma'\backslash N)$, then $A\psi$ is given by a sum over all characters in the Γ'-orbit $\chi_i \cdot \Gamma' \subseteq (\chi \cdot N)_\#$,

$$A\psi(\Gamma' x) = \sum_{\delta'\in U'=\Gamma_0\backslash\Gamma'} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i\cdot\delta'(m)}\, \psi(\Gamma' m x)\, d\dot{m}\right), \tag{25}$$

the sum being absolutely convergent as in (2). Though this formula might not be valid for $\psi \in C^0(\Gamma'\backslash N)$, Proposition 5.1 insures that A maps $C^0(\Gamma'\backslash N) \cap \mathscr{H}_\sigma'$ into itself, and that there is a finite measure μ' on $\Gamma'\backslash N$ such that

$$j(A\psi) = \mu' * \psi \qquad (A\psi(\Gamma' x) = \langle \mu, x\cdot\psi\rangle)$$

for all $\psi \in C^0(\Gamma'\backslash N) \cap \mathscr{H}_\sigma'$, where j lifts functions on $\Gamma'\backslash N$ to functions on N.

Define operators $(s \in S)$ $A^{ys} : C^0(\Gamma\backslash N) \cap \mathscr{H}_\sigma \to C^0(N)$ via

$$A^{ys} : C^0 \cap \mathscr{H}_\sigma \xrightarrow{i} C^0(\Gamma'\backslash N) \xrightarrow{A} C^0(\Gamma'\backslash N) \xrightarrow{j} C^0(N) \xrightarrow{I^{ys}} C^0(N) \quad (26)$$

where i is the obvious injection of $L^2(\Gamma\backslash N)$ into $L^2(\Gamma'\backslash N)$ and $I^{ys}f(n) = f(ysn)$. The composition (26) makes sense because i maps $C^0 \cap \mathscr{H}_\sigma$ into $C^0 \cap \mathscr{H}_\sigma'$, so that $j(A(i(\phi))) = \mu' * i(\phi) \in C^0(N)$. In fact, let $X = i(\mathscr{H}_\sigma)$. Then X is a closed subspace of $L^2(\Gamma'\backslash N)$; but i obviously intertwines the right actions of N, so N acts irreducibly on X. Hence X lies within a single primary summand of $L^2(\Gamma'\backslash N)$. Since i intertwines and is injective, the action of N on X is a copy of σ, so $X \subseteq \mathscr{H}_\sigma'$ and i maps $\mathscr{H}_\sigma$ into $\mathscr{H}_\sigma'$. It also maps $C^0 \cap \mathscr{H}_\sigma$ into $C^0 \cap \mathscr{H}_\sigma'$ since it certainly carries continuous functions to continuous functions.

Next we make A^{ys} into a map V^{ys}: $C^0(\Gamma\backslash N) \cap \mathscr{H}_\sigma \to L^2(\Gamma\backslash N)$ by taking a bounded, measurable fundamental domain K for $\Gamma\backslash N$ and defining

$$V^{ys}\phi(\Gamma n) = A^{ys}\phi(n) \qquad \text{for} \quad n \in K.$$

Evidently $V^{ys}\phi$ is a bounded Borel function on $\Gamma\backslash N$, even if $A^{ys}\phi$ is not left Γ-invariant on N. We assert that V^{ys} is a bounded linear map with respect to the L^2 norms. In fact, if dn = Haar measure on N normalized so that K has mass one, and if $\phi \in C^0 \cap \mathscr{H}_\sigma$, we get

$$\begin{aligned} \| V^{ys}\phi \|_2^2 &= \int_K | A^{ys}\phi(n)|^2 \, dn \\ &= \int_K | \mu' * \phi'(ysn)|^2 \, dn \\ &= \int_{yK} | \mu' * \phi'(ysy^{-1}n)|^2 \, dn. \end{aligned}$$

Since $\bar{K}$ is compact, only finitely many Γ-translates $K_1, \ldots, K_p$ meet yK (a bounded set), and these cover yK. Furthermore, the elements $ySy^{-1} = S'$ are coset representatives for $\Gamma'\backslash\Gamma$, so that $K_j' = S'K_j$ $(1 \leqslant j \leqslant p)$ are each fundamental domains for $\Gamma'\backslash N$. Therefore, the last expression above is dominated by

$$\begin{aligned} &\sum_{j=1}^{p} \int_{K_j} | \mu' * \phi'(s' \cdot n)|^2 \, dn \\ &\leqslant \sum_{s' \in S'} \sum_{j=1}^{p} \int_{s'K_j} | \mu' * \phi'(n)|^2 \, dn \\ &= \sum_{j=1}^{p} \int_{K_j'} | \mu' * \phi'(n)|^2 \, (| S |/| S |) \, dn, \end{aligned}$$

where $|S| = |S'| =$ cardinality of S'. Obviously $d'n = |S|^{-1}dn$ is Haar measure on N normalized so that fundamental domains of $\Gamma'\backslash N$ have mass 1. Thus the last expression becomes

$$|S| \sum_{j=1}^{p} \int_{K_j'} |\mu' * \phi'(n)|^2 \, d'n$$

$$\leqslant p\,|S| \int_{\Gamma'\backslash N} |\mu' * \phi'|^2 \, d'\dot{n}$$

$$\leqslant p\,|S|\, \|\mu'\|^2\, C_K \int_{\Gamma'\backslash N} |\phi'|^2 \, d'\dot{n}$$

$$= p\,|S|\, \|\mu'\|^2\, C_K \int_{\Gamma\backslash N} |\phi|^2 \, d\dot{n} = pC_K\,|S|\, \|\mu'\|^2 \cdot \|\phi\|_2^2$$

as required.

Now define

$$A_{ji}\phi(n) = \sum_{s\in S} A^{ys}\phi(n) \qquad \text{for} \quad \phi \in C^0(\Gamma\backslash N) \cap \mathscr{H}_\sigma .$$

Obviously $A_{ji}\phi \in C^0(N)$, but it is actually left Γ-invariant and hence is a function on $C^0(\Gamma\backslash N)$. This is checked by proving it first for $\phi \in C^\infty \cap \mathscr{H}_\sigma$. In this case, if $\gamma_0 \in \Gamma$ we may reverse the steps in (24) to get

$$\begin{aligned} A_{ji}\phi(\gamma_0 x) &= \sum_{s\in S} \sum_{\delta'\in U'} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i \cdot \delta'(m)}\, \phi(\Gamma m y s \gamma_0 x)\, d\dot{m} \right) \\ &= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i(m)}\, \phi(\Gamma m y \gamma\gamma_0 x)\, d\dot{m} \right) \\ &= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i(m)}\, \phi(\Gamma m y \gamma x)\, d\dot{m} \right) \\ &= A_{ji}\phi(x) \end{aligned} \tag{27}$$

since the sum over $\Gamma_0\backslash\Gamma$ is independent of the choice of coset representatives. By writing

$$A_{ji}\psi = \sum_{s\in S} I^{ys}(j(\mu' * \psi')), \qquad \psi' = i(\psi),$$

which is valid for $\psi \in C^0 \cap \mathscr{H}_\sigma$, it is obvious that $\|\psi_n - \psi\|^\infty \to 0$ in $C^0 \cap \mathscr{H}_\sigma$ implies that $A_{ji}(\psi_n) \to A_{ji}\psi$ pointwise on N, so that (27) remains valid for sup-norm limits of $C^\infty \cap \mathscr{H}_\sigma$ within $C^0 \cap \mathscr{H}_\sigma$. But $C^\infty \cap \mathscr{H}_\sigma$ is sup-norm dense in $C^0 \cap \mathscr{H}_\sigma$.

We conclude that $V_{ji} = \sum_{s\in S} V^{ys}$ is L^2-norm continuous and maps

$C^0 \cap \mathscr{H}_\sigma$ into $C^0(\Gamma\backslash N)$. But for smooth functions $\phi \in C^\infty \cap \mathscr{H}_\sigma$ we have $V_{ji}\phi = T_{ji}\phi$: if $x \in K$,

$$\begin{aligned} V_{ji}\phi(\Gamma x) &= \sum_{s\in S} A^{ys}\phi(x) \\ &= \sum_{s\in S} (A\phi')(ysx) \\ &= \sum_{s\in S}\sum_{\delta'\in U'} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i \cdot \delta'(m)}\, \phi'(\Gamma' mysx)\, d\dot{m}\right) \\ &= \sum_{s\in S}\sum_{\delta'\in U'} \left(\int_{\Gamma_0\backslash M} \overline{\chi_i \cdot \delta'(m)}\, \phi(\Gamma mysx)\, d\dot{m}\right) \\ &= T_{ji}\phi(\Gamma x). \end{aligned}$$

By L^2-norm continuity, $T_{ji}\phi = V_{ji}\phi$ for all $\phi \in C^0 \cap \mathscr{H}_\sigma$, so that T_{ji} maps $C^0 \cap \mathscr{H}_\sigma$ into itself, and the theorem is proved. Q.E.D.

Note. If $y \in N$ is rational, it can be shown that there is a subgroup Θ such that: (i) Θ is normal in Γ and of finite index, (ii) $y^k\Theta y^{-k} \subseteq \Theta$ for all k. The calculatiuns in this section can be simplified by replacing Δ, Γ', $\Gamma_0\backslash\Gamma'$, $\Gamma_0\Delta\backslash\Gamma$ with the simpler objects Θ, Θ, $\Gamma_0\backslash\Theta$, $\Theta\backslash\Gamma$.

7. A Few Examples

7.1. Example. Take the usual basis X, Y, Z in the Lie algebra $\mathfrak{n}$ of the Heisenberg group $N = N_3$ = upper triangular real 3×3 matrices with ones on the diagonal, and let $\Gamma = N_3 \cap SL(3, \mathbf{Z})$. Let X^*, Y^*, Z^* be the dual basis in $\mathfrak{n}^*$. We shall consider the orbits in $\mathfrak{n}^*/Ad^*(N)$ of maximal dimension, which are hyperplanes $\mathcal{O}_c = \mathbf{R}X^* + \mathbf{R}Y^* + cZ^*$ $(c \neq 0)$. Now $M = \exp(\mathbf{R}Y + \mathbf{R}Z)$ is normal, and is maximal subordinate for all $f \in \mathfrak{n}^*$ such that $\langle f, Z\rangle \neq 0$. It is rational since M is abelian and $\log(\Gamma \cap M) = \mathbf{Z}Y + \mathbf{Z}Z$. The only orbits $\mathcal{O}_c$ $(c \neq 0)$ contributing multiplicity in $U^1 = \mathrm{Ind}(\Gamma \uparrow N, 1)$ are those such that $c \in \mathbf{Z}$, the multiplicity being $|\, c\,|$. Identify $M^\wedge = \mathfrak{m}^*$ by taking $\chi_l = e^{2\pi i l}$. If $c \in \mathbf{Z} \sim (0)$, the irreducible representation $\sigma(c)$ associated with $\mathcal{O}_c$ is induced from the following integral maximal character on M, $\chi_l = \chi_{cZ^*}$. Its N-orbit in $M^\wedge$ is identified in $\mathfrak{m}^*$ with the N-orbit $(cZ^*) \cdot N$, which is just the line $cZ^* + \mathbf{R}Y^*$. The integral points $(\chi_c \cdot N)_\#$ identify with the points on this line which belong to $\mathfrak{m}_{\mathbf{Z}}{}^* = \{l \in \mathfrak{m}^*: l(\log(\Gamma \cap M)) \subseteq \mathbf{Z}\} = \mathbf{Z}Y^* + \mathbf{Z}Z^*$; these are precisely

$$\mathfrak{m}_{\mathbf{Z}}{}^* \cap (cZ^* + \mathbf{R}Y^*) = cZ^* + \mathbf{Z}Y^*. \tag{28}$$

The individual Γ-orbits within $(\chi_c \cdot N)_{\#}$ identify with the Γ-orbits in $\mathfrak{m}_{\mathbf{Z}}{}^*$. These Γ-orbits have representatives

$$l_q = cZ^* + qY^* \qquad 0 \leqslant q < |c|, \qquad q \in \mathbf{Z},$$

and the Γ-orbits themselves have the form

$$l_q \cdot \Gamma = cZ^* + qY^* + |c| \, \mathbf{Z}Y^*. \tag{29}$$

Now $T^k = \Gamma \cap M \backslash M = T^2$ and $\hat{T}^2$ identifies with $\mathfrak{m}_{\mathbf{Z}}{}^* = \mathbf{Z}^2$. On T^2 the distributions D_σ and D_q corresponding to P_σ and $P_{(\chi_q, M)}$ ($\chi_q = e^{2\pi i l_q}$) are sums of the characters in the sets (28) and (29). These sets lie in the coset ring $\mathrm{COS}(\mathbf{Z}^2)$, so D_σ and the D_q are measures. For $D_{\sigma(c)}$ we may actually identify the measure on $\Gamma \backslash N$ by Poisson summation. The action of the center $Z(N) = \exp(\mathbf{R}Z)$ fibers $\Gamma \backslash N$ into one-dimensional tori, the orbit through the origin Γe being $\Gamma \backslash \Gamma Z(N) \approx \Gamma \cap Z(N) \backslash Z(N)$. The character χ_c annihilates $\Gamma \cap Z(N)$ when restricted to give a character $\chi_c(\exp(tZ)) = e^{2\pi i c t}$ on $Z(N)$; thus we may identify χ_c as a continuous function on $T^1 = \Gamma \cap Z(N) \backslash Z(N)$. If μ is normalized invariant measure on T^1, then $D_{\sigma(c)}$ is precisely the measure $\nu_c(dt) = \overline{\chi_c(t)} \, \mu(dt)$ on T^1, lifted over to a measure on $\Gamma \backslash \Gamma Z(N) \approx \Gamma \cap Z(N) \backslash Z(N)$. This measure ν_c on $\Gamma \backslash N$ is right Γ-invariant, and

$$P_{\sigma(c)} f(\Gamma n) = \nu_c * f(\Gamma n) = \langle \nu_c \, , n \cdot f \rangle \qquad \text{all} \quad n \in N, f \in C^0(\Gamma \backslash N).$$

Thus the value of $P_{\sigma(c)} f$ at $\zeta \in \Gamma \backslash N$ is an average of the values of f along the fiber $\zeta \cdot Z(N)$,

$$\begin{aligned} P_{\sigma(c)} f(\Gamma n) &= \langle D_{\sigma(c)} \, , n \cdot f \rangle \\ &= \int_0^1 e^{-2\pi i c t} f(\exp(tZ) \cdot n) \, dt \\ &= \int_0^1 e^{-2\pi i c t} f(\Gamma n \cdot \exp(tZ)) \, dt. \end{aligned}$$

In effect, $P_{\sigma(c)}$ washes out the behavior of f along each fiber $\zeta \cdot Z(N)$; $P_{\sigma(c)} f$ is covariant like $\overline{\chi_c}$ along each fiber. $P_{\sigma(c)}$ has no effect on the behavior transverse to these fibers. It is also possible to work out the measures on $\Gamma \backslash N$ associated with the constructible irreducible projections $P \leqslant P_{\sigma(c)}$, but we omit this.

7.2. Example. Let $\mathfrak{n}$ have basis W, X, Y, Z such that $[W, X] =$

$2Y$ and $[W, Y] = 2Z$. This defines a Campbell–Hausdorff multiplication

$$A * B = A + B + \tfrac{1}{2}[A, B] + \tfrac{1}{12}[A, [A, B]] + \tfrac{1}{12}[B, [A, B]].$$

Take $N = (\mathfrak{n}, *)$; then $\Gamma = 3\mathbf{Z}W * (\mathbf{Z}X + \mathbf{Z}Y + \mathbf{Z}Z)$ is a discrete uniform subgroup, and $\mathfrak{m} = \mathbf{R}X + \mathbf{R}Y + \mathbf{R}Z$ a rational abelian ideal. This ideal happens to be maximal subordinate for all functionals $f \in \mathfrak{n}^*$ such that $\langle f, Z\rangle \neq 0$; hence $\mathfrak{m}$ is maximal subordinate for all orbits of highest dimension ($\dim \mathcal{O} = 2$) in $\mathfrak{n}^*$. We shall consider the orbits of functionals $f = aW^* + bX^* + cY^* + dZ^*$ such that $d \neq 0$ ($a, b, c \in \mathbf{R}$); this gives almost all highest-dimensional orbits. The corresponding orbits have orbit representatives of the form $bX^* + dZ^*$; for $b \in \mathbf{R}$, $d \in \mathbf{R} \sim (0)$ write $\mathcal{O}_{b,d} = (bX^* + dZ^*) \cdot N$ for the (distinct) N-orbits of these functionals. Identifying $M^\wedge$ with $\mathfrak{m}^* = \mathbf{R}X^* + \mathbf{R}Y^* + \mathbf{R}Z^*$, we find that the corresponding orbit in $M^\wedge$ is a parabola

$$\mathcal{O}'_{b,d} = \{(b + 2s^2 d)\, X^* + 2sdY^* + dZ^* : s \in \mathbf{R}\}$$

lying in the hyperplane $\mathbf{R}X^* + \mathbf{R}Y^* + dZ^* \subseteq \mathfrak{m}^*$. All N-orbits in $\mathfrak{m}^* \sim \{l \in \mathfrak{m}^*: l(Z) = 0\}$ have this form.

Now $\log(\Gamma \cap M) = \mathbf{Z}X + \mathbf{Z}Y + \mathbf{Z}Z$, and the N-orbit $\mathcal{O}'_{b,d}$ is the orbit of an integral maximal character on $M \Leftrightarrow$ it meets the integral points $\mathfrak{m}_{\mathbf{Z}}^* = \{l \in \mathfrak{m}^*: l(\log(\Gamma \cap M)) \subseteq \mathbf{Z}\} = \mathbf{Z}X^* + \mathbf{Z}Y^* + \mathbf{Z}Z^*$. This happens $\Leftrightarrow$ (i) $d \in \mathbf{Z} \sim (0)$ and (ii) there exists an integer k such that

$$b + (k^2/2d) \equiv 0 \qquad (\text{mod } 1).$$

Fix a pair b, d satisfying conditions (i) and (ii). Let $\chi_{b,d} = e^{2\pi i(bX^* + dZ^*)}$ (an integral maximal character on M), and let $\sigma = \sigma_{b,d} = \mathrm{Ind}(M \uparrow N, \chi_{b,d}) \in (N : \Gamma)^\wedge$. The primary projection P_σ corresponds to the distribution on $T^3 = \Gamma \cap M \backslash M$ corresponding to the sum of all integral characters in $\mathcal{O}'_{b,d} \cap \mathfrak{m}_{\mathbf{Z}}^*$, the integral points on the parabola $\mathcal{O}'_{b,d} \subseteq \mathbf{R}^3$. This orbit has no saturating linear variety $\mathscr{V}' \subseteq \mathfrak{m}^*$; by Theorem 1.1 and (14), we conclude that $\mathrm{order}(D_\sigma) \leqslant \dim(\mathcal{O}'_{b,d}) = 1$. It is not hard to show that the integral points are sufficiently numerous on the parabola that $\mathcal{O}'_{b,d} \cap \mathfrak{m}_{\mathbf{Z}}^*$ is not in the coset ring of $\mathbf{Z}^3 = \mathfrak{m}_{\mathbf{Z}}^*$. Therefore, $\mathrm{order}(D_\sigma) = 1$. By the Zero–One law, all irreducible projections $P \leqslant P_\sigma$ have order 1 too. The distribution D_σ, which we may think of as living on T^3, is being described by specifying its Fourier transform $D_\sigma{}^\wedge$ = characteristic function of $\mathcal{O}'_{b,d} \cap \mathfrak{m}_{\mathbf{Z}}^* \subseteq \mathbf{Z}^3 \cong \hat{T}^3$. In this example it seems to be a difficult classical problem to give a geometric description of D_σ back on the torus T^3, as we did in the last example.

7.3. EXAMPLE. Let $N = N_4$ = upper triangular real 4×4 matrices with ones on the diagonal, and let $\Gamma = N \cap SL(4, \mathbf{Z})$. Realize $\mathfrak{n}$ as upper triangular 4×4 matrices, and take basis vectors A, B, W, X, Y, Z such that

$$\begin{bmatrix} 0 & a & x & z \\ & 0 & w & y \\ & & 0 & b \\ 0 & & & 0 \end{bmatrix} = aA + bB + wW + xX + yY + zZ.$$

Let $A^*,\ldots, Z^*$ be the dual basis in $\mathfrak{n}^*$. We shall consider the representation $\sigma \in (N:\Gamma)^\wedge$ associated with the orbit $\mathcal{O} = Z^* \cdot N \subseteq \mathfrak{n}^*$. Straightforward calculations show that if

$$n = \begin{bmatrix} 1 & a & x & z \\ & 1 & w & y \\ & & 1 & b \\ 0 & & & 1 \end{bmatrix} \in N \qquad X = \begin{bmatrix} 0 & A & X & Z \\ & 0 & W & Y \\ & & 0 & B \\ 0 & & & 0 \end{bmatrix} \in \mathfrak{n}$$

then

$$Ad^*(n)\, Z^* = Z^* \cdot n = Z^* + (bw - y)\, A^* + xB^* - abW^* - bX^* + aY^*$$

and

$$\langle Z^*, [U_1, U_2]\rangle = A_1Y_2 - A_2Y_1 + X_1B_2 - X_2B_1, \qquad U_1, U_2 \in \mathfrak{n}.$$

From this we see that the radical of $l = Z^*$ is $\mathfrak{r} = \mathbf{R}W + \mathbf{R}Z$, and that $\mathfrak{m} = \mathbf{R}W + \mathbf{R}X + \mathbf{R}Y + \mathbf{R}Z$ is an abelian ideal maximal subordinate to Z^*. Both $\mathfrak{r}$ and $\mathfrak{m}$ are rational since $\log(\Gamma \cap M) = \mathbf{Z}W + \cdots + \mathbf{Z}Z$, so all of our previous results apply. Let $\chi = e^{2\pi i l}$, an integral character on M. Since M is abelian, we identify $M^\wedge = \mathfrak{m}^*$, and integral maximal characters on M with

$$\mathfrak{m}_{\mathbf{Z}}^* = \{l' \in \mathfrak{m}^* : l'(\log(\Gamma \cap M)) \subseteq \mathbf{Z}\} = \mathbf{Z}W^* + \cdots + \mathbf{Z}Z^*.$$

The orbit $\chi \cdot N$ in $M^\wedge$ is identified with the hyperbolic surface

$$\mathcal{O}' = \{Z^* + t_1X^* + t_2Y^* + t_1t_2W^* : t_1, t_2 \in \mathbf{R}\}.$$

The integral points $(\chi \cdot N)_{\#}$ correspond to $\mathcal{O}_{\#}' = \mathcal{O}' \cap \mathfrak{m}_{\mathbf{Z}}^* = \{Z^* + t_1X^* + t_2Y^* + t_1t_2W^* : t_1, t_2 \in \mathbf{Z}\}$. (It turns out that there is just one Γ-orbit, so σ has multiplicity 1.) Now $\Gamma_0\backslash M$ is a torus T^4, and the primary distribution D_σ is the liftback from T^4 of the sum of characters in $\mathcal{O}_{\#}' \subseteq \mathfrak{m}_{\mathbf{Z}}^* \cong \mathbf{Z}^4 = \hat{T}^4$. Since $\mathcal{O}_{\#}'$ is not in $\mathrm{COS}(\mathbf{Z}^4)$, D_σ is a not a measure on T^4. Since $\mathcal{O}'$ has only $\mathscr{V}' = (0)$ as a saturating subspace, our order estimate (14) says that $0 < \mathrm{order}(D_\sigma) \leqslant 2$. One can show that $\mathrm{order}(D_\sigma) = 1$, so that the order estimate in Theorem 1.1 is not always the best possible.

Notes added in proof. In 4.1 we originally proved order $\leqslant d$, but Jon Brezin pointed out to us that a sharper estimate, roughly: order $\leqslant d/2$, should work; we have modified the proof, and strengthened the estimate (14) accordingly, and thank him for this comment.

The Zero–One Law has been proved in full generality by R. Penney (unpublished manuscript). R. Howe also has an unpublished proof.

The projection formula (7) has been proved without the normality hypothesis in a forthcoming paper by R. Penney and the authors.

Corollary 3.2 has been proved without normality hypotheses by R. Penney in a paper "Central idempotent measures on a nilmanifold" (to appear).

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