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SHORT COMMUNICATION

TRANSIENT BEHAVIOUR OF AN M/M/1/*N* QUEUE

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For a simple queue with finite waiting space the difference equations satisfied by the Laplace transforms of the state probabilities at finite time are solved and the state probabilities have been obtained. The method economizes in algebra and the simple closed form of the state probabilities is used to obtain important parameters.

Transient behaviour	state probabilities
finite waiting space	queue length

1. Introduction

The transient behaviour of the M/M/1/N queue for a general N has been discussed by Takács [6] and Morse [4]. The expressions obtained by them are so complex that these cannot be used to obtain parameters of the queue length such as the mean in the explicit form. In this paper a very simple and elegant algebraic method developed by the first author is used to obtain the closed form solution of this important problem.

2. Main results

The inter-arrival and service times are taken to be negative exponentially distributed with mean λ^{-1} and μ^{-1}, respectively. The waiting room capacity is limited to $N-1$ places, that is, the maximum number of customers in the system is N. The system is taken to be empty at $t=0$.

Let $p_n(t)$ be the probability that there are n customers in the system at the time t. Then $p_0(0)=1$ and $p_n(0)=0$ for $n\neq 0$. Writing the difference-differential equations

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of the system and taking Laplace transform of these equations, we get

$$\begin{aligned}
&(\lambda+\theta)\psi(0,\theta)=\mu\psi(1,\theta),\\
&(\lambda+\mu+\theta)\psi(n,\theta)=\mu\psi(n+1,\theta)+\lambda\psi(n-1,\theta),\quad 1\leq n\leq N-1,\\
&(\mu+\theta)\psi(N,\theta)=\lambda\psi(N-1,\theta)
\end{aligned} \tag{2.1}$$

where

$$\psi(n,\theta)=\int_0^\infty e^{-\theta t}p_n(t)\,dt.$$

These equations have a solution,

$$\psi(n,\theta)=A\alpha^n+B\beta^n, \tag{2.2}$$

where

$$\alpha=\frac{\theta+\lambda+\mu+\sqrt{(\theta+\lambda+\mu)^2-4\lambda\mu}}{2\mu},\qquad \beta=\frac{\theta+\lambda+\mu-\sqrt{(\theta+\lambda+\mu)^2-4\lambda\mu}}{2\mu}. \tag{2.3}$$

Using the fact that $\sum_{n=0}^N\psi(n,\theta)=\theta^{-1}$ and the boundary condition $(\mu+\theta)\psi(N,\theta)=\lambda\psi(N-1,\theta)$, we obtain the values of A and B. We thus have

$$\psi(n,\theta)=\frac{(\alpha\beta)^n[\alpha^{N-n+1}-\beta^{N-n+1}]-(\alpha\beta)^{n+1}[\alpha^{N-n}-\beta^{N-n}]}{\theta[\alpha^{N+1}-\beta^{N+1}]} \tag{2.4}$$

$$=\frac{(\alpha\beta)^n\sum_{k=0}^{N-n}\alpha^k\beta^{N-n-k}-(\alpha\beta)^{n+1}\sum_{k=0}^{N-n-1}\alpha^k\beta^{N-n-1-k}}{\theta\sum_{k=0}^{N}\alpha^k\beta^{N-k}}. \tag{2.5}$$

It is easily observed that for integral values of n

$$\sum_{k=0}^{n}x^ky^{n-k}=\sum_{k=0}^{[n/2]}(-1)^k\,{}^{n-k}C_k(xy)^k(x+y)^{n-2k}. \tag{2.6}$$

Writing

$$\sum_{k=0}^{n}x^ky^{n-k}=\prod_{k=1}^{n}(x+y+\alpha_{nk}\sqrt{xy}) \tag{2.7}$$

and comparing it with (2.6) we find that α_{nk} are n roots of the nth degree Chebychev's polynomial of the first kind

$$g_n(x)=\sum_{k=0}^{[n/2]}(-1)^k\,{}^{n-k}C_kx^{n-2k}. \tag{2.8}$$

Let us define

$$\varphi_n(\theta)=\prod_{k=1}^{n}(\theta+\lambda+\mu+\alpha_{nk}\sqrt{\lambda\mu}) \tag{2.9}$$

where α_{nk} are as defined in (2.8) and $\varphi_0(\theta)=1$, $\varphi_n(\theta)=0$ for $n<0$. Making use of (2.7) and (2.9) in (2.5) and remembering that $\alpha+\beta=(\theta+\lambda+\mu)/\mu$, $\alpha\beta=\lambda/\mu$, we get

$$\psi(n,\theta)=\frac{\lambda^n[\varphi_{N-n}(\theta)-\lambda\varphi_{N-n-1}(\theta)]}{\theta\varphi_N(\theta)}. \tag{2.10}$$

Because of (2.8), $\varphi_N(\theta)$ has distinct real factors. Making use of partial fractions and taking the inverse Laplace transform of (2.10), we get after some simplification

$$p_n(t)=\frac{(1-\lambda/\mu)(\lambda/\mu)^n}{1-(\lambda/\mu)^{N+1}}-\mathrm{e}^{-(\lambda+\mu)t}\sum_{j=1}^{N}A_{nj}\,\mathrm{e}^{-\alpha_{Nj}t\sqrt{\lambda\mu}} \tag{2.11}$$

where

$$A_{nj}=(-1)^{N-n}\left(\frac{\lambda}{\mu}\right)^{n/2}\frac{g_{N-n}(\alpha_{Nj})+\sqrt{\lambda/\mu}\,g_{N-n-1}(\alpha_{Nj})}{(\sqrt{\lambda/\mu}+\sqrt{\mu/\lambda}+\alpha_{Nj})b_{Nj}} \tag{2.12}$$

and

$$b_{Nj}=\prod_{\substack{k=1\\k\neq j}}^{N}(\alpha_{Nk}-\alpha_{Nj}),\quad n=0,1,\ldots,N,\ j=1,2,\ldots,N,$$

for $\lambda\neq\mu$, and

$$p_n(t)=\frac{1}{N+1}-\mathrm{e}^{-2\lambda t}\sum_{j=1}^{N}(-1)^{N-n}\frac{g_{N-n}(\alpha_{Nj})+g_{N-n-1}(\alpha_{Nj})}{(2+\alpha_{Nj})b_{Nj}}\,\mathrm{e}^{-\alpha_{Nj}\lambda t} \tag{2.13}$$

for $\lambda=\mu$.

It is easily seen that as $t\to\infty$ (2.11) and (2.13) reduce to the well known steady state distributions. Also, when $t\to\infty$, $N\to\infty$, and if it is assumed that $\lambda<\mu$, we get the well known steady state distribution given by

$$p_n=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n,\quad n=0,1,2,\ldots. \tag{2.14}$$

3. Infinite waiting space

Letting $N\to\infty$ we see from (2.3) that $0<\beta<\alpha$. If we divide the numerator and denominator on the right-hand side of (2.4) by α^{N+1} and take the limit as $N\to\infty$, we get

$$\psi(n,\theta)=\frac{(1-\beta)\beta^n}{\theta}=\frac{(1-\beta)(\lambda/\mu)^n}{\theta\alpha^n}. \tag{3.1}$$

But $\theta = -\mu(1-\alpha)(1-\beta)$, so we get

$$\psi(n, \theta) = -\frac{(\lambda/\mu)^n}{\mu(1-\alpha)\alpha^n}. \tag{3.2}$$

This agrees with [5, eq. (4.10), p. 89]. The probabilities $p_n(t)$ can now be obtained by the method developed in [5].

4. Further results

Let $Q(t)$ be the queue length at the time t. After some algebra we get

$$\mathrm{E}\{Q(t)\} = \sum_{n=0}^{N} n p_n(t)$$

$$= \begin{cases} \dfrac{\rho\{1-(N+1)\rho^N + N\rho^{N+1}\}}{(1-\rho)(1-\rho^{N+1})} - \mathrm{e}^{-(\lambda+\mu)t} \\ \quad \times \displaystyle\sum_{j=1}^{N}\sum_{n=1}^{N} \frac{(-1)^{N-n}\rho^{n/2} g_{N-n}(\alpha_{Nj})}{(\rho^{1/2}+\rho^{-1/2}+\alpha_{Nj})b_{Nj}} \mathrm{e}^{-\alpha_{Nj}t\sqrt{\lambda\mu}}, & \lambda \neq \mu, \rho = \lambda/\mu \quad (4.1) \\ \tfrac{1}{2}N - \mathrm{e}^{-2\lambda t} \displaystyle\sum_{j=1}^{N}\sum_{n=1}^{N} \frac{(-1)^{N-n} g_{N-n}(\alpha_{Nj})\,\mathrm{e}^{-\alpha_{Nj}\lambda t}}{(2+\alpha_{Nj})b_{Nj}}, & \lambda = \mu. \quad (4.2) \end{cases}$$

Let us now find the probability that the queue length exceeds a given number. We have that

$$\sum_{n=r}^{N} p_n(t) =$$

$$= \begin{cases} \dfrac{(\lambda/\mu)^r\{1-(\lambda/\mu)^{N-r+1}\}}{1-(\lambda/\mu)^{N+1}} - \mathrm{e}^{-(\lambda+\mu)t} \\ \quad \times \displaystyle\sum_{j=1}^{N} \frac{(\lambda/\mu)^{r/2}(-1)^{N-r} g_{N-r}(\alpha_{Nj})}{\{\sqrt{\lambda/\mu}+\sqrt{\mu/\lambda}+\alpha_{Nj}\}b_{Nj}} \mathrm{e}^{-\alpha_{Nj}t\sqrt{\lambda\mu}}, & \lambda \neq \mu, \quad (4.3) \\ \dfrac{N-r+1}{N+1} - \mathrm{e}^{-2\lambda t} \displaystyle\sum_{j=1}^{N} \frac{(-1)^{N-r} g_{N-r}(\alpha_{Nj})}{(2+\alpha_{Nj})b_{Nj}} \mathrm{e}^{-\alpha_{Nj}\lambda t}, & \lambda = \mu. \quad (4.4) \end{cases}$$

In a similar way other parameters, involving $p_n(t)$, can be easily calculated.

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