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Non-neutrality of the Stiefel manifolds $V_{n,k}$ II

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Abstract

The Stiefel manifolds $V_{2^m-1,k}$ are shown to be non-neutral for $m \geqslant 5$, $2^{m-1}+2 \leqslant k = 2\ell < 2^m - 2$.

Keywords: Stiefel manifolds

1. Introduction

Let $V_{n,k}$ denote the Stiefel manifold of orthogonal k-frames in $\mathbf{R}^n$. Thus $V_{n,k} = \{(y_1, \ldots, y_k) \mid y_i \in \mathbf{R}^n, \|y_i\| = 1, y_i \perp y_j \text{ for } i \neq j\}$ where $\| \;\|$ is the usual Euclidean norm on $\mathbf{R}^n$. Each $(y_1, \ldots, y_k)$ can be viewed as an $n \times k$ matrix. Note that $V_{n,1} = S^{n-1}$ and $V_{n,n} = O(n)$.

The orthogonal group $O(k)$ acts on $V_{n,k}$ via the matrix multiplication

$$(y_1, \ldots, y_k) \cdot g, \quad g \in O(k).$$

Thus each $g \in O(k)$ defines a self-map $\hat{g}: V_{n,k} \to V_{n,k}$. Consider the homotopy class $[\hat{g}]$ in the semi-group of homotopy classes of self-maps of $V_{n,k}$. If $g \in SO(k)$, the rotation group, then $[\hat{g}] = 1$ since $SO(k)$ contains the identity matrix and is path connected. The set $\{[\hat{g}] \mid g \in O(k) - SO(k)\}$ also consists of only one homotopy class since $O(k) - SO(k)$ is the other path component of $O(k)$. Denote this class by λ. λ contains the self-maps which change the sign of any column. It is clear that $\lambda^2 = 1$. Following James [4], we say $V_{n,k}$ is neutral if $\lambda = 1$.

The neutrality problem on $V_{n,k}$ is

$$\text{to determine, for what } n \text{ and } k, V_{n,k} \text{ is neutral.} \qquad (*)$$

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So far, the following are known.

(1) If n is even and k is odd then $V_{n,k}$ is neutral.
(2) If $n-k$ is even then $V_{n,k}$ is non-neutral.
(3) $V_{n,n-1}$ is neutral for all $n \geqslant 2$.
(4) If n is odd and k is even then $V_{n,k}$ is non-neutral for $n+1 \neq 2^m$ and for $(n,k) \neq (n,n-1)$.

(1) and (2) are not difficult and are proved in [4]. A summary of these proofs is given in [7]. (3) is also fairly easy and a proof of it is given in [7]. (4) is non-trivial. James proves it in [4] for the case $n \leqslant 2k-1$ and the cases $n \geqslant 2k+1$ with $k = 2,4$ or 8. The remaining cases for $n \geqslant 2k+1$ are proved in [7].

It remains to consider the problem $(*)$ for $V_{2^m-1,\,k}$ with $k = 2\ell < 2^m - 2$. For $k = 2$, the problem is connected to a problem in the homotopy theory of spheres. James proves in [4] that $V_{2^m-1,2}$ is neutral if and only if the Whitehead product $[\iota_{2^m-1}, \iota_{2^m-1}] \in \pi_{2^{m+1}-3}(S^{2^m-1})$ can be halved. In fact, he proves this for all $V_{n,2}$ with n odd. Whether $[\iota_{2^m-1}, \iota_{2^m-1}]$ can be halved is an important problem in the homotopy theory of spheres. This problem is known as the strong Kervaire invariant conjecture [2]. The conjecture is known true for $m \leqslant 6$. The cases $m = 1,2$, and 3 are trivial since the corresponding $[\iota_{2^m-1}, \iota_{2^m-1}]$ are zero. The case $m = 4$ is due to Toda [11], the case $m = 5$ is due to Mahowald and Tangora [8] and the case $m = 6$ is due to Mahowald (see [6]). These imply $V_{2^m-1,2}$ is neutral for $m \leqslant 6$. The conjecture for $m \geqslant 7$ is presently unknown. Equivalently, the neutrality problem on $V_{2^m-1,2}$ for $m \geqslant 7$ is still open.

The problem $(*)$ for $V_{2^m-1,2}$ is a difficult one as just described. It is conceivable that the problem for $V_{2^m-1,\,k=2\ell}$ is also difficult for small k. The purpose of this paper is to show that one can give a definite answer to the problem when k is large enough. The result is the following.

Theorem 1.1. *For $m \geqslant 5$, $V_{2^m-1,\,k}$ is non-neutral for $2^{m-1} + 2 \leqslant k = 2\ell < 2^m - 2$.*

This solves approximately "*a half*" of the problem $(*)$ on the remaining $V_{2^m-1,\,k}$ for $m \geqslant 5$. For $m = 2$, $V_{3,2}$ is neutral as remarked above. For $m = 3$, $V_{7,k}$ ($k = 2,4,6$) is known to be neutral since it is an equivariant retract of $V_{8,k+1}$, see [4]. We conjecture Theorem 1.1 is also true for $m = 4$. The method to prove Theorem 1.1 for $m \geqslant 5$ in this paper probably can be refined to cover the case $m = 4$ also.

The proof of Theorem 1.1 will be a contradiction proof. For $1 \leqslant \ell < n$ let P^n_ℓ denote the stunted real projective space $P^n/P^{\ell-1}$. Assuming $V_{2^m-1,k}$ is neutral, for m, k as in Theorem 1.1, we will show that there are space maps $\Sigma P^{2^m-8}_{2^{m-1}-3} \xrightarrow{\phi} V_{2^m,2^{m-1}+3}$ and $\Sigma P^{2^m-3}_{2^m-7} \xrightarrow{g_1} C_\phi = V_{2^m,2^{m-1}+3} \cup_\phi C\Sigma P^{2^m-8}_{2^{m-1}-3}$ with $\phi^* = 0$, $g_1^* = 0$ in mod 2 cohomology $\tilde{H}^*(\ )$ such that $Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \Sigma^2 x_{2^m-3}$ in $\tilde{H}^*(C_{g_1})$ where $x_{2^{m-1}-1}$ is a nonzero class in $\tilde{H}^{2^{m-1}-1}(V_{2^m,2^{m-1}+3}) \subset \tilde{H}^{2^{m-1}-1}(C_{g_1})$ and $\Sigma^2 x_{2^m-3}$ is the generator of $\tilde{H}^{2^m-1}(\Sigma^2 P^{2^m-3}_{2^m-7}) = \mathbf{Z}/2 \subset \tilde{H}^{2^m-1}(C_{g_1})$. This is a contradiction to the fact that in $\tilde{H}^*(X)$ of any space X, $Sq^n x = 0$ if $|x| < n$. This contradiction proves $V_{2^m-1,k}$ is non-neutral.

In Section 2 we recall some basic facts about $V_{n,k}$. In Section 3 we recall, from [7], some other facts on $V_{n,k}$ and prove some more results that we will need. In Section 4 we show $P^n_\ell \bar{\times} S^1 = P^n_\ell \times S^1/S^1$, for $2 \leqslant \ell \leqslant n/2$ and $n \leqslant 3\ell - 3$, is the cofiber of a space map $P^n_{2\ell-1} \xrightarrow{f} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell$. The construction of the space C_{g_1} above will depend on this recognition of $P^n_\ell \bar{\times} S^1$ as the cofiber C_f. The proof of Theorem 1.1 will be given in Section 5.

All cohomology and homology of spaces in this paper have mod 2 coefficients except in Section 4 where integral cohomology will also be considered.

2. Some basic facts about the Stiefel manifolds $V_{n,k}$

In this section we recall from [4,10] some basic facts about $V_{n,k}$. We need only consider $k < n$, and from now on we assume this.

Let P^n denote the n-dimensional real projective space. For each pair (n,k) of positive integers with $k < n$ there is a standard inclusion $P_{n,k} = P^{n-1}/P^{n-k-1} \overset{i}{\hookrightarrow} V_{n,k}$ such that the pair $(V_{n,k}, P_{n,k})$ is $(2n-2k)$-connected. These inclusions have the following compatibility properties:

$$\begin{array}{ccc} P_{n,k} & \xrightarrow{i} & V_{n,k} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle p} \\ P_{n,k'} & \xrightarrow{i} & V_{n,k'} \end{array} \qquad \begin{array}{ccc} P_{n,k} & \xrightarrow{i} & V_{n,k} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle q} \\ P_{n+1,k+1} & \xrightarrow{i} & V_{n+1,k+1} \end{array} \tag{2.1}$$

where $n > k > k'$, τ is the collapsing map $P^{n-1}/P^{n-k-1} \to P^{n-1}/P^{n-k'-1}$, p is the map obtained by taking the last k' vectors in each k-frame, ρ is the inclusion $P^{n-1}/P^{n-k-1} \hookrightarrow P^n/P^{n-k-1}$ and q is the inclusion defined by $q(y_1, \dots, y_k) = (y_1, \dots, y_k, e_{n+1})$ where $e_{n+1} = (0, \dots, 0, \underset{n+1}{1}) \in \mathbf{R}^{n+1}$. The sequence

$$V_{n,k} \xrightarrow{q} V_{n+1,k+1} \xrightarrow{p} V_{n+1,1} = S^n$$

is a fibration. The reduced cohomology $\tilde{H}^*(P_{n,k})$ has $\{x_{n-k}, \dots, x_{n-1}\}$ as a $\mathbf{Z}/2$-base where x_ℓ is the nonzero class of $\tilde{H}^\ell(P_{n,k}) = \mathbf{Z}/2$. The mod 2 Steenrod algebra A acts on $\tilde{H}^*(P_{n,k})$ by

$$Sq^j x_\ell = \binom{\ell}{j} x_{\ell+j}. \tag{2.2}$$

We refer to [10, Chapter IV] for the following.

Theorem 2.1. *For each pair (n,k) of positive integers with $k < n$ there is a canonical choice of an A-submodule $\bar{H}^*(P_{n,k})$ of $\tilde{H}^*(V_{n,k})$ which is A-isomorphic to $\tilde{H}^*(P_{n,k})$. The generator of $\bar{H}^\ell(P_{n,k}) = \mathbf{Z}/2$ for $n-k \leqslant \ell \leqslant n-1$ is also denoted by x_ℓ. These classes are called cohomology normal classes (of length one) in $\tilde{H}^*(V_{n,k})$. They have the following properties, where i, q and p are as in* (2.1).

(1) $\tilde{H}^*(V_{n,k}) \xrightarrow{i^*} \tilde{H}^*(P_{n,k})$ *has* $i^*(x_\ell) = x_\ell$ *for* $n-k \leqslant \ell \leqslant n-1$.

(2) $\tilde{H}^*(V_{n+1,k+1}) \xrightarrow{q^*} \tilde{H}^*(V_{n,k})$ *has* $q^*(x_\ell) = x_\ell$ *for* $n-k \leqslant \ell \leqslant n-1$ *and* $q^*(x_n) = 0$.

(3) $\tilde{H}^*(V_{n,k'}) \xrightarrow{p^*} \tilde{H}^*(V_{n,k})$ *has* $p^*(x_\ell) = x_\ell$ *for* $n-k' \leqslant \ell \leqslant n-1$.

(4) *The cohomology algebra* $H^*(V_{n,k})$ *is generated multiplicatively by* $\{x_\ell \mid n-k \leqslant \ell \leqslant n-1\}$ *subject to the relations*

$$x_\ell^2 = \begin{cases} x_{2\ell}, & 2\ell \leqslant n-1, \\ 0, & 2\ell > n-1. \end{cases}$$

So $\{x_{i_1} x_{i_2} \cdots x_{i_r} \mid r \geqslant 1, n-k \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}$ *is a* $\mathbf{Z}/2$*-base for* $\tilde{H}^*(V_{n,k})$. *The A-module* $H^*(V_{n,k})$ *is determined by* (2.2) *and the Cartan formula.*

We recall from [4] the following.

Theorem 2.2. $P_{n,k}$ *is a stable retract of* $V_{n,k}$, *that is, there is a stable map* $V_{n,k} \xrightarrow{r} P_{n,k}$ *such that the composite of stable maps*

$$P_{n,k} \xrightarrow{i} V_{n,k} \xrightarrow{r} P_{n,k}$$

is homotopic to the identity map on $P_{n,k}$. *Furthermore, r can be chosen so that* $\tilde{H}^*(P_{n,k}) \xrightarrow{r^*} \tilde{H}^*(V_{n,k})$ *maps* $\tilde{H}^*(P_{n,k})$ *onto* $\bar{H}^*(P_{n,k})$ *isomorphically.*

Let $\lambda \in [V_{n,k}, V_{n,k}]$ be as defined in Section 1. It has the following properties.

λ contains the self-maps which change the sign of any vector in each k-frame. It also contains the self-map $(y_1, \ldots, y_i, \ldots, y_j, \ldots, y_k) \to (y_1, \ldots, y_j, \ldots, y_i, \ldots, y_k)$ for any i,j with $1 \leqslant i < j \leqslant k$ $(k \geqslant 2)$. (∗)

The following fact is proved in [7].

$$\lambda^*: H^*(V_{n,k}) \to H^*(V_{n,k}) \quad \text{is the identity map.} \tag{2.3}$$

A special case of the facts above is the following. For i with $1 \leqslant i \leqslant k$ let $p_i: V_{n,k} \to V_{n,1} = S^{n-1}$ be the map $p_i(y_1, \ldots, y_i, \ldots, y_k) = y_i$; so $p_k = p$ where p is as in (2.1).

For $k \geqslant 2$, $\tilde{H}^{n-1}(V_{n,1} = S^{n-1}) = \mathbf{Z}/2 \xrightarrow{p_i^*} \tilde{H}^{n-1}(V_{n,k})$ maps the nonzero class $x_{n-1} \in \tilde{H}^{n-1}(S^{n-1})$ to the class $x_{n-1} \in \bar{H}^{n-1}(P_{n,k}) \subset \tilde{H}^{n-1}(V_{n,k})$ for each i with $1 \leqslant i \leqslant k$. (2.4)

This follows from Theorem 2.1(3), (∗) and Eq. (2.3).

3. Some other facts on $V_{n,k}$

Let $\mathbf{Z}_2 = \{1, g\}$ act on $V_{n,k}$ (resp. S^1) by letting $g: V_{n,k} \to V_{n,k}$ (resp. $g: S^1 \to S^1$) be the map which changes the sign of the last vector in each k-frame (resp. the antipodal map $e^{i\theta} \to -e^{i\theta}$). Consider the resulting space $V_{n,k} \times_{\mathbf{Z}_2} S^1$. In this section we recall, from [7], the construction of a space map

$V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\varphi} V_{n+1,k+1}$ for n odd, k even and the behavior of the induced map φ^* in mod 2 cohomology. In addition, we will also show here that if $V_{n,k}$ is neutral then there is a space map $f: V_{n,k} \times S^1 \to V_{n,k} \times_{\mathbf{Z}_2} S^1$ that induces a specific isomorphism $f^*: H^*(V_{n,k} \times_{\mathbf{Z}_2} S^1) \to H^*(V_{n,k} \times S^1)$. The main conclusion of the section, which is based on these facts, is Lemma 3.6 of which we want to make use to prove Theorem 1.1.

In this paragraph a topological space is meant a compact Hausdorff space. Let X be such a space. We suppose X is a $\mathbf{Z}_2$-space. Thus if g is the generator of $\mathbf{Z}_2$ then $g: X \to X$ is an involution. The mapping torus of $g: X \to X$ is defined to be the quotient space

$$T(g) = X \times I/(x,0) \sim (g(x),1)$$

where I denotes the closed interval [0,1]. $T(g)$ can be identified with $X \times_{\mathbf{Z}_2} S^1$ as follows. Elements of $T(g)$ are denoted by $\langle x, t \rangle$ and elements of $X \times_{\mathbf{Z}_2} S^1$ are denoted by $[x, e^{i\theta}]$. Then the map $\langle x, t \rangle \to [x, e^{i\pi t}]$ is a homeomorphism from $T(g)$ onto $X \times_{\mathbf{Z}_2} S^1$ which is easy to see. Let Y be another $\mathbf{Z}_2$-space and $f: X \to Y$ be an equivariant $\mathbf{Z}_2$-map. Then f induces a map $\bar{f}: X \times_{\mathbf{Z}_2} S^1 \to Y \times_{\mathbf{Z}_2} S^1$ given by

$$\bar{f}([x, e^{i\theta}]) = [f(x), e^{i\theta}].$$

Suppose $1_X \simeq g: X \to X$ and let $H: X \times I \to X$ be such a homotopy. Then H induces a map $\bar{H}: X \times_{\mathbf{Z}_2} S^1 \to X$ given by

$$\bar{H}([x, e^{i\pi t}]) = H(x,t) \qquad \text{for } 0 \leqslant t \leqslant 1. \tag{3.1}$$

Finally we note that the map $j: X \to X \times_{\mathbf{Z}_2} S^1$ given by

$$j(x) = [x, e^{i2\pi}] \tag{3.2}$$

is an embedding. Also, if $H: 1_X \simeq g$ then the composite $X \xrightarrow{j} X \times_{\mathbf{Z}_2} S^1 \xrightarrow{\bar{H}} X$ is the identity map.

For $n > k > 0$ there is a commutative diagram

$$\begin{array}{ccc} V_{n,k} & \xrightarrow{p} & V_{n,1} = S^{n-1} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_1} \\ V_{n+1,k+1} & \xrightarrow{p_k} & V_{n+1,1} = S^n \end{array} \tag{3.3}$$

where p, q are as in (2.1), $p_k(y_1, \ldots, y_k, y_{k+1}) = y_k$ and q_1 is induced by $\mathbf{R}^n \hookrightarrow \mathbf{R}^n \oplus \mathbf{R}^1 = \mathbf{R}^{n+1}$. Let $\mathbf{Z}_2 = \{1, g\}$ act on $V_{n,k}$ (resp. $V_{n+1,k+1}$) by letting g be the self-map which changes the sign of the last vector in each k-frame (resp. the second last vector in each $(k+1)$-frame) and act on S^{n-1} and S^n by letting g be the antipodal map. Then (3.3) is commutative diagram of $\mathbf{Z}_2$-maps. This results in a commutative diagram of induced maps:

$$\begin{array}{ccc} V_{n,k} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\bar{p}} & S^{n-1} \times_{\mathbf{Z}_2} S^1 \\ \downarrow{\scriptstyle \bar{q}} & & \downarrow{\scriptstyle \bar{q}_1} \\ V_{n+1,k+1} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\bar{p}_k} & S^n \times_{\mathbf{Z}_2} S^1 \end{array} \tag{3.4}$$

Now assume n is odd and k is even; so $n+1$ is even and $k+1$ is odd. Then $V_{n+1,k+1}$ and $S^n = V_{n+1,1}$ are neutral (by (1) in Section 1). So there are homotopies $H_k : 1 \simeq g : V_{n+1,k+1} \times I \to V_{n+1,k+1}$ and $H : 1 \simeq g : S^n \times I \to S^n$. It is shown in [7] that H_k and H can be chosen so that they are compatible with respect to the map p_k in (3.3), that is, there is a commutative diagram

$$\begin{array}{ccc} V_{n+1,k+1} \times I & \xrightarrow{p_k \times id} & S^n \times I \\ \downarrow{\scriptstyle H_k} & & \downarrow{\scriptstyle H} \\ V_{n+1,k+1} & \xrightarrow{p_k} & S^n \end{array}$$

This results in a commutative diagram

$$\begin{array}{ccc} V_{n+1,k+1} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\bar{p}_k} & S^n \times_{\mathbf{Z}_2} S^1 \\ \downarrow{\scriptstyle \bar{H}_k} & & \downarrow{\scriptstyle \bar{H}} \\ V_{n+1,k+1} & \xrightarrow{p_k} & S^n \end{array} \tag{3.5}$$

where $\bar{H}_k, \bar{H}$ are defined from H_k, H as in (3.1). Composing (3.5) with (3.4) we get a commutative diagram

$$\begin{array}{ccc} V_{n,k} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\bar{p}} & S^{n-1} \times_{\mathbf{Z}_2} S^1 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} \\ V_{n+1,k+1} & \xrightarrow{p_k} & S^n \end{array} \tag{3.6}$$

where $\varphi = \bar{H}_k \bar{q}, \varphi_1 = \bar{H}\bar{q}_1$. Note that $q = \varphi j$, $q_1 = \varphi_1 j_1$ where $j : V_{n,k} \to V_{n,k} \times_{\mathbf{Z}_2} S^1$, $j_1 : S^{n-1} \to S^{n-1} \times_{\mathbf{Z}_2} S^1$ are as given by (3.2).

We stress that we get φ and φ_1 only under the assumption that $n+1$ is even and $k+1$ is odd. The map $\bar{p}$ is defined for all $n > k > 0$.

For arbitrary n and k with $n > k > 0$ there is a fibration

$$V_{n,k} \xrightarrow{j} V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\pi} S^1$$

where $\pi([y, e^{i\theta}]) = e^{i2\theta}$. Also, we have a map of fiber spaces

$$\begin{array}{ccccc} V_{n,k} & \xrightarrow{j} & V_{n,k} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\pi} & S^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} & & \| \\ S^{n-1} & \xrightarrow{j_1} & S^{n-1} \times_{\mathbf{Z}_2} S^1 & \xrightarrow{\pi_1} & S^1 \end{array} \tag{3.7}$$

The following is proved in [7].

Lemma 3.1. (1) *For any* (n,k) *with* $n>k>0$, $j^*: H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)\to H^*(V_{n,k})$ *is onto, and there is a splitting map* $\sigma: H^*(V_{n,k})\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$, *which is an algebra homomorphism over* $\mathbf{Z}/2$, *that is,* $j^*\sigma = 1_{H^*(V_{n,k})}$.

(2) *For any such* σ, *the map* $\mu: H^*(V_{n,k})\otimes H^*(S^1)\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ *given by*

$$\mu(x\otimes y) = \sigma(x)\cdot\pi^*(y)$$

is an algebra isomorphism.

(3) *If we denote by* σ_1 *and* μ_1 *the maps for the case* $V_{n,1}=S^{n-1}$, *then we can choose* σ (*for* $k\geqslant 2$) *and* σ_1 *so that* $\bar{p}^*\sigma_1=\sigma p^*$; *thus* $\mu(p^*\otimes 1)=\bar{p}^*\mu_1$.

We fix a σ and a σ_1 such that $\bar{p}^*\sigma_1=\sigma p^*$ as in Lemma 3.1(3). Simply denote the class $\sigma(x)\cdot\pi^*(y)\in H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ by $\sigma(x)y$. Since the $\mathbf{Z}/2$-module $H^*(S^1)$ is generated by $\{1,\gamma\}$ where $|\gamma|=1$, basis elements in $H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ are either $\sigma(x)=\sigma(x)\cdot 1$ or $\sigma(x)\gamma$ where x is a basis element in $H^*(V_{n,k})$ as in Theorem 2.1(4). Here, and also in later sections, if $y\in H^{\ell}(Y)$ then $|y|$ denotes the number ℓ.

For n odd and k even, consider the map $\varphi: V_{n,k}\times_{\mathbf{Z}_2}S^1\to V_{n+1,k+1}$ in (3.6).

Proposition 3.2. $\varphi^*: H^n(V_{n+1,k+1})\to H^n(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ *maps* x_n *to* $\sigma(x_{n-1})\gamma$ *where* $x_n\in H^n(V_{n+1,k+1})$ (*resp.* $x_{n-1}\in H^{n-1}(V_{n,k})$) *is the normal class* $x_n\in\bar{H}^n(P_{n+1,k+1})\subset\tilde{H}^n(V_{n+1,k+1})$ (*resp. the normal class* $x_{n-1}\in\bar{H}^{n-1}(P_{n,k})\subset H^{n-1}(V_{n,k})$) *as in Theorem* 2.1.

To prove Proposition 3.2 we need the following fact proved in [7].

Lemma 3.3. *Let* $\bar{q}_1: S^{n-1}\times_{\mathbf{Z}_2}S^1\to S^n\times_{\mathbf{Z}_2}S^1$ *be as in* (3.4). *Then,*

$$\bar{q}_1^*\neq 0: H^n(S^n\times_{\mathbf{Z}_2}S^1)=\mathbf{Z}/2\to H^n(S^{n-1}\times_{\mathbf{Z}_2}S^1)=\mathbf{Z}/2.$$

Proof of Proposition 3.2. Consider the commutative diagram

$$\begin{array}{ccc} V_{n,k}\times_{\mathbf{Z}_2}S^1 & \xrightarrow{\ \bar{p}\ } & S^{n-1}\times_{\mathbf{Z}_2}S^1 \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi_1} \\ V_{n+1,k+1} & \xrightarrow{\ p_k\ } & S^n \end{array}$$

as in (3.6). Recall, from Theorem 2.1, that the generator of $H^n(S^n=P_{n+1,1}=V_{n+1,1})=\mathbf{Z}/2$ (resp. $H^{n-1}(S^{n-1}=P_{n,1}=V_{n,1})=\mathbf{Z}/2$) is also denoted by x_n (resp. x_{n-1}). By (2.4), $p_k^*(x_n)=x_n$ and $p^*(x_{n-1})=x_{n-1}$ where p is as in (3.3). Also recall that σ and σ_1 are chosen to satisfy $\bar{p}^*\sigma_1=\sigma p^*$. φ_1 above is the composite $S^{n-1}\times_{\mathbf{Z}_2}S^1\xrightarrow{\bar{q}_1}S^n\times_{\mathbf{Z}_2}S^1\xrightarrow{\bar{H}}S^n$ where $\bar{H}$ is as in (3.5). The composite $S^n\xrightarrow{j}S^n\times_{\mathbf{Z}_2}S^1\xrightarrow{\bar{H}}S^n$ is the identity map on S^n. So $j^*\bar{H}^*(x_n)=x_n$ and this implies $\bar{H}^*(x_n)$ is the generator of $H^n(S^n\times_{\mathbf{Z}_2}S^1)=\mathbf{Z}/2$ which by Lemma 3.3, is mapped by $\bar{q}_1^*$ to the generator of

$H^n(S^{n-1}\times_{\mathbf{Z}_2}S^1)=\mathbf{Z}/2$ which, by Lemma 3.1, is the class $\sigma_1(x_{n-1})\gamma$. So $\varphi_1^*(x_n)=\bar{q}_1^*\bar{H}^*(x_n)=\sigma_1(x_{n-1})\gamma$. Then

$$\begin{aligned}\varphi^*(x_n)=\varphi^*p_k^*(x_n)=\bar{p}^*\varphi_1^*(x_n)&=\bar{p}^*(\sigma_1(x_{n-1})\gamma)\\&=(\bar{p}^*\sigma_1(x_{n-1}))\gamma\\&=\sigma p^*(x_{n-1})\gamma\\&=\sigma(x_{n-1})\gamma.\end{aligned}$$

This proves Proposition 3.2.

We remark that Proposition 3.2 here, if imposed with the additional condition $n\geqslant 2k+1$, is Proposition 3.13 of [7]. What we have shown above for Proposition 3.2 is that Proposition 3.13 of [7] actually is also true without the condition $n\geqslant 2k+1$ if $x_n\in H^n(V_{n+1,k+1})$ and $x_{n-1}\in H^{n-1}(V_{n,k})$ are chosen to be the normal classes as in Proposition 3.2.

Note that the splitting map $\sigma:H^*(V_{n,k})\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ in Proposition 3.2 is chosen to satisfy $\bar{p}^*\sigma_1=\sigma p^*$ for some splitting map $\sigma_1:H^*(S^{n-1})\to H^*(S^{n-1}\times_{\mathbf{Z}_2}S^1)$ and Proposition 3.2 is stated and proved for such splitting maps. We will show below that Proposition 3.2 is also true for any splitting map σ' not necessarily satisfying $\bar{p}^*\sigma_1=\sigma' p^*$ for some σ_1. This will be relevant in the main conclusion Lemma 3.6 that follows.

Corollary 3.4. *Still assume n is odd and k is even. Let $\sigma':H^*(V_{n,k})\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ be any splitting map to j^* as in Lemma* 3.1(1). *Then $\varphi^*:H^n(V_{n+1,k+1})\to H^n(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ maps x_n to $\sigma'(x_{n-1})\gamma$ where x_n,x_{n-1} are as in Proposition* 3.2.

Proof. Let $\sigma:H^*(V_{n,k})\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ be a splitting map as in Proposition 3.2 so that $\varphi^*(x_n)=\sigma(x_{n-1})\gamma$. It suffices to show $\sigma(x_{n-1})\gamma=\sigma'(x_{n-1})\gamma$. Since both the composites

$$H^{n-1}(V_{n,k})\xrightarrow{\sigma'}H^{n-1}(V_{n,k}\times_{\mathbf{Z}_2}S^1)\xrightarrow{j^*}H^{n-1}(V_{n,k})$$

$$H^{n-1}(V_{n,k})\xrightarrow{\sigma}H^{n-1}(V_{n,k}\times_{\mathbf{Z}_2}S^1)\xrightarrow{j^*}H^{n-1}(V_{n,k})$$

are equal to the identity map it follows that $\sigma'(x_{n-1})-\sigma(x_{n-1})\in\ker j^*$. $\ker j^*$ is the $\mathbf{Z}/2$-submodule $\{y\gamma\mid y\in H^{n-2}(V_{n,k})\}$ by Lemma 3.1(2). So $\sigma'(x_{n-1})=\sigma(x_{n-1})+y\gamma$ for some y. Then $\sigma'(x_{n-1})\gamma=\sigma(x_{n-1})\gamma+y\gamma^2=\sigma(x_{n-1})\gamma$ since $\gamma^2=0$. This proves Corollary 3.4. □

In the remainder of this section we discuss some results on $V_{n,k}$ under the assumption that $V_{n,k}$ is neutral. We also assume $n-k\geqslant 2$ so that $V_{n,k}$ is simply connected.

Since $V_{n,k}$ is neutral there is a homotopy $H:1\simeq g:V_{n,k}\times I\to V_{n,k}$. Consider the induced map $\bar{H}:V_{n,k}\times_{\mathbf{Z}_2}S^1\to V_{n,k}$ as in (3.1). Since the composite $V_{n,k}\xrightarrow{j}V_{n,k}\times_{\mathbf{Z}_2}S^1\xrightarrow{\bar{H}}V_{n,k}$ is $1_{V_{n,k}}$, the induced map $\bar{H}^*:H^*(V_{n,k})\to H^*(V_{n,k}\times_{\mathbf{Z}_2}S^1)$ is a splitting map to j^*. Let $\sigma'=\bar{H}^*$.

Consider the product space $V_{n,k}\times S^1$. We have $H^*(V_{n,k}\times S^1)=H^*(V_{n,k})\otimes H^*(S^1)$. Thus basis elements in $H^*(V_{n,k}\times S^1)$ are either $x=x\otimes 1$ or $x\gamma=x\otimes\gamma$ where x is a basis element in $H^*(V_{n,k})$ as in Theorem 2.1(4). Note that, since $V_{n,k}$ is simply connected, $H^1(V_{n,k}\times S^1)\cong\mathbf{Z}_2$ generated by

$\gamma = 1 \otimes \gamma$. Let $S^1 \xrightarrow{i_2} V_{n,k} \times S^1$, $V_{n,k} \xrightarrow{i_1} V_{n,k} \times S^1$ be the maps defined by $i_2(e^{i\theta}) = (*, e^{i\theta})$, $i_1(y) = (y, e^{i2\pi})$ where $*$ is a base point in $V_{n,k}$. If $V_{n,k} \times S^1 \xrightarrow{q} S^1$ is a map such that $qi_2 : S^1 \to S^1$ is a degree 1 map then $q^* : H^1(S^1) \to H^1(V_{n,k} \times S^1)$ maps the generator γ of $H^1(S^1) = \mathbf{Z}/2$ to γ. This is clear. Note that $qi_1 \simeq 0 : V_{n,k} \to S^1$ since $V_{n,k}$ is simply connected.

Lemma 3.5. *With the assumptions and notations above, there is a space map $f : V_{n,k} \times S^1 \to V_{n,k} \times_{\mathbf{Z}_2} S^1$ such that $f^* : H^*(V_{n,k} \times_{\mathbf{Z}_2} S^1) \to H^*(V_{n,k} \times S^1)$ has $f^*(\sigma'(x)) = x$ and $f^*(\sigma'(x)\gamma) = x\gamma$.*

Proof. Let $q_1 : V_{n,k} \times I \to V_{n,k} \times S^1$ and $q_2 : V_{n,k} \times I \to V_{n,k} \times_{\mathbf{Z}_2} S^1$ be the quotient maps defined by $q_1(y, t) = (y, e^{i2\pi t})$ and $q_2(y, t) = [y, e^{i\pi t}]$. Define a map $f' : V_{n,k} \times I \to V_{n,k} \times I$ by

$$f'(y,t) = \begin{cases} (y, 2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ (H(y, 2t-1), 1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

f' induces a map $f : V_{n,k} \times S^1 \to V_{n,k} \times_{\mathbf{Z}_2} S^1$ such that the square

$$\begin{array}{ccc} V_{n,k} \times I & \xrightarrow{f'} & V_{n,k} \times I \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ V_{n,k} \times S^1 & \xrightarrow{f} & V_{n,k} \times_{\mathbf{Z}_2} S^1 \end{array}$$

is commutative. It is easy to see that the composite $V_{n,k} \xrightarrow{i_1} V_{n,k} \times S^1 \xrightarrow{f} V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\bar{H}} V_{n,k}$ is the identity map. The composite $S^1 \xrightarrow{i_2} V_{n,k} \times S^1 \xrightarrow{f} V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\bar{H}} V_{n,k}$ is homotopically trivial since $V_{n,k}$ is simply connected. These imply $f^*(\bar{H}^*(x)) = f^*(\sigma'(x)) = x$ for $x \in H^*(V_{n,k})$. Let $V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\pi} S^1$ be as in (3.7). By Lemma 3.1(2), $\pi^*(\gamma) = \sigma'(1) \otimes \gamma = 1 \otimes \gamma = 1 \cdot \gamma = \gamma$. The composite $S^1 \xrightarrow{i_2} V_{n,k} \times S^1 \xrightarrow{f} V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\pi} S^1$ is the map

$$\begin{cases} e^{i\pi t} \to e^{i2\pi t}, & 0 \leqslant t \leqslant 1, \\ e^{i\pi t} \to e^{i2\pi}, & 1 \leqslant t \leqslant 2, \end{cases}$$

and so is a degree one map. Then $f^*(\gamma) = f^*(\pi^*(\gamma)) = \gamma$. So $f^*(\sigma'(x)\gamma) = f^*(\sigma'(x)) f^*(\gamma) = x\gamma$. This proves Lemma 3.5.

Now assume further that n is odd and k is even so that we can consider the map $\varphi : V_{n,k} \times_{\mathbf{Z}_2} S^1 \to V_{n+1,k+1}$ as in (3.6). Let $f : V_{n,k} \times S^1 \to V_{n,k} \times_{\mathbf{Z}_2} S^1$ be the map in Lemma 3.5. Consider the composite

$$\psi : P_{n,k} \times S^1 \xrightarrow{i \times id} V_{n,k} \times S^1 \xrightarrow{f} V_{n,k} \times_{\mathbf{Z}_2} S^1 \xrightarrow{\varphi} V_{n+1,k+1} \tag{3.8}$$

where i is as in (2.1). Note that $H^{n-k}(P_{n,k} \times S^1) = \mathbf{Z}/2(x_{n-k})$, $H^{\ell}(P_{n,k} \times S^1) = \mathbf{Z}/2(x_{\ell}) \oplus \mathbf{Z}/2(x_{\ell-1}\gamma)$ for $n-k+1 \leqslant \ell \leqslant n-1$ and $H^n(P_{n,k} \times S^1) = \mathbf{Z}/2(x_{n-1}\gamma)$. Here if $\{y_1, y_2, \ldots, y_m\}$ is a $\mathbf{Z}/2$-base for

$$H^{\ell}(X) = \underbrace{\mathbf{Z}/2 \oplus \cdots \oplus \mathbf{Z}/2}_{m}$$

then we write $H^\ell(X) = \mathbf{Z}/2(y_1) \oplus \ldots \oplus \mathbf{Z}/2(y_m)$. Consider the set of normal classes $\{x_\ell \mid n-k \leqslant \ell \leqslant n\}$ in $H^*(V_{n+1,k+1})$. Then $\psi^*(x_\ell) = \delta_\ell x_\ell$ for $\ell = n-k$, $\psi^*(x_\ell) = \delta_\ell x_\ell + \varepsilon_\ell x_{\ell-1}\gamma$ for $n-k+1 \leqslant \ell \leqslant n-1$ and $\psi^*(x_n) = \varepsilon_\ell x_{n-1}\gamma$ for $\ell = n$ where $\delta_\ell, \varepsilon_\ell$ are either 0 or 1.

Lemma 3.6. (i) $\psi^*(x_{n-k}) = x_{n-k}$, $\psi^*(x_\ell) = x_\ell + \varepsilon_\ell x_{\ell-1}\gamma$ *for* $n-k+1 \leqslant \ell \leqslant n-1$ *and* $\psi^*(x_n) = x_{n-1}\gamma$, *that is*, $\delta_\ell = 1$ *for* $n-k \leqslant \ell \leqslant n-1$ *and* $\varepsilon_n = 1$.

(ii) *If* n *and* k *satisfy* $n-k \geqslant 3$ *and* $2k-1 > n$ *then* $\psi^*(x_\ell) = x_\ell$, *that is*, $\varepsilon_\ell = 0$ *for* $n-k+1 \leqslant \ell \leqslant n-1$.

To prove Lemma 3.6 we recall a notion due to James [4]. For a space X, a pair of cohomology classes $x, y \in \tilde{H}^*(X)$ are said to be evenly connected if $Sq^t x = y$ for some even $t \geqslant 0$. This nonsymmetric relation generates an equivalent relation on $\tilde{H}^*(X)$; we describe x, y are evenly related if they are equivalent in this sense. James observes the following in [4]. Recall that n is odd and k is even.

All the normal classes x_s with s odd and $n-k < s < n$ are evenly related in $\tilde{H}^*(V_{n+1,k+1})$. (3.9)

This follows from the following relations (by (2.2)) when defined:

$$Sq^4 x_{8i-1} = x_{8i+3}, \qquad Sq^2 x_{8i+3} = x_{8i+5},$$

$$Sq^4 x_{8i+5} = x_{8i+9}, \qquad Sq^2 x_{8i+7} = x_{8i+9}. \tag{3.10}$$

We also need the following which is easy to see from (2.2).

(3.11)

(i) For t even and s odd with $n-k < s < n$, $Sq^t x_s \neq 0$ in $\tilde{H}^*(V_{n+1,k+1})$ if and only if $Sq^t x_{s-1} \neq 0$ in $\tilde{H}^*(P_{n,k})$.

(ii) $Sq^1 x_\ell = 0$ for ℓ even and $Sq^1 x_{\ell-1} = x_\ell$ for $n-k+1 \leqslant \ell = 2q < n$ in $\tilde{H}^*(V_{n+1,k+1})$ and also in $\tilde{H}^*(P_{n,k})$.

Proof of Lemma 3.6. First we prove (i). Consider the diagram

$$\begin{array}{ccccccc}
P_{n,k} & \xrightarrow{\ i\ } & V_{n,k} & & & & \\
\big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle i_1} & & & & \\
P_{n,k}\times S^1 & \xrightarrow{\ i\times id\ } & V_{n,k}\times S^1 & \xrightarrow{\ f\ } & V_{n,k}\times_{\mathbf{Z}_2} S^1 & \xrightarrow{\ \varphi\ } & V_{n+1,k+1}
\end{array}$$

where $i_1(y) = (y, e^{i2\pi})$ as in Lemma 3.5 and $i_2 = i_1 \mid P_{n,k}$. It is clear that $i_2^*(x_\ell) = x_\ell$ and $i_2^*(x_{\ell-1}\gamma) = 0$. From the construction of f in the proof of Lemma 3.5 we see fi_1 is the embedding $j: V_{n,k} \to V_{n,k} \times_{Z_2} S^1$ as in (3.2), and recall that the composite $V_{n,k} \xrightarrow{j} V_{n,k} \times_{Z_2} S^1 \xrightarrow{\varphi} V_{n+1,k+1}$ is the map $V_{n,k} \xrightarrow{q} V_{n+1,k+1}$ in (2.1). So for $n-k \leqslant \ell \leqslant n-1$, $i^* i_1^* f^* \varphi^*(x_\ell) = i^* j^* \varphi^*(x_\ell) = i^* q^*(x_\ell) = x_\ell$. Then

$$
\begin{aligned}
x_\ell = i^* i_1^* f^* \varphi^*(x_\ell) = i_2^*(i \times id)^* f^* \varphi^*(x_\ell) &= i_2^* \psi^*(x_\ell) \\
&= i_2^*(\delta_\ell x_\ell + \varepsilon_\ell x_{\ell-1}\gamma) \\
&= \delta_\ell x_\ell
\end{aligned}
$$

for $n-k \leqslant \ell \leqslant n-1$ where ε_{n-k} is set to be zero. Thus $\delta_\ell = 1$ for $n-k \leqslant \ell \leqslant n-1$. This proves the first two conclusions of Lemma 3.6(i).

By Theorem 2.1(1), $i^*(x_{n-1}) = x_{n-1}$. By Lemma 3.5, $f^*(\sigma'(x_{n-1})\gamma) = x_{n-1}\gamma$. By Corollary 3.4, $\varphi^*(x_n) = \sigma'(x_{n-1})\gamma$. Then

$$
\begin{aligned}
\psi^*(x_n) = (i \times id)^* f^* \varphi^*(x_n) = (i \times id)^* f^*(\sigma'(x_{n-1})\gamma) &= (i \times id)^*(x_{n-1}\gamma) \\
&= i^*(x_{n-1})\gamma = x_{n-1}\gamma.
\end{aligned}
$$

This proves Lemma 3.6(i).

Next we prove Lemma 3.6(ii). First we show

$$x_{n-k} \text{ is evenly related to some } x_{s'} \text{ with } s' \text{ odd and } n-k < s' < n. \tag{$*$}$$

If $n-k = 8i+3$ or $8i+7$ then $Sq^2 x_{n-k} = x_{n-k+2}$ by (3.10), and $n-k+2$ is odd with $n-k+2 < n$ (since $2k-1 > n \geqslant k+3$ implies $k > 3$). If $n-k = 8i+5$ then $Sq^4 x_{n-k} = x_{n-k+4}$ again by (3.10), and $n-k+4$ is odd with $n-k+4 < k-1+4 = k+3 \leqslant n$ (as $2k-1 > n$). If $n-k = 8i+1$ then $i \geqslant 1$ since $n-k \geqslant 3$, and if we let $n-k = 2^{p+1}q + 2^p + 1$ then $q \geqslant 0$, $p \geqslant 3$. Note that $k > 2^p$ since $2k-1 > n$. We have $Sq^{2^p} x_{n-k} = x_{n-k+2^p}$ (by (2.2)). $n-k+2^p$ is also odd and has $n-k+2^p < n$ since $k > 2^p$. This proves ($*$).

For odd s and s_1 with $n-k < s < s_1 < n$ suppose $Sq^t x_s = x_{s_1}$ in $\tilde{H}^*(V_{n+1,k+1})$ for some even t. Then $Sq^t x_s = x_{s_1}$ and $Sq^t x_{s-1} = x_{s_1-1}$ in $\tilde{H}^*(P_{n,k})$ by (3.11)(i). From Lemma 3.6(i) we have

$$
\begin{aligned}
x_{s_1} + \varepsilon_{s_1} x_{s_1-1}\gamma = \psi^*(x_{s_1}) = \psi^*(Sq^t x_s) &= Sq^t \psi^*(x_s) \\
&= Sq^t(x_s + \varepsilon_s x_{s-1}\gamma) \\
&= Sq^t x_s + \varepsilon_s Sq^t(x_{s-1})\gamma \\
&= x_{s_1} + \varepsilon_s x_{s_1-1}\gamma.
\end{aligned}
\tag{$**$}
$$

So $\varepsilon_s = \varepsilon_{s_1}$. It follows, then, from the "evenly related" equivalence relation and from (3.9), that all ε_s with s odd and $n-k < s < n$ are equal. By ($*$), x_{n-k} is evenly related to some $x_{s'}$ with s' odd and $n-k < s' < n$, say $Sq^{t'} x_{n-k} = x_{s'}$. By making a similar calculation as ($**$) we see $\varepsilon_{s'} = 0$ since $\psi^*(x_{n-k}) = x_{n-k}$ by Lemma 3.6(i). Thus $\varepsilon_s = 0$ for all odd s with $n-k < s < n$. This proves Lemma 3.6(ii) for x_ℓ with ℓ odd and $n-k < \ell < n$.

If ℓ is even and $n-k+1 \leqslant \ell \leqslant n-1$ then $\ell - 1$ is odd and $n-k \leqslant \ell - 1 < n$. We have just shown above that $\psi^*(x_{\ell-1}) = x_{\ell-1}$. By (3.11)(ii), $Sq^1 x_{\ell-1} = x_\ell$ in $\tilde{H}^*(V_{n+1,k+1})$ and also in $\tilde{H}^*(P_{n,k})$. Then $\psi^*(x_\ell) = \psi^*(Sq^1 x_{\ell-1}) = Sq^1\psi^*(x_{\ell-1}) = Sq^1 x_{\ell-1} = x_\ell$. This proves Lemma 3.6(ii). This completes the proof of Lemma 3.6.

From the proof of Lemma 3.6 above we see the map $\psi: P_{n,k} \times S^1 \to V_{n+,k+1}$ in (3.8) has the following property.

$$\text{The composite } P_{n,k} \xrightarrow{i_1} P_{n,k} \times S^1 \xrightarrow{\psi} V_{n+1,k+1} \text{ is the composite } P_{n,k} \xrightarrow{\rho} P_{n+1,k+1} \xrightarrow{i} V_{n+1,k+1} \text{ where } i_1(y) = (y, e^{i2\pi}) \text{ and } \rho \text{ is the inclusion map } P_{n,k} = P^{n-1}_{n-k} \hookrightarrow P^n_{n-k} = P_{n+1,k+1}. \tag{3.12}$$

Note that the map i_1 here is the map i_2 in the proof of Lemma 3.6 above.

4. $P^n_\ell \bar{\times} S^1$ as a cofiber

For a space X with base point $*$ we use i_1, i_2, p_1 to denote the inclusions $X \xrightarrow{i_1} X \times S^1$, $S^1 \xrightarrow{i_2} X \times S^1$ and the projection $X \times S^1 \xrightarrow{p_1} X$ defined by $i_1(x) = (x, e^{i2\pi})$, $i_2(e^{i\theta}) = (*, e^{i\theta})$ and $p_1(x, e^{i\theta}) = x$. So $p_1 i_2 = *$. X and S^1 are considered as subspaces of $X \times S^1$ via these inclusions. The quotient space $X \times S^1/S^1$ is denoted by $X \bar{\times} S^1$. Elements in $X \bar{\times} S^1$ are denoted by $[x, e^{i\theta}]$. Let $X \times S^1 \xrightarrow{q} X \bar{\times} S^1$ be the quotient map. So $q(x, e^{i\theta}) = [x, e^{i\theta}]$. The composite $X \xrightarrow{i_1} X \times S^1 \xrightarrow{q} X \bar{\times} S^1$ is denoted by $\bar{i}_1$. Define $\bar{p}_1 : X \bar{\times} S^1 \to X$ by $\bar{p}_1[x, e^{i\theta}] = x$. Since $p_1 i_2 = *$, $\bar{p}_1$ is well defined. Let $\bar{q}_1 : X \bar{\times} S^1 \to X \wedge S^1 = \Sigma X$ be the quotient map defined by $\bar{q}_1([x, e^{i\theta}]) = x \wedge e^{i\theta}$. Then

$$\begin{aligned} &\text{(i) } X \xrightarrow{\bar{i}_1} X \bar{\times} S^1 \xrightarrow{\bar{p}_1} X \text{ is the identity map } 1_X, \\ &\text{(ii) } X \xrightarrow{\bar{i}} X \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma X \text{ is a cofibration.} \end{aligned} \tag{4.1}$$

These are clear. We have $H^*(X \times S^1) = H^*(X) \otimes H^*(S^1)$. Thus basis elements of $H^*(X \times S^1)$ are either $x = x \otimes 1$ or $x\gamma = x \otimes \gamma$ where x is a basis element in $H^*(X)$ and γ is the generator of $H^1(S^1) = \mathbf{Z}/2$. $H^*(X \bar{\times} S^1) \xrightarrow{q^*} H^*(X \times S^1)$ is monomorphic. Elements in $H^*(X \bar{\times} S^1)$ will be identified with the corresponding elements in $H^*(X \times S^1)$ via q^*. Thus basis elements of $H^*(X \bar{\times} S^1)$ are either of the form $x = x \otimes 1$ for $x \in H^*(X)$ or of the form $x\gamma$ for $x \in \tilde{H}^*(X)$. The induced maps $H^*(X \bar{\times} S^1) \xrightarrow{\bar{i}_1^*} H^*(X)$ and $H^*(X) \xrightarrow{\bar{p}_1^*} H^*(X \bar{\times} S^1)$ satisfy $\bar{i}_1^*(x) = x$, $\bar{i}_1^*(x\gamma) = 0$, $\bar{p}_1^*(x) = x$. The induced map $\tilde{H}^*(\Sigma X) \xrightarrow{\bar{q}_1^*} \tilde{H}^*(X \bar{\times} S^1)$ has $\bar{q}_1^*(\Sigma x) = x\gamma$ where $\Sigma x \in \tilde{H}^*(\Sigma X)$ is the image of $x \in \tilde{H}^*(X)$ under the suspension isomorphism $\tilde{H}^*(X) \xrightarrow{\Sigma} \tilde{H}^*(\Sigma X)$. These are also clear.

We specialize to $X = P^n_\ell = P_{n+1,n+1-\ell}$, the stunted projective space $P^n/P^{\ell-1}$. In this section we use the notation P^n_ℓ for $P_{n+1,n+1-\ell}$. We will only consider P^n_ℓ for $n \geqslant \ell \geqslant 2$.

It is well known that $\Sigma(P^n_\ell \times S^1) \simeq S^2 \vee \Sigma P^n_\ell \vee \Sigma^2 P^n_\ell$. Clearly this implies $\Sigma(P^n_\ell \bar{\times} S^1) \simeq \Sigma P^n_\ell \vee \Sigma^2 P^n_\ell$.

Lemma 4.1. *For $n \leqslant 2\ell - 2$, $P_\ell^n \bar{\times} S^1$ is already the wedge of P_ℓ^n and ΣP_ℓ^n, that is, there is a homotopy equivalence $P_\ell^n \bar{\times} S^1 \xrightarrow{h} P_\ell^n \vee \Sigma P_\ell^n$. And we can choose an h such that $P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1 \xrightarrow{h} P_\ell^n \vee \Sigma P_\ell^n$ is the obvious inclusion map, the composite $P_\ell^n \bar{\times} S^1 \xrightarrow{h} P_\ell^n \vee \Sigma P_\ell^n \xrightarrow{q_2} \Sigma P_\ell^n$ is homotopic to $P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P_\ell^n$ and the composite $P_\ell^n \bar{\times} S^1 \xrightarrow{h} P_\ell^n \vee \Sigma P_\ell^n \xrightarrow{q_1} P_\ell^n$ is homotopic to $P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{p}_1} P_\ell^n$ where $\bar{q}_1, \bar{p}_1$ are as in* (4.1) *and q_i, $i = 1,2$, are projection maps.*

This follows from the fact $\Sigma(P_\ell^n \bar{\times} S^1) \simeq \Sigma P_\ell^n \vee \Sigma^2 P_\ell^n$ and the fact that a $(k-1)$-connected finite CW-complex $Y = Y^m$, with $k \geqslant 2, 2k - 2 \geqslant m$, can be homotopically desuspended uniquely.

Note that if $n \geqslant 2\ell$ then $P_\ell^n \bar{\times} S^1$ is not the wedge of P_ℓ^n and ΣP_ℓ^n as $x_\ell \cdot x_\ell \gamma = x_{2\ell}\gamma \neq 0$ in $H^*(P_\ell^n \bar{\times} S^1)$.

Consider the cofibration $P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P_\ell^n$ as in (ii) of (4.1) for $X = P_\ell^n$. If $n \leqslant 2\ell - 2$ then $P_\ell^n \bar{\times} S^1$ is the cofiber of any homotopically trivial map $P_\ell^n \xrightarrow{t} P_\ell^n$ as shown by Lemma 4.1 so that there is a cofiber sequence

$$P_\ell^n \xrightarrow{t} P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P_\ell^n.$$

If $n \geqslant 2\ell$ then it is unlikely that $P_\ell^n \bar{\times} S^1$ is the cofiber of any self-map of P_ℓ^n. This is a part of what we will discuss next. Indeed, there are two main themes in this section. The first one is to show that, for $2 \leqslant \ell \leqslant n/2$ and $n \leqslant 3\ell - 3$, $P_\ell^n \bar{\times} S^1$ is the cofiber of a map $P_{2\ell-1}^n \xrightarrow{f} P_\ell^n \vee \Sigma P_\ell^{2\ell-2}$ so that there is a cofiber sequence

$$P_{2\ell-1}^n \xrightarrow{f} P_\ell^n \vee \Sigma P_\ell^{2\ell-2} \xrightarrow{j} P_\ell^n \bar{\times} S^1 \xrightarrow{\delta} \Sigma P_{2\ell-1}^n$$

where j, when restricted to P_ℓ^n, is $P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1$ and δ is the composite

$$P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P_\ell^n \xrightarrow{p} \Sigma P_{2\ell-1}^n$$

with p the collapsing map. Precise description of the result will be given in Proposition 4.7. The second one is to assume a map $P_\ell^n \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell}$ having certain properties with respect to the cofiber sequence above from which we want to construct a space map $\Sigma P_{2\ell-1}^n \xrightarrow{g} C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P_\ell^{2\ell-2}$, where $\phi = \psi_1(j|\Sigma P_\ell^{2\ell-2})$, such that $\tilde{H}^*(C_\phi) = \tilde{H}^*(V_{n+2,n+2-\ell}) \oplus \tilde{H}^*(\Sigma^2 P_\ell^{2\ell-2}) \xrightarrow{g^*} \tilde{H}^*(\Sigma P_{2\ell-1}^n)$ maps all of $\tilde{H}^*(V_{n+2,n+2-\ell})$ to zero for $* \leqslant n+1$ except the normal class x_{n+1} on which $g^*(x_{n+1}) = \Sigma x_n$. Details are given in Assumption 4.8 and Proposition 4.10. These will be relevant to our proof of Theorem 1.1 in Section 5.

Before discussing these two main themes we give some preliminaries on maps from cofibrations that we need.

All spaces to be considered are pointed spaces with base points denoted by $*$ and have the homotopy types of CW-complexes. All maps between two such spaces will be base-point-preserving maps. If $f, g: X \to Y$ are two maps such that $f \underset{H}{\simeq} g$ then the homotopy H is understood to be a base-point- preserving homotopy.

For a space X the cone CX on X will be the reduced cone defined to be the quotient space $CX = X \times I/(X \times 1 \cup * \times I)$ where I is the closed interval [0, 1]. The reduced suspension ΣX is the quotient space $CX/X\times 0$. Elements in CX or in ΣX are denoted by $[x,t]$. For a map $X \xrightarrow{f} Y$, its cofiber $C_f = Y\cup_f CX$ is the quotient space $Y\cup CX/f(x)\sim[x,0]$ where "$\cup$" is the disjoint union. The image of $[x,t]\in CX$ in C_f under the quotient map $CX\to C_f$ is also denoted by $[x,t]$. We take $[*,t]$ to be the base point for CX or ΣX or $Y\cup_f CX$. Let $Y\cup_f CX \xrightarrow{q} \Sigma X$ be the quotient map defined by $q(Y)=*, q([x,t])=[x,t]$ for $x\in X$. Call q the natural pinching map. Then

$$X \xrightarrow{f} Y \xrightarrow{i} Y\cup_f CX \xrightarrow{q} \Sigma X \simeq C_i$$

is a cofibration sequence where i is the inclusion map.

Let $Y\cup_f CX$ be as above. Given two maps $Y\cup_f CX \xrightarrow{g'} Z$ and $\Sigma X \xrightarrow{h} Z$. Define a map $g'+h: Y\cup_f CX\to Z$ by

$$(g'+h)(y) = g'(y) \quad \text{for } y\in Y\subset Y\cup_f CX,$$

$$(g'+h)([x,t]) = \begin{cases} g'([x,2t]) & 0\leqslant t\leqslant \frac{1}{2} \\ h([x,2t-1]) & \frac{1}{2}\leqslant t\leqslant 1 \end{cases} \quad \text{for } x\in X. \tag{$*$}$$

Definition 4.2. Given two maps $g_0, g_1: Y\cup_f CX\to Z$. If there is a map $\Sigma X\xrightarrow{g} Z$ such that $g_1\simeq g_0+g: Y\cup_f CX\to Z$ then call g a homotopy difference of g_0 and g_1 and we denote this relation by $g\simeq g_1-g_0$.

Lemma 4.3 below is straightforward. (1) and (2) in Lemma 4.4 are well-known elementary facts.

Lemma 4.3. (1) *Let $Y\cup_f CX \xrightarrow{g'} Z$, $\Sigma X\xrightarrow{h} Z$ be as in $(*)$ and let $Z\xrightarrow{\alpha} W$ be a map from Z to another space W. Then*

(i) $(g'+h)\,|\,Y = g'\,|\,Y$, (ii) $\alpha(g'+h) = \alpha g' + \alpha h$.

(2) *Let*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{j} & Y\cup_f CX & \xrightarrow{q} & \Sigma X \\
\| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\bar\beta} & & \| \\
X & \xrightarrow{f'} & Y' & \xrightarrow{j'} & Y'\cup_{f'} CX & \xrightarrow{q'} & \Sigma X
\end{array}$$

be a map of cofiber sequences. If $g_0, g_1: Y'\cup_{f'} CX\to Z$ and $g:\Sigma X\to Z$ have the relation $g_1\simeq g_0+g$ then $g_1\bar\beta\simeq g_0\bar\beta+g$.

Lemma 4.4. (1) $(g'+h)^* = (g')^* + q^*h^*: \tilde H^*(Z,R)\to \tilde H^*(Y\cup_f CX, R)$ *for any coefficient ring R.*

(2) *Given two maps $Y\cup_f CX \xrightarrow{g_0, g_1} Z$. Let $\bar g_i = g_i\,|\,Y$, $i=0,1$. If $\bar g_0\simeq \bar g_1: Y\to Z$ then there is a map $\Sigma X\xrightarrow{g} Z$ such that $g\simeq g_1-g_0$.*

Let $H: Y\times I\to Z$ be a homotopy from $\bar{g}_0$ to $\bar{g}_1$. We recall the construction of a homotopy difference $g\simeq g_1-g_0$ for Lemma 4.4(2) with respect to H as follows.

$$g([x,t])=\begin{cases} g_0([x,-3t+1]) & 0\leqslant t\leqslant \frac{1}{3}, \\ H(f(x),3t-1) & \frac{1}{3}\leqslant t\leqslant \frac{2}{3}, \quad x\in X. \\ g_1([x,3t-2]) & \frac{2}{3}\leqslant t\leqslant 1 \end{cases} \tag{2$'$}$$

Next we will consider a special situation which is somewhat complicated. Again, given a map $X\xrightarrow{f} Y$. We suppose $Y=Y_1\vee Y_2$. Let $Y_1\vee Y_2\xrightarrow{p_i} Y_i$, for $i=1,2$, be the projection maps. Assume there is a map $Y\cup_f CX=(Y_1\vee Y_2)\cup_f CX\xrightarrow{r} Y_1$ such that the square

$$\begin{array}{ccc} Y_1\vee Y_2 & \xrightarrow{j} & (Y_1\vee Y_2)\cup_f CX \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle r} \\ Y_1 & = & Y_1 \end{array}$$

is commutative. Note that r is a retraction and $r|Y_2=*$. We further suppose given two maps $W=(Y_1\vee Y_2)\cup_f CX\xrightarrow{\psi_1} Z$ and $Y_1\xrightarrow{\phi_1} Z$ such that $\psi_1|Y_1\simeq\phi_1: Y_1\to Z$. Let $W\xrightarrow{\psi_2} Z$ be the composite

$$W=(Y_1\vee Y_2)\cup_f CX\xrightarrow{r} Y_1\xrightarrow{\phi_1} Z.$$

Clearly, $\psi_1\,|\,Y_1\simeq\psi_2|Y_1=\phi_1: Y_1\to Z$. Let $\phi=\phi_2=\psi_1\,|\,Y_2: Y_2\to Z$ and consider $C_\phi=Z\cup_\phi CY_2$. Let $W\xrightarrow{\bar{\psi}_i} C_\phi$, for $i=1,2$, be the maps defined as follows. $\bar{\psi}_1$ is the composite

$$W=(Y_1\vee Y_2)\cup_f CX\xrightarrow{\psi_1} Z\xrightarrow{j_1} C_\phi=Z\cup_\phi CY_2$$

where j_1 is the inclusion map. $\bar{\psi}_2$ is the composite

$$W=(Y_1\vee Y_2)\cup_f CX\xrightarrow{r} Y_1\xrightarrow{\phi_1} Z\xrightarrow{j_1} C_\phi.$$

It is clear that $\bar{\psi}_1|Y_1\simeq\bar{\psi}_2|Y_1$, $\bar{\psi}_1|Y_2\simeq 0$ and $\bar{\psi}_2|Y_2=*$. So $\bar{\psi}_1|Y\simeq\bar{\psi}_2|Y\colon Y\to C_\phi$. Let $Y_1\times I\xrightarrow{H} Z$ be a homotopy from $\psi_1\,|\,Y_1$ to ϕ_1. Then a homotopy $(Y=Y_1\vee Y_2)\times I\xrightarrow{\bar{H}} C_\phi=Z\cup_\phi CY_2$ from $\bar{\psi}_1\,|\,Y$ to $\bar{\psi}_2\,|\,Y$ can be given by

$$\text{(a)}\quad \bar{H}(y,t)=\begin{cases} H(y,t) & y\in Y_1 \\ [y,t] & y\in Y_2 \end{cases}\quad 0\leqslant t\leqslant 1.$$

By Lemma 4.4(2) there is a homotopy difference $g\simeq\bar{\psi}_2-\bar{\psi}_1:\Sigma X\to C_\phi$.

Lemma 4.5. *Under the above assumptions, if we choose the homotopy difference* $g \simeq \bar{\psi}_2 - \bar{\psi}_1$ *to be the map, as constructed in* (2)′, *with respect to the homotopy* $\bar{H}$ *in* (a), *then the composite*

$$\Sigma X \xrightarrow{g} C_\phi = Z \cup_\phi CY_2 \xrightarrow{q} \Sigma Y_2$$

is homotopic to the composite

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y = \Sigma Y_1 \vee \Sigma Y_2 \xrightarrow{\Sigma p_2} \Sigma Y_2.$$

Proof. It is easy to check that the homotopy difference g in (2)′ of Lemma 4.4 for $g_0 = \bar{\psi}_1$, $g_1 = \bar{\psi}_2$ with respect to the homotopy $\bar{H}$ in (a) is given by

$$g([x,t]) = \begin{cases} \psi_1([x, -3t+1]), & 0 \leqslant t \leqslant \frac{1}{3}, \\ \left.\begin{array}{ll} H(f(x), 3t-1) & \text{for } f(x) \in Y_1 \\ [f(x), 3t-1] & \text{for } f(x) \in Y_2 \end{array}\right\}, & \frac{1}{3} \leqslant t \leqslant \frac{2}{3},\ x \in X. \\ \phi_1(r([x, 3t-2])), & \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

So $qg : \Sigma X \xrightarrow{g} C_\phi = Z \cup_\phi CY_2 \xrightarrow{q} \Sigma Y_2$ is given by

$$qg([x,t]) = \begin{cases} * & \text{for } 0 \leqslant t \leqslant \frac{1}{3} \text{ and } \frac{2}{3} \leqslant t \leqslant 1 \\ [p_2 f(x), 3t-1] & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}. \end{cases}$$

Let $G : \Sigma X \times I \to \Sigma Y_2$ be the homotopy defined by

$$G([x,t],s) = \begin{cases} * & \text{for } 0 \leqslant t \leqslant s/3 \text{ and } (3-s)/3 \leqslant t \leqslant 1 \\ [p_2 f(x), \frac{3t-s}{3-2s}] & \text{for } s/3 \leqslant t \leqslant (3-s)/3. \end{cases}$$

Then $\Sigma(p_2 f) = \Sigma p_2 \Sigma f \underset{G}{\simeq} qg$. This proves Lemma 4.5.

We proceed to discuss the two main themes. In these discussions we assume

(b) n and ℓ satisfy $2 \leqslant \ell \leqslant n/2$ and $n \leqslant 3\ell - 3$.

By Lemma 4.1, $P_\ell^{2\ell-2} \bar{\times} S^1 \simeq P_\ell^{2\ell-2} \vee \Sigma P_\ell^{2\ell-2}$. We will identify $P_\ell^{2\ell-2} \bar{\times} S^1$ with $P_\ell^{2\ell-2} \vee \Sigma P_\ell^{2\ell-2}$ via a homotopy equivalence h having the properties in Lemma 4.1. Thus $P_\ell^{2\ell-2} \xrightarrow{\bar{i}_1} P_\ell^{2\ell-2} \bar{\times} S^1 = P_\ell^{2\ell-2} \vee \Sigma P_\ell^{2\ell-2}$ is the obvious inclusion map.

$P_\ell^{2\ell-2} \bar{\times} S^1$ is a subcomplex of $P_\ell^n \bar{\times} S^1$ via the embedding $P_\ell^{2\ell-2} \bar{\times} S^1 \xrightarrow{\rho \bar{\times} id} P_\ell^n \bar{\times} S^1$ where ρ is the standard inclusion map as in (2.1). Consider the subcomplex

(c) $P_\ell^n \bar{\times} * \cup P_\ell^{2\ell-2} \bar{\times} S^1 \overset{j}{\hookrightarrow} P_\ell^n \bar{\times} S^1$

where $P_\ell^n \bar{\times} * = P_\ell^n$ is the subcomplex of $P_\ell^n \bar{\times} S^1$ via the embedding $P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1$.

Lemma 4.6. $P_\ell^n \bar{\times} * \cup P_\ell^{2\ell-2} \bar{\times} S^1 \simeq P_\ell^n \vee \Sigma P_\ell^{2\ell-2}$.

Proof. Let $Y = P^n_\ell \bar{\times} * \cup P^{2\ell-2}_\ell \bar{\times} S^1$. Let $P^n_\ell \xrightarrow{i_3} Y, P^{2\ell-2}_\ell \bar{\times} S^1 \xrightarrow{i_4} Y, \Sigma P^{2\ell-2}_\ell \xrightarrow{i_5} P^{2\ell-2}_\ell \bar{\times} S^1 = P^{2\ell-2}_\ell \vee \Sigma P^{2\ell-2}_\ell$ be the obvious inclusion maps. Then the composite

$$P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{i_3 \vee i_4 i_5} Y \vee Y \xrightarrow{F} Y = P^n_\ell \bar{\times} * \cup P^{2\ell-2}_\ell \bar{\times} S^1$$

is easily seen to be a homotopy equivalence where F is the folding map. This proves Lemma 4.6. □

Indentify $P^n_\ell \bar{\times} * \cup P^{2\ell-2}_\ell \bar{\times} S^1$ with $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell$ via the homotopy equivalence in the proof of Lemma 4.6 above. Then the inclusion map j in (c) can be considered as an inclusion map $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j} P^n_\ell \bar{\times} S^1$. It has the following properties.

(d) Let $\bar{p}_1 : P^n_\ell \bar{\times} S^1 \to P^n_\ell$ be as in (i) of (4.1). Then

(i) $\bar{p}_1 j \,|\, \Sigma P^{2\ell-2}_\ell = *$,
(ii) $(j \,|\, P^n_\ell) = \bar{i}_1 : P^n_\ell \to P^n_\ell \bar{\times} S^1$, and so the composite $P^n_\ell \xrightarrow{j|P^n_\ell} P^n_\ell \bar{\times} S^1 \xrightarrow{\bar{p}_1} P^n_\ell$ is the identity map.

Furthermore, if we let $P^n_\ell \bar{\times} S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}$ be the composite

$$P^n_\ell \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P^n_\ell \xrightarrow{\Sigma\tau} \Sigma P^n_{2\ell-1}$$

with $\bar{q}_1$ as in (ii) of (4.1) and τ as in (2.1), then

$$P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j} P^n_\ell \bar{\times} S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}$$

is a cofibration. This is clear. The following (e) and (f) are also clear (for (f) with $R = \mathbf{Z}$ note that $2\ell - 1$ is an odd integer).

(e) The diagram

$$\begin{array}{ccc} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad j \quad} & P^n_\ell \bar{\times} S^1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \bar{q}_1} \\ \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad \Sigma\rho \quad} & \Sigma P^n_\ell \end{array}$$

is homotopy commutative where p_2 is the projection map and ρ is the inclusion map.

(f) There is a short exact sequence of cohomology groups

$$0 \to \tilde{H}^*(\Sigma P^n_{2\ell-1}, R) \xrightarrow{\delta^*} \tilde{H}^*(P^n_\ell \bar{\times} S^1, R) \xrightarrow{j^*} \tilde{H}^*(P^n_\ell \vee \Sigma P^{2\ell-2}_\ell, R) \to 0$$

where R is either $\mathbf{Z}$ or $\mathbf{Z}/2$.

Proposition 4.7. *There is a map* $P^n_{2\ell-1} \xrightarrow{f} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell$ *such that*

(i) *the sequence* $P^n_{2\ell-1} \xrightarrow{f} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j} P^n_\ell \,\overline{\times}\, S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}$ *is a cofiber sequence,*

and

(ii) *the cofiber of the composite* $P^n_{2\ell-1} \xrightarrow{f} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{p_2} \Sigma P^{2\ell-2}_\ell$ *has the homotopy type of* ΣP^n_ℓ *where* p_2 *is as in* (*e*).

Proof. Let F be the homotopy theoretical fiber of δ so that we have a fiber sequence

$$\Omega\Sigma P^n_{2\ell-1} \xrightarrow{\bar f} F \xrightarrow{j_1} P^n_\ell \,\overline{\times}\, S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}. \qquad (*)$$

Since $\delta j \simeq 0$, there is a map $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j_2} F$ such that $j_1 j_2 \simeq j$. We may assume $j = j_1 j_2$. We may also assume that F is a CW-complex and that j_2 is cellular. From the integral Serre spectral sequence of the fibration $F \xrightarrow{j_1} P^n_\ell \,\overline{\times}\, S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}$, the short exact sequence (f) for $R = \mathbf{Z}$ and the condition $n \leqslant 3\ell - 3$ (in (b)), it is not difficult to see that $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell$ can be considered as a subcomplex of F via j_2 and that $(F, P^n_\ell \vee \Sigma P^{2\ell-2}_\ell)$ is $(3\ell-2)$-connected, that is, the CW-complex F is of the form

$$F = (P^n_\ell \vee \Sigma P^{2\ell-2}_\ell) \cup e^{3\ell-1} \cup \cdots.$$

Recall, by James construction [5,12], that $\Omega\Sigma P^n_{2\ell-1}$ has the homotopy type of a CW-complex $JP^n_{2\ell-1}$. We will identify $\Omega\Sigma P^n_{2\ell-1}$ with $JP^n_{2\ell-1}$ via a suitable homotopy equivalence. Then $P^n_{2\ell-1}$ is a subcomplex of $\Omega\Sigma P^n_{2\ell-1}$ via the canonical embedding $P^n_{2\ell-1} \xrightarrow{i} \Omega\Sigma P^n_{2\ell-1}$ and $\Omega\Sigma P^n_{2\ell-1} = P^n_{2\ell-1} \cup e^{4\ell-2} \cup \cdots$. We may assume that the map $\bar f$ in $(*)$ is cellular. Since $n \leqslant 3\ell - 3$, the restriction map

$$P^n_{2\ell-1} \xrightarrow{\bar f \mid P^n_{2\ell-1}} F = (P^n_\ell \vee \Sigma P^{2\ell-2}_\ell) \cup e^{3\ell-1} \cup \cdots$$

can be consider as the composite $P^n_{2\ell-1} \xrightarrow{f} P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j_2} F$ for some f. It is clear that $\tilde H^*(P^n_\ell \vee \Sigma P^{2\ell-2}_\ell, \mathbf{Z}) \xrightarrow{f^*} \tilde H^*(P^n_{2\ell-1}, \mathbf{Z})$ is zero.

Let Y and Ω denote the spaces $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell$ and $\Omega\Sigma P^n_{2\ell-1}$ respectively. Since the cofiber C_j of $Y \xrightarrow{j} P^n_\ell \,\overline{\times}\, S^1$ is $\Sigma P^n_{2\ell-1}$, the cofiber C_{j_1} of $F = Y \cup e^{3\ell-1} \cup \cdots \xrightarrow{j_1} P^n_\ell \,\overline{\times}\, S^1$ has the homotopy type of a CW-complex of the form $\Sigma P^n_{2\ell-1} \cup e^{3\ell} \cup \cdots$ (noting again that $n \leqslant 3\ell - 3$). Since $j_1 \bar f \simeq 0$, there is a map $C_{\bar f} \xrightarrow{h} P^n_\ell \,\overline{\times}\, S^1$ such that the square in the diagram

$$\begin{array}{ccccc} \Omega & \xrightarrow{\bar f} & F & \xrightarrow{i_7} & C_{\bar f} = F \cup_{\bar f} C\Omega \\ & & \Vert & & \downarrow h \\ & & F & \xrightarrow{j_1} & P^n_\ell \,\overline{\times}\, S^1 \end{array}$$

is commutative where i_7 is the inclusion map. This commutative square defines a map of cofibrations

$$\begin{array}{ccccc}
F & \xrightarrow{i_7} & C_{\bar{f}} & \xrightarrow{j_7} & C_{i_7} \simeq \Sigma\Omega = \Sigma P^n_{2\ell-1} \cup e^{4\ell-1} \cup \cdots \\
\Vert & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} \\
F & \xrightarrow{j_1} & P^n_\ell \bar{\times} S^1 & \xrightarrow{j_8} & C_{j_1} \simeq \Sigma P^n_{2\ell-1} \cup e^{3\ell} \cup \cdots
\end{array}$$

where j_7, j_8 are inclusion maps. The map $\bar{h}$ is a homotopy equivalence through dimension $3\ell - 1 \geqslant n+2$ (see [9, pp. 153]). Consider the diagram of maps of cofiber sequences

$$\begin{array}{ccccccc}
P^n_{2\ell-1} & \xrightarrow{f} & Y & \xrightarrow{i_6} & C_f & \xrightarrow{j_6} & C_{i_6} \simeq \Sigma P^n_{2\ell-1} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle \bar{j}_2} & & \downarrow{\scriptstyle \Sigma i} \\
\Omega & \xrightarrow{\bar{f}} & F & \xrightarrow{i_7} & C_{\bar{f}} & \xrightarrow{j_7} & C_{i_7} \simeq \Sigma\Omega \\
 & & \Vert & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} \\
 & & F & \xrightarrow{j_1} & P^n_\ell \bar{\times} S^1 & \xrightarrow{j_8} & C_{j_1}
\end{array}$$

where i_6, j_6 are inclusion maps. Let $h_1 = h\bar{j}_2 : C_f \to P^n_\ell \bar{\times} S^1$ and $\bar{h}_1 = \bar{h}\Sigma i : \Sigma P^n_{2\ell-1} \to C_{j_1}$. Then we have a map of cofibrations

$$\begin{array}{ccccc}
Y & \xrightarrow{i_6} & C_f & \xrightarrow{j_6} & \Sigma P^n_{2\ell-1} \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \bar{h}_1} \\
F & \xrightarrow{j_1} & P^n_\ell \bar{\times} S^1 & \xrightarrow{j_8} & C_{j_1}
\end{array}$$

Since $\tilde{H}^*(Y, \mathbf{Z}) \xrightarrow{f^*=0} \tilde{H}^*(P^n_{2\ell-1}, \mathbf{Z})$, there is a short exact sequence of integral cohomology groups

$$0 \leftarrow \tilde{H}^*(Y, \mathbf{Z}) \overset{i_6^*}{\leftarrow} \tilde{H}^*(C_f, \mathbf{Z}) \overset{j_6^*}{\leftarrow} \tilde{H}^*(\Sigma P^n_{2\ell-1}, \mathbf{Z}) \leftarrow 0.$$

Note that $j_1 j_2 = j$: $Y \to P^n_\ell \bar{\times} S^1$ and that, by (f), $\tilde{H}^*(P^n_\ell \bar{\times} S^1, \mathbf{Z}) \xrightarrow{j^* = j_2^* j_1^*} \tilde{H}^*(Y, \mathbf{Z})$ is onto with ker $j^* \cong \tilde{H}^*(\Sigma P^n_{\ell-1}, \mathbf{Z})$. Also note that $\bar{h}_1$ is a homotopy equivalence through dimension $3\ell - 1 \geqslant n+2$ too, and this implies

$$\tilde{H}^*(C_{j_1} \simeq \Sigma P^n_{2\ell-1} \cup e^{3\ell} \cup \cdots, \mathbf{Z}) \overset{\bar{h}_1^*}{\to} \tilde{H}^*(\Sigma P^n_{2\ell-1}, \mathbf{Z})$$

is iso for $* \leqslant 3\ell - 1$. From these we deduce that $\tilde{H}^*(P^n_\ell \bar{\times} S^1, \mathbf{Z}) \overset{h_1^*}{\to} \tilde{H}^*(C_f, \mathbf{Z})$ is iso for all $*$. So $C_f \overset{h_1}{\to} P^n_\ell \bar{\times} S^1$ is a homotopy equivalence. This proves the first conclusion (i) of Proposition 4.7.

To prove (ii), consider the diagram

$$\begin{array}{ccccccc}
P^n_{2\ell-1} & \xrightarrow{\quad f \quad} Y = & P^n_\ell \vee \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad j \quad} & P^n_\ell \bar{\times} S^1 & \xrightarrow{\quad \delta \quad} & \Sigma P^n_{2\ell-1} \\
\Vert & (1) & \downarrow p_2 & (2) & \downarrow \bar{p}_2 & (3) & \Vert \\
P^n_{2\ell-1} & \xrightarrow{f_1 = p_2 f} & \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad j_9 \quad} & C_{f_1} & \xrightarrow{\quad \delta' \quad} & \Sigma P^n_{2\ell-1} \\
 & & \Vert & (4) & \downarrow \rho_1 & & \\
 & & \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad \Sigma\rho \quad} & \Sigma P^n_\ell & &
\end{array}$$

described as follows. The map from the first row to the second row is a map of cofiber sequences defined by the commutative square (1). We recall from (e) that the square in the diagram

$$\text{(e)}' \qquad \begin{array}{ccccc}
P^n_{2\ell-1} \xrightarrow{\ f\ } & P^n_\ell \vee \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad j \quad} & P^n_\ell \bar{\times} S^1 \\
 & \downarrow p_2 & (5) & \downarrow \bar{q}_1 \\
 & \Sigma P^{2\ell-2}_\ell & \xrightarrow{\quad \Sigma\rho \quad} & \Sigma P^n_\ell
\end{array}$$

is homotopy commutative. Since $jf \simeq 0$ it follows that $\Sigma\rho f_1 = \Sigma\rho p_2 f \simeq \bar{q}_1 jf \simeq 0$. So there is a map $C_{f_1} \overset{\rho_1}{\to} \Sigma P^n_\ell$ that makes square (4) homotopy commutative. Thus $\rho_1 j_9 \simeq \Sigma\rho$. We claim

$$\begin{array}{l} \text{there is a choice of } \rho_1 \text{ with } \rho_1 j_9 \simeq \Sigma\rho \text{ such that} \\ \rho_1 \bar{p}_2 \simeq \bar{q}_1 \colon P^n_\ell \bar{\times} S^1 \to \Sigma P^n_\ell. \end{array} \qquad (**)$$

To see this, take any ρ_1 with $\rho_1 j_9 \simeq \Sigma\rho$. From squares (2), (4) and (5) we see

$$\rho_1 \bar{p}_2 | Y = \rho_1 \bar{p}_2 j \simeq \Sigma\rho p_2 \simeq \bar{q}_1 j = \bar{q}_1 | Y.$$

Since $P^n_\ell \bar{\times} S^1 \simeq C_f = Y \cup_f CP^n_{2\ell-1}$, by (2) of Lemma 4.4, this implies that there is a map $\Sigma P^n_{2\ell-1} \overset{g}{\to} \Sigma P^n_\ell$ such that $g \simeq \bar{q}_1 - \rho_1 \bar{p}_2$, that is, $\bar{q}_1 \simeq \rho_1 \bar{p}_2 + g$. Let $\rho'_1 = \rho_1 + g\colon C_{f_1} \to \Sigma P^n_\ell$. By Lemma 4.3(2), $\rho'_1 \bar{p}_2 \simeq \rho_1 \bar{p}_2 + g \simeq \bar{q}_1$. By (i) of Lemma 4.3(1)

$$\rho'_1 j_9 = \rho'_1 | \Sigma P^{2\ell-2}_\ell = (\rho_1 + g) | \Sigma P^{2\ell-2}_\ell = \rho_1 | \Sigma P^{2\ell-2}_\ell = \rho_1 j_9 \simeq \Sigma\rho.$$

This proves (∗∗). Let $C_{f_1} \xrightarrow{\rho_1} \Sigma P^n_\ell$ be a map with the properties in (∗∗). Consider the following diagram of the induced maps in integral cohomology of squares (2)–(4):

$$\begin{array}{ccccccccc}
0 \longleftarrow & \tilde{H}^*(P^n_\ell \vee \Sigma P^{2\ell-2}_\ell, \mathbf{Z}) & \xleftarrow{j^*} & \tilde{H}^*(P^n_\ell \bar{\times} S^1, \mathbf{Z}) & \xleftarrow{\delta^*} & \tilde{H}^*(\Sigma P^n_{2\ell-1}, \mathbf{Z}) & \longleftarrow 0 \\
& \uparrow p_2^* & & \uparrow \bar{p}_2^* & & \| & \\
0 \longleftarrow & \tilde{H}^*(\Sigma P^{2\ell-2}_\ell, \mathbf{Z}) & \xleftarrow{j_9^*} & \tilde{H}^*(C_{f_1}, \mathbf{Z}) & \xleftarrow{\delta'^*} & H^*(\Sigma P^n_{2\ell-1}, \mathbf{Z}) & \longleftarrow 0 \\
& \| & & \uparrow \rho_1^* & & & \\
& \tilde{H}^*(\Sigma P^{2\ell-2}_\ell, \mathbf{Z}) & \xleftarrow{\Sigma\rho^*} & \tilde{H}^*(\Sigma P^n_\ell, \mathbf{Z}) & & &
\end{array}$$

The first row is short exact by (f) and so is the second row since $\tilde{H}^*(\Sigma P^{2\ell-2}_\ell, \mathbf{Z}) \xrightarrow{f_1^* = f^* p_2^* = 0} \tilde{H}^*(P^n_{2\ell-1}, \mathbf{Z})$. We want to show ρ_1^* is iso. By (∗∗), $\bar{p}_2^* \rho_1^* = \bar{q}_1^*$: $\tilde{H}^*(\Sigma P^n_\ell, \mathbf{Z}) \to \tilde{H}^*(P^n_\ell \bar{\times} S^1, \mathbf{Z})$. Since $\bar{q}_1^*$ is monomorphic (recall $\Sigma(P^n_\ell \bar{\times} S^1) \simeq \Sigma P^n_\ell \vee \Sigma^2 P^n_\ell$ with $\Sigma^2 P^n_\ell$ coming from $\Sigma \bar{q}_1$: $\Sigma^2 P^n_\ell \to \Sigma(P^n_\ell \bar{\times} S^1)$) it follows that ρ_1^* is monomorphic. Recall $P^n_\ell \bar{\times} S^1 \xrightarrow{\delta} \Sigma P^n_{2\ell-1}$ is the composite $P^n_\ell \bar{\times} S^1 \xrightarrow{\bar{q}_1} \Sigma P^n_\ell \xrightarrow{\Sigma\tau} \Sigma P^n_{2\ell-1}$. So $im\delta^* \subset im\bar{q}_1^*$. Note that $\Sigma\rho^*$ is onto (since $2\ell - 2$ is even) and $\bar{p}_2^*$ is monomorphic (since p_2^* is monomorphic). By chasing diagram it is easy to see from these that ρ_1^* is onto. Thus ρ_1^* is an isomorphism. So $C_{f_1} \xrightarrow{\rho_1} \Sigma P^n_\ell$ is a homotopy equivalence. This proves Proposition 4.7(ii). This completes the proof of Proposition 4.7.

Next we discuss the second theme of this section which is Proposition 4.10 that follows. We will consider the Stiefel manifold $V_{n+2,n+2-\ell}$ where ℓ and n satisfy the conditions in (b). Let $P^{n+1}_\ell = P_{n+2,n+2-\ell} \xrightarrow{i} V_{n+2,n+2-\ell}$ and $P^n_\ell \xrightarrow{\rho} P^{n+1}_\ell$ be the inclusion maps as in (2.1). To discuss Proposition 4.10 we make the following assumption. We recall that $\tilde{H}^*(X)$ means $\tilde{H}^*(X, \mathbf{Z}/2)$.

Assumption 4.8. There is a space map $P^n_\ell \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell}$ having the following two properties.

(i) $\tilde{H}^*(V_{n+2,n+2-\ell}) \xrightarrow{\psi_1^*} \tilde{H}^*(P^n_\ell \bar{\times} S^1)$ has $\psi_1^*(x_{n+1}) = x_n\gamma$ and $\psi_1^*(x_j) = x_j$ for $\ell \leqslant j \leqslant n$.

(ii) The composite $P^n_\ell \xrightarrow{\bar{i}_1} P^n_\ell \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell}$ is homotopic to the composite $P^n_\ell \xrightarrow{h} P^n_\ell \xrightarrow{\rho} P^{n+1}_\ell \xrightarrow{i} V_{n+2,n+2-\ell}$ for some map h with $h^* =$ the identity map on $\tilde{H}^*(P^n_\ell)$.

Note. The map ψ_1 in Assumption 4.8 will be connected to the map ψ in (3.8) later in Section 5 in the following content. Assume $V_{2^m-1,k}$ is neutral for $2^{m-1} + 2 \leqslant k = 2j < 2^m - 2$ $(m \geqslant 5)$. Then by (3.8) there is a space map $P_{2^m-1,k} \times S^1 = P^{2^m-2}_{2^m-k-1} \times S^1 \xrightarrow{\psi} V_{2^m,k+1}$ having the properties in Lemma 3.6. We will derive from this that there is a space map $P^{2^m-2}_{2^{m-1}-3} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$ having the properties in Assumption 4.8 (Proposition 5.1). Thus n in Assumption 4.8 is $n+1$ in (3.8) in this content.

Let $P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j} P^n_\ell \bar{\times} S^1$ be as in Proposition 4.7. Let $\Sigma P^{2\ell-2}_\ell \xrightarrow{\phi} V_{n+2,n+2-\ell}$ be the composite $\Sigma P^{2\ell-2}_\ell \xrightarrow{\bar{j}=j|\Sigma P^{2\ell-2}_\ell} P^n_\ell \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell}$. By Lemma 4.6 and (f) for $R = \mathbf{Z}/2$ the induced map $\tilde{H}^*(P^n_\ell \bar{\times} S^1) \xrightarrow{\bar{j}^*} \tilde{H}^*(\Sigma P^{2\ell-2}_\ell)$ has $\bar{j}^*(x_j) = 0$ for $\ell \leqslant j \leqslant n$ and $\bar{j}^*(x_n\gamma) = 0$ (since $2\ell \leqslant n$). So $\tilde{H}^*(V_{n+2,n+2-\ell}) \xrightarrow{\phi^*=0} \tilde{H}^*(\Sigma P^{2\ell-2}_\ell)$. Thus

$$\tilde{H}^*(C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P^{2\ell-2}_\ell) \cong \tilde{H}^*(V_{n+2,n+2-\ell}) \oplus \tilde{H}^*(\Sigma^2 P^{2\ell-2}_\ell)$$

as a $\mathbf{Z}/2$-module. We describe such a $\mathbf{Z}/2$-module decomposition more precisely as follows.

Convention 4.9. *Suppose $X \xrightarrow{f} Y$ is a map such that $f^* = 0$ in* mod 2 *cohomology. Consider the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} C_f = Y \cup_f CX \xrightarrow{q} \Sigma X$. Then we have a short exact sequence of A-modules*

$$0 \leftarrow \tilde{H}^*(Y) \xleftarrow{i^*} \tilde{H}^*(C_f) \xleftarrow{q^*} \tilde{H}^*(\Sigma X) \leftarrow 0.$$

Indentify $\tilde{H}^*(\Sigma X)$ with $\operatorname{im} q^* \subset \tilde{H}^*(C_f)$. Let V be a $\mathbf{Z}/2$-submodule of $\tilde{H}^*(C_f)$ which is mapped isomorphically onto $\tilde{H}^*(Y)$ under i^*. Then $\tilde{H}(C_f) \cong V \oplus \tilde{H}^*(X)$ as a $\mathbf{Z}/2$-module. We will identify V with $\tilde{H}^*(Y)$ so that there is a $\mathbf{Z}/2$-module decomposition $\tilde{H}^*(C_f) \cong \tilde{H}^*(Y) \oplus \tilde{H}^*(\Sigma X)$. The names of classes in $\tilde{H}^*(Y)$ will be the names of the corresponding classes in $V \subset \tilde{H}^*(C_f)$.

Proposition 4.10. *Under Assumption* 4.8 *there is a space map $\Sigma P^n_{2\ell-1} \xrightarrow{g} C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P^{2\ell-2}_\ell$ having the following properties.*

(i) *The cofiber of the composite*

$$\Sigma P^n_{2\ell-1} \xrightarrow{g} C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P^{2\ell-2}_\ell \xrightarrow{q} \Sigma^2 P^{2\ell-2}_\ell$$

has the homotopy type of $\Sigma^2 P^n_\ell$ where q is the natural pinching map.

(ii) $\tilde{H}^*(C_\phi) \cong \tilde{H}^*(V_{n+2,n+2-\ell}) \oplus \tilde{H}^*(\Sigma^2 P^{2\ell-2}_\ell) \xrightarrow{g^*} \tilde{H}^*(\Sigma P^n_{2\ell-1})$ *has* $g^*(x_{n+1}) = \Sigma x_n$, $g^*(x_j) = 0$ *for* $\ell \leqslant j \leqslant n$ *on the normal classes* $x_j \in \tilde{H}^*(V_{n+2,n+2-\ell})$ *and* $g^*(\tilde{H}^*(\Sigma^2 P^{2\ell-2}_\ell)) = 0$.

Proof. We shall use Lemma 4.5 and Proposition 4.7 to prove the proposition. For this purpose we describe relevant data as those assumed in Lemma 4.5 as follows.

By Proposition 4.7(i) there is a cofiber sequence

$$X = P^n_{2\ell-1} \xrightarrow{f} Y = P^n_\ell \vee \Sigma P^{2\ell-2}_\ell \xrightarrow{j} P^n_\ell \bar{\times} S^1 \simeq Y \cup_f CX \xrightarrow{\delta} \Sigma X = \Sigma P^n_{2\ell-1} \quad (*)$$

where δ is equivalent to the natural pinching map $Y \cup_f CX \to \Sigma X$. The map j has the two properties in (d) and these are equivalent to (d)′ below.

(d)′ The square

$$\begin{array}{ccc}
Y = P_\ell^n \vee \Sigma P_\ell^{2\ell-2} & \xrightarrow{\quad j \quad} & P_\ell^n \bar{\times} S^1 \\
\Big\downarrow p_1 & & \Big\downarrow r = \bar{p}_1 \\
P_\ell^n & =\!=\!=\!= & P_\ell^n
\end{array}$$

is commutative with $j|P_\ell^n = \bar{i}_1$ where p_1 is the projection, $\bar{i}_1$ and $\bar{p}_1$ are as in (i) of (4.1).

Let $P_\ell^n \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell} = Z$ be as assumed in Assumption 4.8. By Assumption 4.8(ii) the composite $P_\ell^n \xrightarrow{\bar{i}_1} P_\ell^n \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell}$ is homotopic to the composite $P_\ell^n \xrightarrow{h} P_\ell^n \xrightarrow{\rho} P_\ell^{n+1} \xrightarrow{i} V_{n+2,n+2-\ell}$ with $h^* =$ the identity map on $\tilde{H}^*(P_\ell^n)$. Let $\phi_1 = i\rho h$: $P_\ell^n \to V_{n+2,n+2-\ell} = Z$. So $\psi_1|P_\ell^n = \psi_1 \bar{i}_1 \simeq \phi_1$: $P_\ell^n \to Z$. Let $P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{\psi}_1} C_\phi$ be the composite

$$P_\ell^n \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell} \xrightarrow{j_1} C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P_\ell^{2\ell-2}$$

where j_1 is the inclusion map, and let $P_\ell^n \bar{\times} S^1 \xrightarrow{\bar{\psi}_2} C_\phi$ be the composite

$$P_\ell^n \bar{\times} S^1 \xrightarrow{r = \bar{p}_1} P_\ell^n \xrightarrow{\phi_1 = i\rho h} V_{n+2,n+2-\ell} \xrightarrow{j_1} C_\phi.$$

With these data, from Lemma 4.5, we conclude that there is a homotopy difference $g \simeq \bar{\psi}_2 - \bar{\psi}_1$: $\Sigma P_{2\ell-1}^n \to C_\phi$ such that the composite

$$\Sigma P_{2\ell-1}^n \xrightarrow{g} C_\phi = V_{n+2,n+2-\ell} \cup_\phi C\Sigma P_\ell^{2\ell-2} \xrightarrow{q} \Sigma^2 P_\ell^{2\ell-2}$$

is homotopic to the composite

$$\Sigma P_{2\ell-1}^n \xrightarrow{\Sigma f} \Sigma P_\ell^n \vee \Sigma^2 P_\ell^{2\ell-2} = \Sigma(Y = P_\ell^n \vee \Sigma P_\ell^{2\ell-2}) \xrightarrow{\Sigma p_2} \Sigma^2 P_\ell^{2\ell-2}.$$

By Proposition 4.7(ii), the cofiber of the latter map has the homotopy type of $\Sigma^2 P_\ell^n$. So $C_{qg} \simeq \Sigma^2 P_\ell^n$. This proves Proposition 4.10(i).

To see Proposition 4.10(ii) we recall that $g \simeq \bar{\psi}_2 - \bar{\psi}_1$ means $\bar{\psi}_2 \simeq \bar{\psi}_1 + g$. By Lemma 4.4(1) $\bar{\psi}_2^* = (\bar{\psi}_1 + g)^* = \bar{\psi}_1^* + \delta^* g^*$ from $\tilde{H}^*(C_\phi) \cong \tilde{H}^*(V_{n+2,n+2-\ell}) \oplus \tilde{H}^*(\Sigma^2 P_\ell^{2\ell-2})$ to $\tilde{H}^*(P_\ell^n \bar{\times} S^1)$ where δ is as in (∗). From the constructions of $\bar{\psi}_2$ and $\bar{\psi}_1$ we see $\bar{\psi}_i^*(x_j) = \psi_i^*(x_j)$, $i = 1,2$, for the normal classes $x_j \in \tilde{H}^*(V_{n+2,n+2-\ell})$ with $\ell \leqslant j \leqslant n+1$ where ψ_2 is the composite $P_\ell^n \bar{\times} S^1 \xrightarrow{r = \bar{p}_1} P_\ell^n \xrightarrow{\phi_1 = i\rho h} V_{n+2,n+2-\ell}$. By Assumption 4.8(i), $\bar{\psi}_1^*(x_{n+1}) = \psi_1^*(x_{n+1}) = x_n \gamma$ and $\bar{\psi}_1^*(x_j) = \psi_1^*(x_j) = x_j$ for $\ell \leqslant j \leqslant n$. Since $h^*(x_j) = x_j$ for $\ell \leqslant j \leqslant n$ we see

$$\bar{\psi}_2^*(x_j) = \psi_2^*(x_j) = \bar{p}_1^* h^* \rho^* i^*(x_j) = x_j \quad \text{for} \quad \ell \leqslant j \leqslant n$$

and

$$\bar{\psi}_2^*(x_{n+1}) = \psi_2^*(x_{n+1}) = \bar{p}_1^* h^* \rho^* i^*(x_{n+1}) = \bar{p}_1^* h^* \rho^*(x_{n+1}) = 0$$

as $\rho^*(x_{n+1}) = 0$. $g^*(x_j)$ are evaluated as follows. For $j = n+1$ we have

$$\delta^* g^*(x_{n+1}) = \bar{\psi}_2^*(x_{n+1}) - \bar{\psi}_1^*(x_{n+1}) = 0 - x_n\gamma = x_n\gamma \neq 0$$

and this implies $g^*(x_{n+1}) \neq 0$ and therefore must be Σx_n. For $\ell \leqslant j \leqslant n$ we have

$$\delta^* g^*(x_j) = \bar{\psi}_2^*(x_j) - \bar{\psi}_1^*(x_j) = x_j - x_j = 0$$

and this implies $g^*(x_j) = 0$ since δ^* is monomorphic by (f) for $R = \mathbf{Z}/2$. The conclusion $g^*(\tilde{H}^*(\Sigma^2 P_\ell^{2\ell-2})) = 0$ follows from Proposition 4.10(i). This proves Proposition 4.10(ii). This completes the proof of Proposition 4.10.

5. Proof of Theorem 1.1

We want to show for Theorem 1.1 that $V_{2^m-1,k}$ is non-neutral for $m \geqslant 5$ and $2^{m-1}+2 \leqslant k = 2\ell < 2^m - 2$. We will prove this by contradiction. From now on we assume $V_{2^m-1,k}$ is neutral with $m \geqslant 5$ and $2^{m-1}+2 \leqslant k = 2\ell < 2^m-2$. All the results that we are going to state and prove hereafter are consequences of this assumption. We will show that this would lead to a contradiction as that described in Section 1.

As in Theorem 2.1, for general n and k, the notations x_ℓ will denote both the normal classes in $H^*(V_{n,k})$ and the generators in $H^*(P_{n,k})$ for $n-k \leqslant \ell \leqslant n-1$. We will freely interchangeably use the notation P_{n-k}^{n-1} and the notation $P_{n,k}$ to denote the same space which is the stunted projective space P^{n-1}/P^{n-k-1}. The maps $P_{n,k} = P_{n-k}^{n-1} \xrightarrow{\rho} P_{n+1,k+1} = P_{n-k}^{n}, P_{n,k} = P_{n-k}^{n-1} \xrightarrow{\tau} P_{n,k'} = P_{n-k'}^{n-1}$ for $n > k > k'$ and $P_{n+1,k+1} \xrightarrow{i} V_{n+1,k+1}$ will be as in (2.1) and the maps $P_{n,k} \xrightarrow{i_1} P_{n,k} \times S^1$, $P_{n,k} \xrightarrow{\bar{i}_1} P_{n,k} \bar{\times} S^1$ will be as in Section 4. In this section the inclusion map $P_{n,k} = P_{n-k}^{n-1} \to P_{m,m-n+k} = P_{n-k}^{m-1}$ for $n < m$ will also be denoted by ρ unless specified otherwise.

Since $V_{2^m-1,k}$ is neutral, 2^m-1 is odd and k is even, from (3.8), Lemma 3.6 and (3.12), we have the following.

> There is a space map $P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1}$ having the following properties.
> (i) The induced map $\tilde{H}^*(V_{2^m,k+1}) \xrightarrow{\psi^*} \tilde{H}^*(P_{2^m-1,k} \times S^1)$ has $\psi^*(x_\ell) = x_\ell$
> for $2^m-1-k \leqslant \ell \leqslant 2^m-2$ and $\psi^*(x_{2^m-1}) = x_{2^m-2}\gamma$. (5.1)
> (ii) The composite $P_{2^m-1,k} \xrightarrow{i_1} P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1}$ is the composite
> $P_{2^m-1,k} \xrightarrow{\rho} P_{2^m,k+1} \xrightarrow{i} V_{2^m,k+1}$.

Here we note the that the conditions $n-k \geqslant 3$ and $2k-1 > n$ in Lemma 3.6(ii) are satisfied for $(n,k) = (2^m-1,k)$ since $2^{m-1}+2 \leqslant k = 2\ell \leqslant 2^m-4$.

From (5.1) we will construct a map

$$P_{2^{m-1}-3}^{2^m-2} \bar{\times} S^1 = P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$$

having the properties in Assumption 4.8. The result is precisely stated as follows.

Proposition 5.1. *There is a space map* $P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$ *having the following properties.*

(i) *The induced map* $\tilde{H}^*(V_{2^m,2^{m-1}+3}) \xrightarrow{\psi_1^*} \tilde{H}^*(P_{2^m-1,2^{m-1}+2} \bar{\times} S^1)$ *has* $\psi_1^*(x_{2^m-1}) = x_{2^m-2}\gamma$ *and* $\psi_1^*(x_\ell) = x_\ell$ *for* $2^{m-1}-3 \leqslant \ell \leqslant 2^m-2$.

(ii) *The composite* $P_{2^m-1,2^{m-1}+2} \xrightarrow{\bar{i}_1} P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$ *is homotopic to the composite*

$$P_{2^m-1,2^{m-1}+2} \xrightarrow{h} P_{2^m-1,2^{m-1}+2} \xrightarrow{\rho} P_{2^m,2^{m-1}+3} \xrightarrow{i} V_{2^m,2^{m-1}+3}$$

for some map h with $h^* =$ identity *map on* $\tilde{H}^*(P_{2^m-1,2^{m-1}+2})$.

The construction of the map ψ_1 in Proposition 5.1 from the map ψ in (5.1) will be discussed separately for the case $k = 2^{m-1}+2$ and the case $k \geqslant 2^{m-1}+4$, and is given in the next several paragraphs.

If $k = 2^{m-1}+2$ then $P_{2^m-1,2^{m-1}+2} \times S^1 = P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1} = V_{2^m,2^{m-1}+3}$ clearly factorizes through the quotient map $P_{2^m-1,2^{m-1}+2} \times S^1 \xrightarrow{q} P_{2^m-1,2^{m-1}+2} \bar{\times} S^1$ yielding a map $P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$ having the property in (i) of Proposition 5.1 by (i) of (5.1) and also the property in (ii) of Proposition 5.1 with $h =$ identity map by (ii) of (5.1).

Next consider the case $2^{m-1}+4 \leqslant k = 2\ell < 2^m-2$. Note that $k \leqslant 2^m-4$. Then $k+1 \geqslant 2^{m-1}+5 > 2^{m-1}+3$ and $2 \leqslant k' = k-2^{m-1}-2 \leqslant 2^{m-1}-6$. Consider the composite

$$\tilde{\psi}: P_{2^{m-1}-3,k'} \times S^1 \xrightarrow{\rho \times id} P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1} \xrightarrow{p} V_{2^m,2^{m-1}+3}$$

where p is the map obtained by taking the last $2^{m-1}+3$ vectors in each $(k+1)$-frame.

Lemma 5.2. $\tilde{\psi} \simeq 0: P_{2^{m-1}-3,k'} \times S^1 \to V_{2^m,2^{m-1}+3}$.

Proof. $P_{2^{m-1}-3,k'} \times S^1$ is a finite complex with

$$P_{2^{m-1}-3,k'} \times S^1 = (P_{2^{m-1}-3,k'} \times S^1)^{2^{m-1}-3} = (P_{2^{m-1}-3,k'} \times S^1)^{2^{m-1}-4} \cup e^{2^{m-1}-3}$$

as $P_{2^{m-1}-3,k'} = P_{2^m-k-1}^{2^{m-1}-4}$. Recall that $(V_{2^m,2^{m-1}+3}, P_{2^m,2^{m-1}+3})$ is (2^m-6)-connected with $P_{2^m,2^{m-1}+3}$ the subspace via the embedding $P_{2^m,2^{m-1}+3} \xrightarrow{i} V_{2^m,2^{m-1}+3}$. Let $P_{2^{m-1}-3,k'} \times S^1 \xrightarrow{q} S^{2^{m-1}-3}$ be the pinching map. Let $P_{2^{m-1}-3}^{2^{m-1}-2} \xrightarrow{j} P_{2^{m-1}-3}^{2^m-1} = P_{2^m,2^{m-1}+3}$ be the inclusion map. Since $2^{m-1}-3 < 2^m-6$ (as $m \geqslant 5$) and $P_{2^{m-1}-3}^{2^m-1}$ is $(2^{m-1}-4)$-connected it follows that the composite

$$\tilde{\psi}: P_{2^{m-1}-3,k'} \times S^1 \xrightarrow{\rho \times id} P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1} \xrightarrow{p} V_{2^m,2^{m-1}+3}$$

is homotopic to the composite

$$P_{2^{m-1}-3,k'} \times S^1 \xrightarrow{q} S^{2^{m-1}-3} \xrightarrow{\tilde{i}} P_{2^{m-1}-3}^{2^{m-1}-2} \xrightarrow{j} P_{2^{m-1}-3}^{2^m-1} \xrightarrow{i} V_{2^m,2^{m-1}+3}$$

for some $\tilde{i}$. To prove $\tilde{\psi} \simeq 0$ is to prove $ij\tilde{i}q \simeq 0$. It suffices to prove $\tilde{i} \simeq 0$.

Note that $P_{2^{m-1}-3}^{2^{m-1}-2}$ is the Moore space $M = S^{2^{m-1}-3} \cup_{2\iota} e^{2^{m-1}-2}$. It is well known that $\tilde{i} \simeq 0 : S^{2^{m-1}-3} \to M$ if and only if the induced map $\tilde{H}^{2^{m-1}-3}(M) = \mathbf{Z}/2 \xrightarrow{\tilde{i}^*} \tilde{H}^{2^{m-1}-3}(S^{2^{m-1}-3}) = \mathbf{Z}/2$ is zero. The generator of $\tilde{H}^{2^{m-1}-3}(M = P_{2^{m-1}-3}^{2^{m-1}-2}) = \mathbf{Z}/2$ is $x_{2^{m-1}-3}$. Let ι be the generator of $\tilde{H}^{2^{m-1}-3}(S^{2^{m-1}-3}) = \mathbf{Z}/2$. Let $\tilde{i}^*(x_{2^{m-1}-3}) = \varepsilon\iota$ where $\varepsilon = 0$ or 1. To prove $\tilde{i} \simeq 0$ is to prove $\varepsilon = 0$. It is clear that $q^*(\iota) = x_{2^{m-1}-4}\gamma \neq 0$. We have $j^*i^*(x_{2^{m-1}-3}) = x_{2^{m-1}-3}$, $p^*(x_{2^{m-1}-3}) = x_{2^{m-1}-3}$ (by Theorem 2.1). By (i) of (5.1), $\psi^*(x_{2^{m-1}-3}) = x_{2^{m-1}-3}$. By dimensional reasons, $\rho^*(x_{2^{m-1}-3}) = 0$ (since $P_{2^{m-1}-3,k'} = P_{2^m-k-1}^{2^{m-1}-4}$); so $(\rho \times id)^*(x_{2^{m-1}-3}) = 0$. Then

$$q^*\tilde{i}^*j^*i^*(x_{2^{m-1}-3}) = q^*\tilde{i}^*(x_{2^{m-1}-3}) = q^*(\varepsilon\iota) = \varepsilon q^*(\iota) = \varepsilon x_{2^{m-1}-4}\gamma$$

which is equal to

$$(\rho \times id)^*\psi^*p^*(x_{2^{m-1}-3}) = (\rho \times id)^*(x_{2^{m-1}-3}) = 0.$$

So $\varepsilon = 0$. This proves $\tilde{i} \simeq 0$ and therefore $\tilde{\psi} \simeq 0$. This completes the proof of Lemma 5.2.

Let q_1 be the composite $P_{2^m-1,k} \times S^1 \xrightarrow{\tau \times id} P_{2^m-1,2^{m-1}+2} \times S^1 \xrightarrow{q} P_{2^m-1,2^{m-1}+2} \bar{\times} S^1$ where q is the quotient map. It is easy to see that

$$P_{2^{m-1}-3,k'} \times S^1 \xrightarrow{\rho \times id} P_{2^m-1,k} \times S^1 \xrightarrow{q_1} P_{2^m-1,2^{m-1}+2} \bar{\times} S^1$$

is a cofibration. From Lemma 5.2 we see the composite

$$P_{2^m-1,k} \times S^1 \xrightarrow{\psi} V_{2^m,k+1} \xrightarrow{p} V_{2^m,2^{m-1}+3}$$

factorizes through q_1 yielding a map $P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 \xrightarrow{\psi_1} V_{2^m,2^{m-1}+3}$ such that

(a)
$$\begin{array}{ccccc} P_{2^m-1,k} \times S^1 & \xrightarrow{\psi} & V_{2^m,k+1} & \xrightarrow{p} & V_{2^m,2^{m-1}+3} \\ \downarrow{\scriptstyle q_1} & & & & \| \\ P_{2^m-1,2^{m-1}+2} \bar{\times} S^1 & & \xrightarrow{\psi_1} & & V_{2^m,2^{m-1}+3} \end{array}$$

is homotopy commutative.

It is clear that

(b) $\tilde{H}^*(P_{2^m-1,2^{m-1}+2} \bar{\times} S^1) \xrightarrow{q_1^*} \tilde{H}^*(P_{2^m-1,k} \times S^1)$ has $q_1^*(x_\ell) = x_\ell, q_1^*(x_\ell\gamma) = x_\ell\gamma$ for $2^{m-1} - 3 \leqslant \ell \leqslant 2^m - 2$; thus q_1^* is monomorphic.

Since $p^*(x_\ell) = x_\ell$ for $2^{m-1} - 3 \leqslant \ell \leqslant 2^m - 1$, from (a), (b) and (i) of (5.1) for ψ^*, we see ψ_1 has the property in (i) of Proposition 5.1.

Next we show ψ_1 also has the property in (ii) of Proposition 5.1. Consider the cofiber sequence

(c) $P_{2^{m-1}-3,k'} \xrightarrow{\rho} P_{2^m-1,k} \xrightarrow{\tau} P_{2^m-1,2^{m-1}+2} \xrightarrow{\delta} \Sigma P_{2^{m-1}-3,k'}$.

Note that $\delta^* = 0$ in mod 2 cohomology. Consider the diagrams

$$
\text{(d)}\qquad
\begin{array}{ccc}
P_{2^m-1,k} & \xrightarrow{\tau} & P_{2^m-1,2^{m-1}+2} \\
\downarrow{\scriptstyle i_1} & (1) & \downarrow{\scriptstyle \bar{i}_1} \\
P_{2^m-1,k}\times S^1 & \xrightarrow{q_1} & P_{2^m-1,2^{m-1}+2}\bar{\times} S^1 \\
\downarrow{\scriptstyle \psi} & (2) & \downarrow{\scriptstyle \psi_1} \\
V_{2^m,k+1} & \xrightarrow{p} & V_{2^m,2^{m-1}+3}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P_{2^m-1,k} & \xrightarrow{\tau} & P_{2^m-1,2^{m-1}+2} \\
\downarrow{\scriptstyle \rho} & (3) & \downarrow{\scriptstyle \rho} \\
P_{2^m,k+1} & \xrightarrow{\tau} & P_{2^m,2^{m-1}+3} \\
\downarrow{\scriptstyle i} & (4) & \downarrow{\scriptstyle i} \\
V_{2^m,k+1} & \xrightarrow{p} & V_{2^m,2^{m-1}+3}
\end{array}
$$

(we recall that we use the same notations ρ, τ, i for the maps among various $P_{n,k}$ and $V_{n,k}$). Squares (3) and (4) are commutative by (2.1). Square (2) is homotopy commutative by (a). Square (1) is also commutative since it is the commutative diagram

$$
\begin{array}{ccccc}
P_{2^m-1,k} & \xrightarrow{\tau} & P_{2^m-1,2^{m-1}+2} & = & P_{2^m-1,2^{m-1}+2} \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \bar{i}_1} \\
P_{2^m-1,k}\times S^1 & \xrightarrow{\tau\times id} & P_{2^m-1,2^{m-1}+2}\times S^1 & \xrightarrow{q} & P_{2^m-1,2^{m-1}+2}\bar{\times} S^1
\end{array}
$$

noting that $q_1 = q(\tau \times id)$. Thus the diagrams in (d) are homotopy commutative. By (ii) of (5.1), the composites ψi_1 and $i\rho$ in the first columns of the two diagrams in (d) are equal. Thus the two maps from $P_{2^m-1,k}$ to $V_{2^m,2^{m-1}+3}$ in diagrams (d) are homotopic and both factorize through τ, one via $\psi_1\bar{i}_1$ and the other via $i\rho$. From the cofiber sequence (c) and from (2) of Lemma 4.4 we see there is a homotopy difference $g \simeq \psi_1\bar{i}_1 - i\rho$: $\Sigma P_{2^{m-1}-3,k'} \to V_{2^m,2^{m-1}+3}$ which, by dimensional reasons is homotopic to the composite

$$\Sigma P^{2^{m-1}-4}_{2^{m-1}-k-1} = \Sigma P_{2^{m-1}-3,k'} \xrightarrow{\bar{g}} P_{2^m-1,2^{m-1}+2} = P^{2^m-2}_{2^{m-1}-3} \xrightarrow{\rho} P_{2^m,2^{m-1}+3} \xrightarrow{i} V_{2^m,2^{m-1}+3}$$

for some $\bar{g}$. Let 1 be the identity map on $P_{2^m-1,2^{m-1}+2}$ and let $h = 1 + \bar{g}$: $P_{2^m-1,2^{m-1}+2} \to P_{2^m-1,2^{m-1}+2}$. Recall $g \simeq \psi_1\bar{i}_1 - i\rho$ means $\psi_1\bar{i}_1 \simeq i\rho + g$. Then

$$\begin{aligned}\psi_1\bar{i}_1 &\simeq i\rho + g \simeq i\rho + i\rho\bar{g} = i\rho 1 + i\rho\bar{g} = i\rho(1+\bar{g}) \quad \text{(by (ii) of Lemma 4.3(1))}\\ &= i\rho h.\end{aligned}$$

By (1) of Lemma 4.4, $h^* = (1+\bar{g})^* = 1^* + \delta^*\bar{g}^*: \tilde{H}^*(P_{2^m-1,2^{m-1}+2}) \to \tilde{H}^*(P_{2^m-1,2^{m-1}+2})$. Since $\delta^* = 0$ it follows that $h^* = 1^*$. Thus ψ_1 also has property (ii) of Proposition 5.1. This completes the proof of Proposition 5.1.

We recall again that the properties in Proposition 5.1 are those in Assumption 4.8 for $P^n_\ell \bar{\times} S^1 = P^{2^m-2}_{2^{m-1}-3} \bar{\times} S^1 \xrightarrow{\psi_1} V_{n+2,n+2-\ell} = V_{2^m,2^{m-1}+3}$. The conditions $2 \leqslant \ell \leqslant n/2$ and $n \leqslant 3\ell - 3$ assumed in Section 4 for the results of Propositions 4.7 and 4.10 are satisfied for $\ell = 2^{m-1} - 3$ and

$n = 2^m - 2$ since $m \geqslant 5$. Thus from Propositions 5.1 and 4.10 we infer the following in which we use g_0 for g in Proposition 4.10.

$$\text{There exist space maps } \Sigma P^{2^m-8}_{2^{m-1}-3} \xrightarrow{\phi} V_{2^m, 2^{m-1}+3} \text{ with } \phi^* = 0 \text{ in mod 2 cohomology, and } \Sigma P^{2^m-2}_{2^m-7} \xrightarrow{g_0} C_\phi \text{ having the following properties.} \tag{5.2}$$

(i) The cofiber of the composite

$$\Sigma P^{2^m-2}_{2^m-7} \xrightarrow{g_0} C_\phi = V_{2^m, 2^{m-1}+3} \cup_\phi C\Sigma P^{2^m-8}_{2^{m-1}-3} \xrightarrow{q} \Sigma^2 P^{2^m-8}_{2^{m-1}-3}$$

has the homotopy type of $\Sigma^2 P^{2^m-2}_{2^{m-1}-3}$ where q is the pinching map.

(ii) $\tilde{H}^*(C_\phi) \cong \tilde{H}^*(V_{2^m, 2^{m-1}+3}) \oplus \tilde{H}^*(\Sigma^2 P^{2^m-8}_{2^{m-1}-3}) \xrightarrow{g_0^*} \tilde{H}^*(\Sigma P^{2^m-2}_{2^m-7})$ has $g_0^*(x_{2^m-1}) = \Sigma x_{2^m-2}, g_0^*(x_j) = 0$ for $2^{m-1} - 3 \leqslant j \leqslant 2^m - 2$ on the normal classes $x_j \in \tilde{H}^*(V_{2^m, 2^{m-1}+3})$ and $g_0^*(\tilde{H}^*(\Sigma^2 P^{2^m-8}_{2^{m-1}-3})) = 0$.

Here we use Convention 4.9 for the $\mathbf{Z}/2$-module decomposition in (ii) of (5.2). This convention will also be used for all similar $\mathbf{Z}/2$-module decompositions hereafter.

Notation. Still assume $m \geqslant 5$. In what follows the numbers $2^m - i$ for $i = 0,1,2,3,4,7,8$ and $2^{m-1} - i$ for $i = 0,1,3$ will be considered. To simplify notations we use n to denote $2^m - 1$ and ℓ to denote $2^{m-1} - 3$. So these numbers are $n - i$ for $i = -1,0,1,2,3,6,7$ and $\ell + i$ for $i = 0,2,3$ respectively. In some instances we will use the original notations for some of these numbers, especially $2^{m-1}, 2^{m-1} - 1$ and $2^m - 3$ in Proposition 5.3 and its equivalent statements given in Propositions 5.3′, 5.3″, and 5.3‴.

Let $g_1 = g_0 | \Sigma P^{n-2}_{n-6} = \Sigma P^{2^m-3}_{2^m-7}$. (5.2) implies the following.

(i) The cofiber of the composite
$\Sigma P^{n-2}_{n-6} \xrightarrow{g_1} C_\phi = V_{n+1, n+1-\ell} \cup_\phi C\Sigma P^{n-7}_{\ell} \xrightarrow{q} \Sigma^2 P^{n-7}_{\ell}$
has the homotopy type of $\Sigma^2 P^{n-2}_{\ell}$. (5.2)′
(ii) $\tilde{H}^*(C_\phi) \cong \tilde{H}^*(V_{n+1, n+1-\ell}) \oplus \tilde{H}^*(\Sigma^2 P^{n-7}_{\ell}) \xrightarrow{g_1^* = 0} \tilde{H}^*(\Sigma P^{n-2}_{n-6})$.

(i) of (5.2)′ is clear. (ii) of (5.2)′ follows from (ii) of (5.2) since $g_1^*(x_j) = 0$ for all the normal classes $x_j \in \tilde{H}^*(V_{n+1, n+1-\ell})$ and these classes are the generators for the cohomology algebra $H^*(V_{n+1, n+1-\ell})$.

Consider the space $C_{g_1} = C_\phi \cup_{g_1} C\Sigma P^{n-2}_{n-6}$. (ii) of (5.2)′ shows that

$$\tilde{H}^*(C_{g_1}) \cong \tilde{H}^*(C_\phi) \oplus \tilde{H}^*(\Sigma^2 P^{n-2}_{n-6}) \cong \tilde{H}^*(V_{n+1, n+1-\ell}) \oplus \tilde{H}^*(\Sigma^2 P^{n-7}_{\ell}) \oplus \tilde{H}^*(\Sigma^2 P^{n-2}_{n-6})$$

as a $\mathbf{Z}/2$-module

Proposition 5.3. *In $\tilde{H}^*(C_{g_1}), Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \Sigma^2 x_{2^m-3}$ where $x_{2^{m-1}-1}$ is the normal class in $\tilde{H}^{2^{m-1}-1}(V_{n+1, n+1-\ell})$ and $\Sigma^2 x_{2^m-3}$ is the generator of $\tilde{H}^{2^m-1}(\Sigma^2 P^{n-2}_{n-6}) = \mathbf{Z}/2$.*

This conclusion derived from the assumption that $V_{2^m-1,k}$ is neutral for $2^{m-1}+2 \leqslant k = 2i < 2^m - 2$ with $m \geqslant 5$ is contradictory to the fact that, in the mod 2 cohomology of any space X, $Sq^j x = 0$ if $|x| < j$. Thus the proof of Theorem 1.1 will be completed if we can show Proposition 5.3.

Remark. $Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \Sigma^2 x_{2^m-3}$ in Proposition 5.3 is well defined. This will be explained from the following general situation. Let $X \xrightarrow{f} Y$ with $f^* = 0$ be as in Convention 4.9 so that there is a short exact sequence of A-modules

$$0 \leftarrow \tilde{H}^*(Y) \xleftarrow{i^*} \tilde{H}^*(C_f) \xleftarrow{q^*} \tilde{H}^*(\Sigma X) \leftarrow 0.$$

Given $y \in \tilde{H}^k(Y)$ with $Sq^j y = 0$ in $\tilde{H}^{k+j}(Y)$. By our convention any element in $i^{*-1}(y)$ is also denoted by y. Thus $y \in \tilde{H}^k(C_f)$ is defined with indeterminancy $\tilde{H}^k(\Sigma X)$. If $\tilde{H}^k(X) = 0$ or $\tilde{H}^{k+j}(\Sigma X) = 0$ which implies $\tilde{H}^k(\Sigma X) \xrightarrow{Sq^j} \tilde{H}^{k+j}(\Sigma X)$ is zero then $Sq^j y$ is a well-defined element in $\tilde{H}^{k+j}(C_f)$. In the case Proposition 5.3 we have to apply this twice. First note $Sq^{2^{m-1}}(x_{2^{m-1}-1}) \in \tilde{H}^{2^m-1}(C_\phi)$ is well-defined and is zero. For g_1, the indeterminancy $\tilde{H}^{2^{m-1}-1}(\Sigma^2 P^{n-2}_{n-6})$ is zero. So $Sq^{2^{m-1}}(x_{2^{m-1}-1}) \in \tilde{H}^{2^m-1}(C_{g_1})$ is well defined. This *Remark* also applies to Propositions 5.3′, 5.3″ and 5.3‴ later.

We shall prove a stable version equivalent to Proposition 5.3.

For the remainder of this section all spaces and maps between them will be in the stable category $\mathscr{S}$. In particular, the maps ϕ, g_0, g_1, q in (5.2) and (5.2)′, will be considered as maps in $\mathscr{S}$ when stabilized. In the following we introduce notions and notations for certain maps that we will need.

Definition 5.4. Given three stable maps $Z \xrightarrow{f} Y, X \xrightarrow{g} Z$ and $W \xrightarrow{h} C_g = Z \cup_g CX$. The canonical induced map $C_g = Z \cup_g CX \xrightarrow{f'} C_{fg} = Y \cup_{fg} CX$ is defined by

$$\begin{cases} f'(z) = f(z), & z \in Z, \\ f'([x,t]) = [x,t], & x \in X. \end{cases}$$

The iterated canonical induced map

$$C_h = C_g \cup_h CW \xrightarrow{(f')'} C_{f'h} = C_{fg} \cup_{f'h} CW$$

will be denoted by f''

By Theorem 2.2 there is a stable retraction map $V_{n+1,n+1-\ell} \xrightarrow{r} P_{n+1,n+1-\ell} = P^n_\ell$. Let $\Sigma P^{n-7}_\ell \xrightarrow{\phi_0} P^n_\ell$ be the composite $\Sigma P^{n-7}_\ell \xrightarrow{\phi} V_{n+1,n+1-\ell} \xrightarrow{r} P^n_\ell$ where ϕ is as in (5.2). Then $\phi_0^* = 0$ in mod 2 cohomology as well. Consider the canonical induced map

$$C_\phi = V_{n+1,n+1-\ell} \cup_\phi C\Sigma P^{n-7}_\ell \xrightarrow{r'} C_{\phi_0} = P^n_\ell \cup_{\phi_0} C\Sigma^{n-7}_\ell.$$

Let $\Sigma P^{n-1}_{n-6} \xrightarrow{\bar{g}_0} C_{\phi_0}$ be the composite $\Sigma P^{n-1}_{n-6} \xrightarrow{g_0} C_{\phi} \xrightarrow{r'} C_{\phi_0}$. From (5.2) we see the cofiber of the composite $\Sigma P^{n-1}_{n-6} \xrightarrow{\bar{g}_0} C_{\phi_0} = P^n_\ell \cup_{\phi_0} C\Sigma P^{n-7}_\ell \xrightarrow{q_0} \Sigma^2 P^{n-7}_\ell$ has the homotopy type of $\Sigma^2 P^{n-1}_\ell$ where q_0 is the pinching map. We will identify $C_{q_0\bar{g}_0}$ with $\Sigma^2 P^{n-1}_\ell$ via a suitable homotopy equivalence. For $i = 1,2$ let $\bar{g}_i = \bar{g}_0|\Sigma P^{n-1-i}_{n-6}$. By the identification $C_{g_0\bar{g}_0} = \Sigma^2 P^{n-1}_\ell$ and from (5.2) we have the following.

(5.2)″

(i) For $i = 0,1,2$ the cofiber of the composite
$\Sigma P^{n-1-i}_{n-6} \xrightarrow{\bar{g}_i} C_{\phi_0} = P^n_\ell \cup_{\phi_0} C\Sigma P^{n-7}_\ell \xrightarrow{q_0} \Sigma^2 P^{n-7}_\ell$
is $\Sigma^2 P^{n-1-i}_\ell$.

(ii) $\tilde{H}^*(C_{\phi_0}) \cong \tilde{H}^*(P^n_\ell) \oplus \tilde{H}^*(\Sigma^2 P^{n-7}_\ell) \xrightarrow{\bar{g}_0^*} \tilde{H}^*(\Sigma P^{n-1}_{n-6})$ has $\bar{g}_0^*(x_n) = \Sigma x_{n-1}$, $\bar{g}_0^*(x_j) = 0$ for $\ell \leqslant j \leqslant n-1$ on $\tilde{H}^*(P^n_\ell)$ and $\bar{g}_0^*(\tilde{H}^*(\Sigma^2 P^{n-7}_\ell)) = 0$.

(iii) $\tilde{H}^*(C_{\phi_0}) \xrightarrow{\bar{g}_i^* = 0} \tilde{H}^*(\Sigma P^{n-1-i}_{n-6})$ for $i = 1,2$.

Consider $C_{\bar{g}_1} = C_{\phi_0} \cup_{\bar{g}_1} C\Sigma P^{n-2}_{n-6}$. (iii) of (5.2)″ for $i = 1$ shows that

$$\tilde{H}^*(C_{\bar{g}_1}) \cong \tilde{H}^*(C_{\phi_0}) \oplus \tilde{H}^*(\Sigma^2 P^{n-2}_{n-6}) \cong \tilde{H}^*(P^n_\ell) \oplus \tilde{H}^*(\Sigma^2 P^{n-7}_\ell) \oplus \tilde{H}^*(\Sigma^2 P^{n-2}_{n-6})$$

as a $\mathbf{Z}/2$-module. It is clear that Proposition 5.3 is equivalent to:

Proposition 5.3′. In $\tilde{H}^*(C_{\bar{g}_1}), Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \Sigma^2 x_{2^m-3}$ where $x_{2^{m-1}-1}$ is the generator of $\tilde{H}^{2^{m-1}-1}(P^n_\ell) = \mathbf{Z}/2$ and $\Sigma^2 x_{2^m-3}$ is the generator of $\tilde{H}^{2^m-1}(\Sigma^2 P^{n-2}_{n-6}) = \mathbf{Z}/2$.

We will give two variants of Proposition 5.3′.

Let $P^{n-3}_{n-6} \xrightarrow{\rho} P^{n-2}_{n-6}$ and $P^{n-1}_{n-6} \xrightarrow{\tau} P^{n-1}_{n-2} = S^{n-2} \cup_{2i} e^{n-1}$ be the inclusion map and the collapsing map respectively as usual. We will use $\rho_0, \rho_1, \rho_2, \rho_3, j$ and τ_1 to denote the inclusion maps $P^{n-1}_\ell \to P^n_\ell, P^{n-2}_{n-6} \to P^{n-1}_{n-6}, P^{n-3}_{n-6} \to P^{n-1}_{n-6}, P^{n-3}_\ell \to P^{n-2}_\ell, S^{n-1} \to S^{n-1} \cup_{2i} e^n$ and the collapsing map $P^{n-2}_{n-6} \to S^{n-2}$ respectively. So $\bar{g}_1 = \bar{g}_0 \Sigma\rho_1$ and $\bar{g}_2 = \bar{g}_0 \Sigma\rho_2$ where $\bar{g}_i, i = 0,1,2$, are as in (5.2)″. Consider the diagram

$$\begin{array}{ccccc}
 & & \Sigma P^{n-2}_{n-6} & \xrightarrow{\Sigma\tau_1} & S^{n-1} \\
 & & \downarrow{\scriptstyle \Sigma\rho_1} & (1) & \downarrow{\scriptstyle j} \\
\Sigma P^{n-3}_{n-6} & \xrightarrow{\Sigma\rho_2} & \Sigma P^{n-1}_{n-6} & \xrightarrow{\Sigma\tau} & S^{n-1} \cup_{2i} e^n \\
\| & (2) & \downarrow{\scriptstyle \bar{g}_0} & (3) & \downarrow{\scriptstyle f} \\
\Sigma P^{n-3}_{n-6} & \xrightarrow{\bar{g}_2} & C_{\phi_0} & \xrightarrow{j_0} & C_{\bar{g}_2}
\end{array}$$

where the map from the second row to the third row is a map of cofibrations defined by the commutative square (2). Square (1) also commutative by definition. Let $S^{n-1} \xrightarrow{f_0} C_{\bar{g}_2}$ be the composite fj. From (iii) of $(5.2)''$ for $i=2$ and the isomorphism $\tilde{H}^n(S^{n-1}\cup_{2\iota}e^n) = \mathbf{Z}/2 \underset{\cong}{\overset{\Sigma\tau^*}{\to}} \tilde{H}^n(\Sigma P_{n-6}^{n-1}) = \mathbf{Z}/2$ we have the following.

$$\begin{aligned}&\text{(i)}\ \tilde{H}^*(C_{\bar{g}_2}) \cong \tilde{H}^*(C_{\phi_0}) \oplus \tilde{H}^*(\Sigma^2 P_{n-6}^{n-3}) \cong \tilde{H}^*(P_\ell^n) \oplus \tilde{H}^*(\Sigma^2 P_\ell^{n-7}) \oplus \\ &\quad \tilde{H}^*(\Sigma^2 P_{n-6}^{n-3}) \overset{f^*}{\to} \tilde{H}^*(S^{n-1}\cup_{2\iota}e^n) \text{ has } f^*(x_n) \neq 0. \\ &\text{(ii)}\ \tilde{H}^*(C_{\bar{g}_2}) \xrightarrow{f_0^*=0} \tilde{H}^*(S^{n-1}).\end{aligned} \tag{5.3}$$

(ii) of (5.3) shows $\tilde{H}^*(C_{f_0} = C_{\bar{g}_2} \cup_{f_0} e^n) \cong \tilde{H}^*(C_{\bar{g}_2}) \oplus \tilde{H}^*(S^n)$ as a $\mathbf{Z}/2$-module. Let ι be the generator of $\tilde{H}^n(S^n) = \mathbf{Z}/2$. Since $\bar{g}_1 = \bar{g}_0 \Sigma\rho_1$ and $\tilde{H}^{n-1}(S^{n-1}) = \mathbf{Z}/2 \underset{\cong}{\overset{\Sigma\tau_1^*}{\to}} \tilde{H}^{n-1}(\Sigma P_{n-6}^{n-2}) = \mathbf{Z}/2$, from the composite of squares (1) and (3) above we see Proposition $5.3'$ is equivalent to:

Proposition 5.3″. *In* $\tilde{H}(C_{f_0}), Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \iota \in \tilde{H}^{2^m-1}(S^{2^m-1}) = \tilde{H}^{2^m-1}(S^n) = \mathbf{Z}/2 \subset \tilde{H}^{2^m-1}(C_{f_0})$ *where* $x_{2^{m-1}-1}$ *is the generator of* $\tilde{H}^{2^{m-1}-1}(P_\ell^n) = \mathbf{Z}/2 \subset \tilde{H}^{2^{m-1}-1}(C_{f_0})$.

We will formulate Proposition $5.3''$ in terms of the $(n-1)$-skeleton of $C_{\bar{g}_2}$.

Recall the map $\Sigma P_\ell^{n-7} \overset{\phi_0}{\to} P_\ell^n$ in $(5.2)''$. By dimensional reasons, ϕ_0 is the composite $\Sigma P_\ell^{n-7} \overset{\phi_1}{\to} P_\ell^{n-1} \overset{\rho_0}{\to} P_\ell^n$ for some ϕ_1. Clearly $\phi_1^* = 0$ in mod 2 cohomology too. Consider the canonical induced map

$$C_{\phi_1} = P_\ell^{n-1} \cup_{\phi_1} C\Sigma P_\ell^{n-7} \overset{\rho_0'}{\to} C_{\phi_0} = P_\ell^n \cup_{\phi_0} C\Sigma P_\ell^{n-7}.$$

Then C_{ϕ_1} is the $(n-1)$-skeleton of C_{ϕ_0} via the embedding ρ'_0 and $C_{\phi_0} = C_{\phi_1} \cup e^n$ where the cell e^n is the top cell of P_ℓ^n. By dimensional reasons again, the map $\Sigma P_{n-6}^{n-3} \overset{\bar{g}_2}{\to} C_{\phi_0}$ is the composite $\Sigma P_{n-6}^{n-3} \overset{\bar{\bar{g}}_2}{\to} C_{\phi_1} \overset{\rho_0'}{\to} C_{\phi_0}$ for some $\bar{\bar{g}}_2$. Consider the iterated canonical induced map

$$C_{\bar{\bar{g}}_2} = C_{\phi_1} \cup_{\bar{\bar{g}}_2} C\Sigma P_{n-6}^{n-3} \overset{\rho_0''}{\to} C_{\bar{g}_2} = C_{\phi_0} \cup_{\bar{g}_2} C\Sigma P_{n-6}^{n-3}.$$

Then $C_{\bar{\bar{g}}_2}$ is the $(n-1)$-skeleton of $C_{\bar{g}_2}$ via the embedding ρ_0'' and $C_{\bar{g}_2} = C_{\bar{\bar{g}}_2} \cup e^n$ where the cell e^n is the top cell of P_ℓ^n which is also the top cell of C_{ϕ_0}. Let $S^{n-1} \overset{\sigma}{\to} P_\ell^{n-1}$ be the attaching map for the top cell of P_ℓ^n. Then the composite

$$\sigma_2 : S^{n-1} \overset{\sigma}{\to} P_\ell^{n-1} \hookrightarrow C_{\phi_1} \hookrightarrow C_{\bar{\bar{g}}_2}$$

is the attaching map for the top cell of $C_{\bar{g}_2}$ so that there is a cofibration sequence

$$S^{n-1} \overset{\sigma_2}{\to} C_{\bar{\bar{g}}_2} \overset{\rho_0''}{\to} C_{\bar{g}_2} \overset{\delta}{\to} S^n$$

where δ is the pinching map. Note that $\tilde{H}^n(C_{\bar{g}_2}) \cong \mathbf{Z}/2 \underset{\cong}{\overset{\delta^*}{\to}} \tilde{H}^n(S^n) = \mathbf{Z}/2$. In fact, $\tilde{H}^*(C_{\bar{g}_2}) \cong \tilde{H}^*(C_{\bar{\bar{g}}_2}) \oplus \tilde{H}^*(S^n)$ as an A-module since $\tilde{H}^*(P_\ell^n) \cong \tilde{H}^*(P_\ell^{n-1}) \oplus \tilde{H}^*(S^n)$ as an A-module as $n = 2^m - 1$. By dimensional reasons, $S^{n-1} \overset{f_0}{\to} C_{\bar{g}_2}$ in Proposition $5.3''$ is the composite $S^{n-1} \overset{f_1}{\to} C_{\bar{\bar{g}}_2} \overset{\rho_0''}{\to} C_{\bar{g}_2}$ for some f_1 .f_1 also has $f_1^* = 0$ in mod 2 cohomology. So

$$\begin{aligned}\tilde{H}^*(C_{f_1} = C_{\bar{\bar{g}}_2} \cup_{f_1} e^n) &\cong \tilde{H}^*(C_{\bar{\bar{g}}_2}) \oplus \tilde{H}^*(S^n)\\ &\cong \tilde{H}^*(C_{\phi_1}) \oplus \tilde{H}^*(\Sigma^2 P^{n-3}_{n-6}) \oplus \tilde{H}^*(S^n)\\ &\cong \tilde{H}^*(P^{n-1}_\ell) \oplus \tilde{H}^*(\Sigma^2 P^{n-7}_\ell) \oplus \tilde{H}^*(\Sigma^2 P^{n-3}_{n-6}) \oplus \tilde{H}^*(S^n)\end{aligned}$$

as a **Z**/2-module. It is clear that Proposition 5.3″ is equivalent to:

Proposition 5.3‴. In $\tilde{H}^*(C_{f_1}), Sq^{2^{m-1}}(x_{2^{m-1}-1}) = \iota \in \tilde{H}^{2^m-1}(S^n) = \mathbf{Z}/2 \subset \tilde{H}^{2^m-1}(C_{f_1})$ *where* $x_{2^{m-1}-1}$ *is the generator of* $\tilde{H}^{2^{m-1}-1}(P^{n-1}_\ell) = \mathbf{Z}/2 \subset \tilde{H}^{2^{m-1}-1}(C_{f_1})$.

The map f_1 in Proposition 5.3‴ has the following property.

Lemma 5.5. *The composite* $S^{n-1} \xrightarrow{2\iota} S^{n-1} \xrightarrow{f_1} C_{\bar{\bar{g}}_2}$ *is an odd multiple of the attaching map* σ_2 *for the top cell of* $C_{\bar{g}_2} = C_{\bar{\bar{g}}_2} \cup e^n$.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
S^{n-1} & \xrightarrow{2l} & S^{n-1} & \xrightarrow{j} & S^{n-1}\cup_{2l} e^n & \xrightarrow{\tau_2} & S^n & \xrightarrow{2l} & S^n \\
\Sigma^{-1}h \downarrow & \Sigma^{-1}(3) & f_1 \downarrow & (1) & f \downarrow & (2) & h \downarrow & (3) & \Sigma f_1 \downarrow \\
S^{n-1} & \xrightarrow{\sigma_2} & C_{\bar{\bar{g}}_2} & \xrightarrow{\rho_0''} & C_{\bar{g}_2} & \xrightarrow{\delta} & S^n & \xrightarrow{\Sigma\sigma_2} & \Sigma C_{\bar{\bar{g}}_2}
\end{array}$$

described as follows. The portion consisting of squares (1), (2) and (3) is a map of cofiber sequences defined by the commutative square (1) (recall $f_0 = fj$) where τ_2 is the pinching map. $\Sigma^{-1}(3)$ is the desuspension of (3). Since f^*,τ_2^* and δ^* are isomorphisms in dimension n (for f^*, this follows from (i) of (5.3)) it follows that deg h is odd. So deg($\Sigma^{-1}h$) is odd. This proves Lemma 5.5.

We shall prove Proposition 5.3‴ by looking at the mod 2 Adams spectral sequence for $\pi_*^s(C_{\bar{\bar{g}}_2})$ and $\pi_*^s(C_{\bar{g}_2})$ using Lemma 5.5 and also Lemma 5.6 below. To describe Lemma 5.6 consider the commutative diagram

$$\begin{array}{ccccc}
\Sigma P^{n-3}_{n-6} & \xrightarrow{\bar{\bar{g}}_2} & C_{\phi_1} = P^{n-1}_\ell \cup_{\phi_1} C\Sigma P^{n-7}_\ell & \xrightarrow{q_1} & \Sigma^2 P^{n-7}_\ell \\
\| & & \rho_0' \downarrow & & \| \\
\Sigma P^{n-3}_{n-6} & \xrightarrow{\bar{g}_2} & C_{\phi_0} = P^{n}_\ell \cup_{\phi_0} C\Sigma P^{n-7}_\ell & \xrightarrow{q_0} & \Sigma^2 P^{n-7}_\ell
\end{array}$$

where q_0, q_1 are pinching maps. By (i) of (5.2)″, $C_{q_0 \bar{g}_2} = \Sigma^2 P^{n-3}_\ell$. So $C_{q_1 \bar{\bar{g}}_2} = \Sigma^2 P^{n-3}_\ell$. Consider the canonical induced map $C_{\bar{\bar{g}}_2} \xrightarrow{q_1'} C_{q_1 \bar{\bar{g}}_2} = \Sigma^2 P^{n-3}_\ell$. Let $P^{n-1}_\ell \xrightarrow{j_2} C_{\bar{\bar{g}}_2}$ be the inclusion map.

Lemma 5.6. *There is a short exact sequence of A-modules*

$$0 \leftarrow \tilde{H}^*(P_\ell^{n-1}) \overset{j_2^*}{\leftarrow} \tilde{H}(C_{\bar{g}_2}) \overset{(q_1')^*}{\leftarrow} \tilde{H}^*(\Sigma^2 P_\ell^{n-3}) \leftarrow 0.$$

Proof. From $\dim_{\mathbf{Z}/2}$ of the $\mathbf{Z}/2$-module $\tilde{H}^*(C_{\bar{g}_2}) \cong \tilde{H}^*(P_\ell^{n-1}) \oplus \tilde{H}^*(\Sigma^2 P_\ell^{n-7}) \oplus \tilde{H}^*(\Sigma^2 P_{n-6}^{n-3})$ we see j_2^* is onto and $(q_1')^*$ is $1-1$. □

For a space or a spectrum X we write $Ext_A^{s,t}(X)$ to denote $Ext_A^{s,t}(\tilde{H}^*(X), \mathbf{Z}/2)$ which is the E_2-term of the mod 2 Adams spectral sequence for the stable homotopy groups ${}_2\pi_*^s(X)$, to be abbreviated as "the ASS the X". $Ext_A^{s,t+k}(S^k) = Ext_A^{s,t}(S^0) = Ext_A^{s,t}(\mathbf{Z}/2, \mathbf{Z}/2)$ will simply be denoted by $Ext_A^{s,t}$. Recall that $Ext_A^{*,*}(X)$ is a right $Ext_A^{*,*}$-module for any X. Note that $Ext_A^{s,t}(P_\ell^n) = Ext_A^{s,t}(P_\ell^{n-1}) \oplus Ext_A^{s,t}(S^n)$ and $Ext_A^{s,t}(C_{\bar{g}_2}) = Ext_A^{s,t}(C_{\bar{g}_2}) \oplus Ext_A^{s,t}(S^n)$ since $\tilde{H}^*(P_\ell^n)$ (resp. $\tilde{H}(C_{\bar{g}_2})$) is isomorphic to $\tilde{H}^*(P_\ell^{n-1}) \oplus \tilde{H}^*(S^n)$ (resp. $\tilde{H}^*(C_{\bar{g}_2}) \oplus \tilde{H}^*(S^n)$) as an A-module.

To prove Proposition 5.3‴ we need only the knowledge of $Ext_A^{s,t(s)}(P_\ell^{n-i})$, for $i = 0,1,3$, $s = 0,1,2$ and certain $t(s)$. These Ext groups are calculated in [3], but not explicitly stated there. In order to describe these groups let $\tilde{H}_*(P_b^a)$ be the reduced mod 2 homology groups of P_b^a, $1 \leqslant b < a$. Let e_k be the generator of $\tilde{H}_k(P_b^a) = \mathbf{Z}/2$ for $b \leqslant k \leqslant a$ and set $e_k = 0$ if $k > a$ or $k < b$. $\tilde{H}^*(P_b^a)$ is a right A-module by

$$e_k Sq^j = \sum_{j>0} \binom{k-j}{j} e_{k-j}$$

which is obtained by dualizing (2.2). If e_k is a primitive element, that is, if $e_k Sq^j = 0$ for all $j > 0$, then let $\bar{e}_k$ denote the corresponding class in $Ext_A^{0,*}(P_b^a)$. The following is easy to see (recall $n = 2^m - 1, \ell = 2^{m-1} - 3$).

$$\begin{aligned}&\{\bar{e}_{2^{m-1}-3}, \bar{e}_{2^{m-1}-1}\} \text{ is a } \mathbf{Z}/2\text{-base for } Ext_A^{0,*}(P_\ell^{n-i}),\ i = 1,3, \text{ and}\\ &\{\bar{e}_{2^{m-1}-3}, \bar{e}_{2^{m-1}-1}, \bar{e}_{2^m-1}\} \text{ is a } \mathbf{Z}/2\text{-base for } Ext_A^{0,*}(P_\ell^n).\end{aligned} \tag{5.4}$$

Let $h_j \in Ext_A^{1,2^j}$ be the class corresponding to the generator $Sq^{2^j} \in A$. Recall [1] that $\{h_j \mid j \geqslant 0\}$ is a $\mathbf{Z}/2$-base for $Ext_A^{2,*}$. Since $Ext_A^{*,*}(P_b^a)$ is a right $Ext_A^{*,*}$-module, for any $\alpha \in Ext_A^{s,*}(P_b^a)$ we may consider $\alpha h_j \in Ext_A^{s+1,*}(P_b^a)$ and $\alpha h_j h_k \in Ext_A^{s+2,*}(P_b^a)$.

The following result is proved in [3] with (1) through (5) implied by the calculations there. Recall $n = 2^{m-1}$ and $\ell = 2^{m-1} - 3$ with $m \geqslant 5$.

Proposition 5.7.

(1) *For $i = 0,1$, $Ext_A^{1,2^m-1}(P_\ell^{n-i}) \cong \mathbf{Z}/2$, generated by $\bar{e}_{2^{m-1}-1} h_{m-1}$.*
(2) $Ext_A^{1,2^m-2}(P_\ell^{n-3}) = 0$.
(3) $Ext_A^{1,2^m-3}(P_\ell^{n-3}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$, *generated by $\bar{e}_{2^{m-1}-3} h_{m-1}$* and $\overline{e_{2^m-5} h_1}$ *with $\bar{e}_{2^{m-1}-3} h_{m-1} h_0 = 0$, $\overline{e_{2^m-5} h_1} h_0 = 0$ in $Ext_A^{2,2^m-2}(P_\ell^{n-3})$.*
(4) *For $i = 0,1$, $Ext_A^{2,2^m}(P_\ell^{n-i}) \cong \mathbf{Z}/2$, generated by $\bar{e}_{2^{m-1}-1} h_{m-1} h_0$.*
(5) $Ext_A^{2,2^m-1}(P_\ell^{n-1}) \cong \mathbf{Z}/2$, *generated by $\bar{e}_{2^{m-1}-3} h_{m-1} h_1$.*
(6) *In the ASS for P_ℓ^n, $d_2(\bar{e}_{2^m-1}) = \bar{e}_{2^{m-1}-1} h_{m-1} h_0 \neq 0$.*

We are not going to describe here the element $\overline{e_{2^m-5}h_1}$ in (3) of Propostion 5.7. It suffices to note that $\dim_{\mathbf{Z}/2}$ of $\mathrm{Ext}_A^{1,2^m-3}(P_\ell^{n-3})$ is 2 and that $\mathrm{Ext}_A^{1,2^m-3}(P_\ell^{n-3})h_0 = 0 \subset \mathrm{Ext}_A^{2,2^m-2}(P_\ell^{n-3})$.

Consider the induced homomorphisms $Ext_A^{s,t}(P_\ell^{n-1}) \xrightarrow{j_{2*}} Ext_A^{s,t}(C_{\bar{g}_2})$ and $Ext_A^{s,t}(P_\ell^n) \xrightarrow{\bar{j}_{2*}} Ext_A^{s,t}(C_{\bar{\bar{g}}_2})$ of the inclusion maps $P_\ell^{n-1} \xrightarrow{j_2} C_{\bar{g}_2}$ and $P_\ell^n \xrightarrow{\bar{j}_2} C_{\bar{\bar{g}}_2}$ respectively. For $\alpha \in Ext_A^{s,t}(P_\ell^{n-1})$ (resp. $Ext_A^{s,t}(P_\ell^n)$) its image in $Ext_A^{s,t}(C_{\bar{g}_2})$ (resp. $Ext_A^{s,t}(C_{\bar{\bar{g}}_2})$) under j_{2*} (resp. $\bar{j}_{2*}$) is also denoted by α. In particular, there are elements

$$\bar{e}_{2^{m-1}-1}h_{m-1} \in Ext_A^{1,2^m-1}(C_{\bar{g}_2}), Ext_A^{1,2^m-1}(C_{\bar{\bar{g}}_2})$$

$$\bar{e}_{2^{m-1}-1}h_{m-1}h_0 \in Ext_A^{2,2^m}(C_{\bar{g}_2}), Ext_A^{2,2^m}(C_{\bar{\bar{g}}_2})$$

and

$$\bar{e}_{2^m-1} \in Ext_A^{0,2^m-1}(C_{\bar{\bar{g}}_2}) = (Ext_A^{0,2^m-1}(C_{\bar{g}_2}) = 0) \oplus Ext_A^{0,2^m-1}(S^n = S^{2^m-1}).$$

$$\bar{e}_{2^m-1} \in Ext_A^{0,2^m-1}(C_{\bar{\bar{g}}_2}) \text{ is the class corresponding to the cell } e^n \text{ of } C_{\bar{\bar{g}}_2} = C_{\bar{g}_2} \cup e^n.$$

The short exact sequence in Lemma 5.6 gives rise to a long exact sequence of Ext groups

$$\text{(a)} \quad \cdots \to Ext_A^{s-1,t}(\Sigma^2 P_\ell^{n-3}) \xrightarrow{\delta_*} Ext_A^{s,t}(P_\ell^{n-1}) \xrightarrow{j_{2*}} Ext_A^{s,t}(C_{\bar{g}_2})$$

$$\xrightarrow{(q_1')_*} Ext_A^{s,t}(\Sigma^2 P_\ell^{n-3}) \xrightarrow{\delta_*} \cdots.$$

By (5.4) and (2) of Proposition 5.7, $Ext_A^{0,2^m-1}(\Sigma^2 P_\ell^{n-3}) = Ext_A^{0,2^m-3}(P_\ell^{n-3}) = 0$ and $Ext_A^{1,2^m}(\Sigma^2 P_\ell^{n-3}) = Ext_A^{1,2^m-2}(P_\ell^{n-3}) = 0$. From these, (a) and (1),(4) of Proposition 5.7, we have the following.

(b) $Ext_A^{2,2^m}(P_\ell^{n-1}) = \mathbf{Z}/2(\bar{e}_{2^{m-1}-1}h_{m-1}h_0) \xrightarrow{j_{2*}} Ext_A^{2,2^m}(C_{\bar{g}_2})$ is 1–1. So $\bar{e}_{2^{m-1}-1}h_{m-1}h_0 \neq 0$ in $Ext_A^{2,2^m}(C_{\bar{g}_2})$. This implies $\bar{e}_{2^{m-1}-1}h_{m-1}h_0 \neq 0$ in $Ext_A^{2,2^m}(C_{\bar{\bar{g}}_2}) = Ext_A^{2,2^m}(C_{\bar{g}_2}) \oplus (Ext_A^{2,2^m}(S^n = S^{2^{m-1}}) = 0)$.

(c) There is a short exact sequence

$$0 \to Ext_A^{1,2^m-1}(P_\ell^{n-1}) = \mathbf{Z}/2(\bar{e}_{2^{m-1}-1}h_{m-1}) \xrightarrow{j_{2*}} Ext_A^{1,2^m-1}(C_{\bar{g}_2})$$

$$\xrightarrow{(q_1')_*} ker\,\delta_* \to 0$$

where δ_* is $Ext_A^{1,2^m-1}(\Sigma^2 P_\ell^{n-3}) \xrightarrow{\delta_*} Ext_A^{2,2^m-1}(P_\ell^{n-1})$. By (3),(5) of Proposition 5.7, $dim_{\mathbf{Z}/2}$ of $Ext_A^{1,2^m-1}(\Sigma^2 P_\ell^{n-3}) = Ext_A^{1,2^m-3}(P_\ell^{n-3})$ is 2, $dim_{\mathbf{Z}/2}$ of $Ext_A^{2,2^m-1}(P_\ell^{n-1})$ is 1 and $Ext_A^{1,2^m-1}(\Sigma^2 P_\ell^{n-3})h_0 = 0$. Thus $dim_{\mathbf{Z}/2}(ker\,\delta_*) = 1$ or 2. Let $\{y_1', \ldots, y_q'\}$ be a $\mathbf{Z}/2$-base for $ker\,\delta_* \subset Ext_A^{1,2^m-1}(\Sigma^2 P_\ell^{n-3}), 1 \leqslant q \leqslant 2$; so $y_j'h_0 = 0$ in $Ext_A^{2,2^m}(\Sigma^2 P_\ell^{n-3})$ for $1 \leqslant j \leqslant q$. Choose $y_j \in Ext_A^{1,2^m-1}(C_{\bar{g}_2})$ for each j such that $(q_1')_*(y_j) = y_j'$. Then (c) shows that $\{\bar{e}_{2^{m-1}-1}h_{m-1}, y_1, \ldots, y_q\}$ is a $\mathbf{Z}/2$-base for $Ext_A^{1,2^m-1}(C_{\bar{g}_2})$. We claim

(d) $\{y_1, \ldots, y_q\} \subset \{\bar{e}_{2^{m-1}-1}h_{m-1}, y_1, \ldots, y_q\}$ can be chosen such that $y_jh_0 = 0$ in $Ext_A^{2,2^m}(C_{\bar{g}_2})$ for $1 \leqslant j \leqslant q$.

To see this, suppose y_j is an element in $\{y_1, \ldots, y_q\}$ such that $y_j h_0 \neq 0$. Since $(q'_1)_*(y_j h_0) = y'_j h_0 = 0$, it follows from (a), (b) that $y_j h_0$ must be $\bar{e}_{2^{m-1}-1} h_{m-1} h_0$. Let $\bar{y}_j = \bar{e}_{2^{m-1}-1} h_{m-1} + y_j$. Then $\bar{y}_j$ also satisfies $(q'_1)_*(\bar{y}_j) = 0$ and has $\bar{y}_j h_0 = 0$. And $\{\bar{e}_{2^{m-1}-1} h_{m-1}, y_1, \ldots, \bar{y}_j, \ldots, y_q\}$ (with y_j replaced by $\bar{y}_j$) is also a $\mathbf{Z}/2$-base for $Ext_A^{1,2^m-1}(C_{\bar{g}_2})$. This proves (d). In what follows, $\{\bar{e}_{2^{m-1}-1} h_{m-1}, y_1, \ldots, y_q\}$ will be a $\mathbf{Z}/2$-base for $Ext_A^{1,2^m-1}(C_{\bar{g}_2})$ with the property in (d).

Let $S^{n-1} \xrightarrow{f_1} C_{\bar{g}_2}$ be as in Proposition 5.3‴. Since $\bar{e}_{2^{m-1}-1}$ is dual to $x_{2^{m-1}-1}$ and h_{m-1} is dual to $Sq^{2^{m-1}}$ it follows that to prove Proposition 5.3‴ is equivalent to proving that f_1 is detected, in the ASS for $C_{\bar{g}_2}$, by an element $\alpha \in Ext_A^{1,2^m-1}(C_{\bar{g}_2})$ of the form $\alpha = \bar{e}_{2^{m-1}-1} h_{m-1} + \sum_{j=1}^{q} \varepsilon_j y_j$ for some ε_j.

By Lemma 5.5, the composite $S^{n-1} \xrightarrow{2\iota} S^{n-1} \xrightarrow{f_1} C_{\bar{g}_2}$ is an odd multiple of the attaching map $S^{n-1} \xrightarrow{\sigma_2} C_{\bar{g}_2}$ for the top cell e^n of $C_{\bar{g}_2} = C_{\bar{g}_2} \cup e^n$. (6) of Proposition 5.7 and (b) imply that, in the ASS for $C_{\bar{g}_2} = C_{\bar{g}_2} \cup e^n$, $d_2(\bar{e}_{2^m-1}) = \bar{e}_{2^{m-1}-1} h_{m-1} h_0 \neq 0$. This in turn implies that $S^{n-1} \xrightarrow{\sigma_2} C_{\bar{g}_2}$ is detected by $\bar{e}_{2^{m-1}-1} h_{m-1} h_0 \neq 0$ in the ASS for $C_{\bar{g}_2}$ as $\bar{e}_{2^m-1}$ corresponds to the cell e^n. Since 2ι is detected by h_0 it follows that f_1 is detected by an element in $Ext_A^{1,2^m-1}(C_{\bar{g}_2})$ of the form $\bar{e}_{2^{m-1}-1} h_{m-1} + \sum_{j=1}^{q} \varepsilon_j y_j$ as $(\sum_{j=1}^{q} \varepsilon_j y_j) h_0 = 0$. This proves Proposition 5.3‴.

This completes the proof of Proposition 5.3 and therefore Theorem 1.1.

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