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A Characterization of (Locally) Uniformly Convex Spaces in Terms of Relative Openness of Quotient Maps on the Unit Ball

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Relative openness of quotient maps on the closed unit ball U of a normed linear

uniformly convex if and only if for any family of linear maps defined on X, equal relative openness on X implies equal relative openness on U. Similarly, uniformly convex spaces can be characterized in terms of equal uniform relative openness of quotient maps on U. © 2000 Academic Press

1. INTRODUCTION

Let X be a real normed linear space and U the closed unit ball of X. The space X is said to be locally uniformly convex if for each $x \in U$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that for each $y \in U$ with $\|x - y\| > \varepsilon$ we have $\|(x+y)/2\| < 1 - \delta$. If, for each $\varepsilon > 0$, such a $\delta > 0$ can be chosen so that it depends only on ε then X is said to be uniformly convex. It should be noted that, following [9], the term (locally) uniformly rotund space is sometimes used for such a space.

Uniformly convex and locally uniformly convex spaces play a central role in the structure theory and renormings of Banach spaces (see e.g. the monographs [2, 9–11]) and some properties of these spaces apply to solving miscellaneous problems of functional analysis (e.g. [1, 3, 12, 15, 20, 24]).

We exhibit connections of (local) uniform convexity of the space X with a certain quality of relative openness of the quotient maps on U. A map T defined on X is said to be relatively open on U if T maps the sets which are relatively open in U onto sets which are relatively open in $T(U)$.

Relative openness of linear maps on convex subsets has been studied in several papers in various contexts ([5–7, 13, 14, 17–19, 21–23, 25, 27]). It

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is somewhat surprising that a linear map on X can fail to be relatively open on U even for a three-dimensional space X. For example, it is easy to check that the map $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T((x, y, z)) = (x + z, y) \qquad \text{for} \quad (x, y, z) \in \mathbb{R}^3$$

is not relatively open on the unit ball

$$U = \{(x, y, z) : (x^2 + y^2)^{1/2} + |z| \leqslant 1\}$$

because T does not map neighbourhoods of the point $u = (0, 0, 1)$ in U onto neighbourhoods of $T(u)$ in $T(U)$.

It follows from [22, Theorem (1)] that if X is a finite-dimensional space then every linear map defined on X is relatively open on U if and only if U has property (P) defined by Wegmann [26]. For instance, U has property (P) whenever U is a finite-dimensional polyhedron, or, whenever X is a strictly convex space. Eifler [13] conjectured that if X is a strictly convex Banach space then any continuous linear open map defined on X is relatively open on U. So, by the above characterization, the conjecture is true if X is finite-dimensional. However, we note that it is false in general; Brown [4] has constructed a strictly convex reflexive space X and a closed linear subspace M of X such that the metric projection of X onto M is discontinuous, thus, by [23, Corollary (4)], the associated quotient map from X onto X/M fails to be relatively open on U.

It follows from the results of the present paper (see Lemma 4.4) that if X is locally uniformly convex, any linear open map defined on X is relatively open on U. Moreover, local uniform convexity of X is equivalent to equal relative openness of the quotient maps on U (by a quotient map we mean the canonical quotient map from X onto X/M associated with a closed linear subspace M of X). Furthermore, X is locally uniformly convex if and only if for any family of linear maps defined on X, equal relative openness on X implies equal relative openness on U. Uniformly convex spaces are characterized in a similar manner (Theorem 3.5).

2. BASIC NOTIONS

Throughout the paper, X stands for a real normed linear space and U for the closed unit ball of X, dim X denotes dimension of X and $\mathbb{R}$ the set of real numbers.

Let $\varepsilon > 0$. The modulus of local convexity $\delta(x, \varepsilon)$, where $x \in U$, and the modulus of convexity $\delta(\varepsilon)$ are defined by

$$\delta(x, \varepsilon) = \inf\{1 - \|(x + y)/2\| : y \in U, \|x - y\| \geqslant \varepsilon\}$$

and

$$\delta(\varepsilon)=\inf\{1-\|(x+y)/2\| : x,\ y\in U,\ \|x-y\|\geqslant\varepsilon\}.$$

It is easily seen that

$$\delta(\varepsilon)=\inf\{\delta(x,\varepsilon): x\in U\}. \tag{2.1}$$

It is also known (see e.g. [8]) that if $\dim X\geqslant 2$ then

$$\delta(\varepsilon)=\inf\{1-\|(x+y)/2\| : x,\ y\in X,\ \|x\|=\|y\|=1,\ \|x-y\|=\varepsilon\},$$

and, for any $x\in X$ of norm one,

$$\delta(x,\varepsilon)=\inf\{1-\|(x+y)/2\| : y\in X,\ \|y\|=1,\ \|x-y\|=\varepsilon\}.$$

Clearly, X is locally uniformly convex if and only if $\delta(x,\varepsilon)>0$ for each $x\in U$ and $\varepsilon>0$, and, X is uniformly convex if and only if $\delta(\varepsilon)>0$ for each $\varepsilon>0$.

The exact values, or their estimates, of the moduli $\delta(x,\varepsilon)$ and $\delta(\varepsilon)$ are known for some classical spaces.

For example, let X be a Hilbert space, $\dim X\geqslant 2$, and let $x\in U$ and $\varepsilon>0$. Using the parallelogram identity in the case of $|\varepsilon-1|\leqslant\|x\|$ and the triangle inequality $\|x+y\|\leqslant\|x\|+1$ for $y\in U$ in the case of $\varepsilon<1-\|x\|$, one gets readily that

$$\delta(x,\varepsilon)=\begin{cases}(1-\|x\|)/2 & \text{for}\quad 0<\varepsilon<1-\|x\|,\\ 1-2^{-1}(2+2\,\|x\|^2-\varepsilon^2)^{1/2} & \text{for}\quad 1-\|x\|\leqslant\varepsilon\leqslant 1+\|x\|,\\ \infty & \text{for}\quad \varepsilon>1+\|x\|,\end{cases}$$

and

$$\delta(\varepsilon)=\begin{cases}1-(1-\varepsilon^2/4)^{1/2} & \text{for}\quad 0<\varepsilon\leqslant 2,\\ \infty & \text{for}\quad \varepsilon>2.\end{cases}$$

The exact value of $\delta(\varepsilon)$ for the space $L_p(\mu)$ can be found in [16]. It depends only on p (but not on the measure μ) and we quote here only its asymptotic estimate for $\varepsilon\to 0$:

$$\delta(\varepsilon)=\begin{cases}(p-1)\,\varepsilon^2/8+o(\varepsilon^2) & \text{for}\quad 1<p\leqslant 2,\\ p^{-1}(\varepsilon/2)^p+o(\varepsilon^p) & \text{for}\quad 2<p<\infty.\end{cases}$$

The moduli of convexity have been studied for some other spaces, see e.g. [11, p. 84 and p. 89] for references.

DEFINITION 2.1. Let $\mathscr{T}$ be a family of maps defined on X and let A be a subset of X. We shall say that the maps from $\mathscr{T}$ are equally relatively open on A if for each $x \in A$ and $\varepsilon > 0$ there is a $\rho > 0$ such that every $T \in \mathscr{T}$ maps the ε-neighbourhood of x in A onto a set containing the ρ-neighbourhood of $T(x)$ in $T(A)$. If such a ρ can be chosen so that it depends only on ε (and not on x), we shall say that the maps from $\mathscr{T}$ are equally uniformly relatively open on A.

We now establish concepts which evaluate equal (uniform) relative openness of quotient maps on U.

DEFINITION 2.2. For any closed linear subspace M of X, let Q_M be the quotient map associated with M, $Q_M: X \to X/M$. Let $\varepsilon > 0$. For any $x \in U$ we define $\rho(x, \varepsilon) = \sup\{r : r \geqslant 0$ and for each closed linear subspace M of X and each $y \in Q_M(U)$ with $\|y - Q_M(x)\| < r$ there is a $u \in U$ such that $\|u - x\| < \varepsilon$ and $Q_M(u) = y\}$, i.e., $\rho(x, \varepsilon) \in [0, \infty]$ is such that every quotient map Q_M maps the ε-neighbourhood of x in U onto a set containing the r-neighbourhood of $Q_M(x)$ in $Q_M(U)$ with $r = \rho(x, \varepsilon)$, and no greater r has this property.

We define further $\rho(\varepsilon) = \inf\{\rho(x, \varepsilon) : x \in U\}$.

Remark 2.1. Let X, Y be normed linear spaces and $T: X \to Y$ an open linear map such that the kernel of T is closed in X. Let $x_0 \in U$. If $\rho(x_0, \varepsilon) > 0$ for each $\varepsilon > 0$ then certain relative openness of T on U at x_0 is guaranteed.

More precisely, let $c > 0$ be such that for each $y \in Y$ of norm $\|y\| < c$ there is $x \in X$ of norm $\|x\| < 1$ such that $T(x) = y$. Then, for any $\varepsilon > 0$, T maps the ε-neighbourhood of x_0 in U onto a set containing the $c\rho(x_0, \varepsilon)$-neighbourhood of $T(x_0)$ in $T(U)$.

To see this, let M denote the kernel of T, $M = \{x \in X : T(x) = 0\}$, and let $Q: X \to X/M$ be the quotient map associated with M. Consider the map S from X/M onto Y defined by the formula

$$S(Q(x)) = T(x) \qquad \text{for} \quad x \in X.$$

Then, clearly, S is well-defined, linear and one-to-one. Furthermore,

$$\|S^{-1}\| \leqslant c^{-1},$$

which yields that S maps the $\rho(x_0, \varepsilon)$-neighbourhood of $Q(x_0)$ in $Q(U)$ onto a set containing the $c\rho(x_0, \varepsilon)$-neighbourhood of $S(Q(x_0)) = T(x_0)$ in $S(Q(U)) = T(U)$.

Since, by the definition, Q maps the ε-neighbourhood of x_0 in U onto a set containing the $\rho(x_0, \varepsilon)$-neighbourhood of $Q(x_0)$ in $Q(U)$ and T is the composition of Q and S, our claim follows immediately.

3. RESULTS

We recall that U is the closed unit ball of a real normed linear space X, $\delta(x, \varepsilon)$ the modulus of local convexity of X, $\delta(\varepsilon)$ the modulus of convexity of X (for definitions see Section 2) and the moduli $\rho(x, \varepsilon)$ and $\rho(\varepsilon)$ were established in Definition 2.2.

THEOREM 3.1. *Let $\varepsilon > 0$ and $x \in U$. Then*

$$\rho(x, \varepsilon) \geqslant \min\left\{\frac{2}{3}\,\delta(x, \varepsilon), \frac{\varepsilon}{2}\right\} \tag{3.1}$$

and, if $\|x\| = 1$,

$$\rho(x, \varepsilon) \geqslant \tfrac{2}{3}\delta(x, \varepsilon). \tag{3.2}$$

THEOREM 3.2. *For any $\varepsilon > 0$,*

$$\rho(\varepsilon) \geqslant \tfrac{2}{3}\delta(\varepsilon).$$

Moreover, there exists a function $g(\varepsilon)$ defined on $(0, 2]$ such that $g(\varepsilon) \to 2$ for positive $\varepsilon \to 0$ and

$$\rho(\varepsilon) \geqslant g(\varepsilon)\,\delta(\varepsilon) \qquad \textit{for} \quad \varepsilon \in (0, 2].$$

If $\dim X \geqslant 2$ *then the function $g(\varepsilon) = 2\varepsilon(\varepsilon + 4\delta(\varepsilon))^{-1}$ has these properties.*

THEOREM 3.3. *Let $x \in U$, $\varepsilon > 0$ and $\lambda \in (1, 3]$ be arbitrary. Then*

$$\rho(x, \varepsilon) \leqslant 4(\lambda - 1)^{-1}\,\delta(x, \lambda\varepsilon) \tag{3.3}$$

and

$$\rho(\varepsilon) \leqslant 4(\lambda - 1)^{-1}\,\delta(\lambda\varepsilon). \tag{3.4}$$

THEOREM 3.4. *The following statements are equivalent*:

(i) *X is locally uniformly convex*;

(ii) *the quotient maps $Q_M\colon X \to X/M$ associated with the closed linear subspaces M of X are equally relatively open on U*;

(iii) *for any family of linear maps defined on X, equal relative openness on X implies equal relative openness on U.*

THEOREM 3.5. *The following statements are equivalent*:

(i) *X is uniformly convex*;

(ii) *the quotient maps $Q_M: X \to X/M$ associated with the closed linear subspaces M of X are equally uniformly relatively open on U*;

(iii) *for any family of linear maps defined on X, equal relative openness on X implies equal uniform relative openness on U.*

4. PROOFS OF THE RESULTS

Since the assertions in Section 3 are obviously true for the trivial space $X = \{0\}$, we assume further that $\dim X \geqslant 1$. We start with simple observations.

Remark 4.1. Let $0 < \varepsilon \leqslant 2$. Then

$$\delta(x, \varepsilon) \leqslant \varepsilon/2 \qquad \text{whenever} \quad x \in X,\ \|x\| = 1; \tag{4.1}$$

$$\delta(\varepsilon) \leqslant \varepsilon/2; \tag{4.2}$$

$$\rho(x, \varepsilon) \leqslant \varepsilon \qquad \text{whenever} \quad x \in U,\ \varepsilon \leqslant 1 + \|x\|; \tag{4.3}$$

$$\rho(\varepsilon) \leqslant \varepsilon. \tag{4.4}$$

To show (4.1), choose $y = (1 - \varepsilon)\, x$; then $\delta(x, \varepsilon) \leqslant 1 - \|(x + y)/2\| = \varepsilon/2$.

To see (4.3), consider the identity map on X (which is the quotient map associated with the trivial subspace $M = \{0\}$) and use the fact that U is not contained in the ε-neighbourhood of x.

Since X is not trivial, there is some $x \in X$ of norm one, hence (4.2) follows from (4.1), and, (4.4) follows from (4.3).

Notation. In Lemma 4.1, Lemma 4.2 and in the proof of Theorem 3.1, let $x \in U$, $\varepsilon > 0$, $\delta = \delta(x, \varepsilon)$, M be a closed linear subspace of X, $Q: X \to X/M$ the quotient map associated with M, $y = Q(x)$ and $\rho_Q = \sup\{r : r \geqslant 0$ and for each $v \in Q(U)$ with $\|v - y\| < r$ there is $u \in U$ such that $\|u - x\| < \varepsilon$ and $Q(u) = v\}$.

LEMMA 4.1. *Let $x_1 \in U$ be such that $Q(x_1) = y$. Then $\rho_Q \geqslant r$, where $r = \min\{1 - \|x_1\|, \varepsilon - \|x_1 - x\|\}$.*

Proof. Let $v \in X/M$ be such that $\|v - y\| < r$. Since Q maps the open unit ball in X onto the open unit ball in X/M, there is $h \in X$ such that $\|h\| < r$ and $Q(h) = v - y$. Define $u = x_1 + h$. Then $Q(u) = v$,

$$\|u\| \leqslant \|x_1\| + \|h\| < \|x_1\| + r \leqslant 1$$

and

$$\|u-x\| = \|x_1 - x + h\| < \|x_1 - x\| + r \leqslant \varepsilon.$$

Therefore, Q maps the ε-neighbourhood of x in U onto a set containing the r-neighbourhood of y in X/M. ▮

LEMMA 4.2. *Let $x_2 \in U$ be such that $Q(x_2) = y$ and $\|x_2 - x\| \geqslant \varepsilon$. Then $\rho_Q \geqslant \min\{\delta, \varepsilon/2\}$.*

Proof. Since $Q(u) = y$ for any u from the segment $[x, x_2]$, we can assume that $\|x_2 - x\| = \varepsilon$. Denote $x_1 = (x_2 + x)/2$. By the definition of $\delta = \delta(x, \varepsilon)$, we have $1 - \|x_1\| \geqslant \delta$. Clearly, $\|x_1 - x\| = \varepsilon/2$. The proof now follows by applying Lemma 4.1. ▮

Proof of Theorem 3.1. If $\delta = \infty$ then $\|u - x\| < \varepsilon$ for each $u \in U$ so that $\rho(x, \varepsilon) = \infty$. Thus assume $\delta < \infty$. Denote $p = 2\delta/3$.

Case 1. Let $\|y\| \geqslant 1 - p$. Then for any $u \in U$ such that $\|Q(u) - y\| < p$ we have

$$\begin{aligned}\|(u+x)/2\| &\geqslant \|Q((u+x)/2\| \\ &= \|y - (y - Q(u))/2\| \\ &\geqslant \|y\| - \|y - Q(u)\|/2 \\ &> 1 - p - p/2 = 1 - \delta.\end{aligned}$$

Thus, by the definition of the modulus of local convexity δ, $\|u - x\| < \varepsilon$. Hence we have shown that $\rho_Q \geqslant p$.

Case 2. Let $\|y\| < 1 - p$. Then there is $x_1 \in X$ such that $\|x_1\| < 1 - p$ and $Q(x_1) = y$. Denote $d = \|x_1 - x\|$. If $d = 0$ then, by Lemma 4.1, $\rho_Q \geqslant \min\{p, \varepsilon\}$. If $d \leqslant \varepsilon - p$, Lemma 4.1 yields $\rho_Q \geqslant p$. If $d \geqslant \varepsilon$, Lemma 4.2 implies $\rho_Q \geqslant \min\{\delta, \varepsilon/2\}$. Therefore, it remains to consider the case of $\varepsilon - p < d < \varepsilon$, $d > 0$. Define $x_2 = x + t(x_1 - x)$ with $t = \varepsilon/d$. We have $t > 1$, $\|x_2 - x\| = td = \varepsilon$ and $Q(x_2) = y$. Further,

$$\begin{aligned}\|x_2\| &= \|x_1 + (t-1)(x_1 - x)\| \\ &\leqslant \|x_1\| + (t-1)\,\|x_1 - x\| \\ &< 1 - p + (t-1)\,d \\ &= 1 - p + \varepsilon - d < 1,\end{aligned}$$

hence, by Lemma 4.2, $\rho_Q \geqslant \min\{\delta, \varepsilon/2\}$.

As Q was an arbitrary quotient map, we have proved the inequality (3.1). Since we assume $\delta < \infty$, we have $\varepsilon \leqslant 2$, whence (3.2) follows from (3.1) and (4.1). ▮

For the proof of Theorem 3.2 we need the following

LEMMA 4.3. *Let M be a closed linear subspace of X, $Q\colon X \to X/M$ the quotient map associated with M, $x_0 \in U$, $x \in X$, let ε, r, q be positive numbers, $K \geqslant 0$ such that $\|x\| \leqslant 1-q$, $\|x-x_0\| \leqslant K$, $\|Q(x)-Q(x_0)\| < r$ and*

$$r(q+K-\varepsilon) < \varepsilon q. \tag{4.5}$$

Then there is $\bar{x} \in U$ such that $\|\bar{x}-x_0\| < \varepsilon$ and $Q(\bar{x}) = Q(x)$.

Proof. Since Q maps the open unit ball of X onto the open unit ball of X/M, there is $h \in X$ such that $\|h\| < r$ and $Q(h) = Q(x) - Q(x_0)$. Thus for the element $x_1 = x_0 + h$ of X we have $\|x_1 - x_0\| < r$ and $Q(x_1) = Q(x)$. Define $\bar{x} = tx_1 + (1-t)\,x$, where $t = q/(r+q)$. Then $Q(\bar{x}) = Q(x)$ and we have

$$\begin{aligned} \|\bar{x}\| &\leqslant t\,\|x_1\| + (1-t)\,\|x\| \\ &< t(1+r) + (1-t)(1-q) \\ &= 1 - q + t(r+q) = 1 \end{aligned}$$

and

$$\begin{aligned} \|\bar{x} - x_0\| &= \|t(x_1 - x_0) + (1-t)(x - x_0)\| \\ &< tr + (1-t)\,K \\ &= r(q+K)(r+q)^{-1}. \end{aligned}$$

However, it follows easily from (4.5) that the last expression is less than ε. ▮

Proof of Theorem 3.2. We set $\delta = \delta(\varepsilon)$ and $\rho = \rho(\varepsilon)$. Since the assertion is trivial for $\delta = 0$, we may assume that $\delta > 0$ and, since $\rho = \infty$ for $\varepsilon > 2$, let $\varepsilon \leqslant 2$. If $\dim X = 1$ then $\rho = \varepsilon$ and $\delta = \varepsilon/2$, so our claim is true. Thus suppose $\dim X \geqslant 2$.

Define $g(\varepsilon) = 2\varepsilon(\varepsilon + 4\delta)^{-1}$ and, for a fixed ε, let $r = g(\varepsilon)\,\delta$. We prove that $\rho \geqslant r$. Let M be a closed linear subspace of X, $Q\colon X \to X/M$ the quotient map associated with M and let $x_0 \in U$ be arbitrary. Denote $y_0 = Q(x_0)$ and let $y \in Q(U)$ be such that $\|y - y_0\| < r$. We show that y has an inverse image in the ε-neighbourhood of x_0 in U. We consider three cases.

Case 1. Suppose that $\|y\| = 1$. Since $y \in Q(U)$, there is $x \in U$ such that $Q(x) = y$. We have

$$\begin{aligned}\|(y + y_0)/2\| &= \|y + (y_0 - y)/2\| \\ &\geqslant \|y\| - \|(y_0 - y)/2\| \\ &> 1 - r/2 > 1 - \delta.\end{aligned}$$

Since $Q((x + x_0)/2) = (y + y_0)/2$ and $\|Q\| \leqslant 1$, we get $\|(x + x_0)/2\| > 1 - \delta$ and, by the definition of the modulus $\delta = \delta(\varepsilon)$, it follows $\|x - x_0\| < \varepsilon$.

Case 2. Let $y = 0$. To show that y has an inverse image in the ε-neighbourhood of x_0 in U, we apply Lemma 4.3 with $x = 0$, $q = 1$ and $K = 1$. Thus we need verify (4.5), i.e., the inequality $r(2 - \varepsilon) < \varepsilon$. However, this can be checked readily because

$$r = g(\varepsilon)\,\delta = 2\varepsilon(\varepsilon + 4\delta)^{-1}\,\delta < \varepsilon/2.$$

Case 3. We now assume that $0 < \|y\| < 1$. Let $\alpha > 1$ be such that

$$1 - \delta < \alpha\,\|y\| < 1. \tag{4.6}$$

Denote $y_1 = \alpha y$. Since $\|y_1\| < 1$, there is $x_1 \in U$ such that $Q(x_1) = y_1$. We define $L = r + (\alpha - 1)\,\|y\|$, $s = 2(\alpha\,\|y\| - 1 + \delta)\,L^{-1}$, $t = \min\{1, s\}$ and $x_2 = x_1 + t(x_0 - x_1)$. We have $t \leqslant 1$ and, by (4.6), $t > 0$, whence $x_2 \in U$. For $y_2 = Q(x_2)$ we have $y_2 = y_1 + t(y_0 - y_1)$, thus

$$\begin{aligned}\|y_2 - y_1\| &= t\,\|y_0 - y_1\| \\ &\leqslant t(\|y_0 - y\| + \|y - y_1\|) < tL.\end{aligned}$$

Using this, we get

$$\begin{aligned}\|(y_2 + y_1)/2\| &= \|y_1 + (y_2 - y_1)/2\| \\ &\geqslant \|y_1\| - \|(y_2 - y_1)/2\| \\ &> \alpha\,\|y\| - tL/2 \\ &\geqslant \alpha\,\|y\| - sL/2 = 1 - \delta.\end{aligned}$$

By the same arguments as in Case 1 (replace y_0, y by y_1, y_2), it follows $\|x_2 - x_1\| < \varepsilon$, thus $\|x_0 - x_1\| < \varepsilon t^{-1}$.

Now denote $x = \alpha^{-1} x_1$ and $q_\alpha = 1 - \alpha^{-1}$. Clearly, $Q(x) = y$, $q_\alpha > 0$ and $\|x\| \leqslant \alpha^{-1} = 1 - q_\alpha$. By Lemma 4.3, it suffices to show that there is an $\alpha > 1$ satisfying (4.6) such that for the corresponding point x the inequality $r(q_\alpha + \|x - x_0\| - \varepsilon) < \varepsilon q_\alpha$ holds. We shall show that this is true for α close to $\|y\|^{-1}$.

Consider the limit case; let α converge to $\|y\|^{-1}$ from the left. Define $q=1-\|y\|$, $t_0=\min\{1, 2\delta(r+q)^{-1}\}$ and $K=q+\varepsilon t_0^{-1}$. Clearly, q_α converges to q. Further,

$$\|x-x_0\| \leqslant \|x-x_1\| + \|x_1-x_0\|$$

and the right side is less than $1-\alpha^{-1}+\varepsilon t^{-1}$, which converges to K. Therefore, it suffices to check that for q and K defined above the inequality (4.5) holds. If $t_0=1$ then

$$r(q+K-\varepsilon)-\varepsilon q = q(2r-\varepsilon),$$

which is negative because $q>0$ and, by the definition, $r<\varepsilon/2$.

Consider now the case $t_0<1$. Then

$$\begin{aligned} r(q+K-\varepsilon)-\varepsilon q &= r[2q+\varepsilon(2\delta)^{-1}(r+q)-\varepsilon]-\varepsilon q \\ &= q[2r+r\varepsilon(2\delta)^{-1}-\varepsilon]+r^2\varepsilon(2\delta)^{-1}-r\varepsilon \\ &= r^2\varepsilon(2\delta)^{-1}-r\varepsilon \\ &= r\varepsilon(2\delta)^{-1}(r-2\delta)<0, \end{aligned}$$

thus (4.5) is satisfied.

In all three cases we have found an inverse image of y in U within the distance ε from x_0, hence Q maps the ε-neighbourhood of x_0 in U onto a set containing the r-neighbourhood of $Q(x_0)$ in $Q(U)$. Since Q was an arbitrary quotient map on X and $x_0 \in U$ an arbitrary point, we have proved that $\rho \geqslant r = g(\varepsilon)\,\delta$.

Observe now that (4.2) yields $g(\varepsilon)\geqslant 2/3$ for each $\varepsilon\in(0, 2]$. Furthermore, as we assume in this part of the proof that $\dim X\geqslant 2$, it follows from the Day–Nordlander theorem (see e.g. [11, p. 60]), that δ is less or equal to the modulus of the two-dimensional Hilbert space, i.e., $\delta\leqslant 1-(1-\varepsilon^2/4)^{1/2}\leqslant\varepsilon^2/4$ for $\varepsilon\in(0, 2]$, thus $2(1+\varepsilon)^{-1}\leqslant g(\varepsilon)\leqslant 2$ for all such ε. ▮

Proof of Theorem 3.3. We denote

$$d=2(\lambda+1)^{-1} \qquad \text{and} \qquad r=\lambda\varepsilon. \tag{4.7}$$

Also, we set $\delta=\delta(x, r)$. We may assume that $\delta<\infty$. For an arbitrary $\alpha\in(0, 1)$, we prove that

$$\rho(x, \varepsilon)<4(\lambda-1)^{-1}(\delta+2\alpha)+2\alpha. \tag{4.8}$$

We consider two cases.

Case 1. Suppose that $d[r-2(\delta+\alpha)]\leqslant\varepsilon$. Using (4.7), we get from this

$$\varepsilon\leqslant 4(\lambda-1)^{-1}(\delta+\alpha). \tag{4.9}$$

Since we assume $\delta<\infty$, there is a $v\in U$ with $\|v-x\|\geqslant\varepsilon$, hence $\varepsilon\leqslant 1+\|x\|$. So, the inequality (4.3) can be applied and in combination with (4.9) it implies (4.8).

Case 2. Suppose that

$$d[r-2(\delta+\alpha)]>\varepsilon. \tag{4.10}$$

It follows from the definition of $\delta=\delta(x,r)$ that there exists $x_1\in U$ such that $\|x_1-x\|\geqslant r$ and

$$\|(x+x_1)/2\|>1-\delta-\alpha.$$

For any $t\in(0,1)$, denote $x_t=x+t(x_1-x)$. Since $d<1$ and $d(1-d)^{-1}=2(\lambda-1)^{-1}$, we can choose $t\in[d,1)$ such that $x_t\neq 0$ and that

$$2t(1-t)^{-1}(\delta+\alpha)<4(\lambda-1)^{-1}(\delta+2\alpha). \tag{4.11}$$

Since $t\geqslant d\geqslant 1/2$, we have

$$\begin{aligned}\|x_t\|&=\|t(x+x_1)-(2t-1)\,x\|\\&\geqslant 2t\,\|(x+x_1)/2\|-|2t-1|\\&>2t(1-\delta-\alpha)-(2t-1)\\&=1-2t(\delta+\alpha).\end{aligned} \tag{4.12}$$

Denote $u=x_t/\|x_t\|$. Then $\|x_t-u\|=1-\|x_t\|$, which combines with (4.12) to yield

$$\|x_t-u\|<2t(\delta+\alpha). \tag{4.13}$$

Using (4.13) and the triangle inequality

$$\|u-x\|\geqslant\|x_t-x\|-\|x_t-u\|,$$

where $\|x_t-x\|=t\,\|x_1-x\|\geqslant tr$, we obtain

$$\|u-x\|\geqslant tr-2t(\delta+\alpha)$$

and, in combination with (4.10) and with the inequality $t\geqslant d$, it implies

$$\|u-x\|>\varepsilon. \tag{4.14}$$

Choose a functional $f \in X^*$ such that $\|f\| = f(u) = 1$. Then $f(x_t) = \|x_t\|$ and $(1-t)\, f(x) + t \geqslant (1-t)\, f(x) + tf(x_1) = f(x_t) = \|x_t\|$, thus $f(x) \geqslant (1-t)^{-1}(\|x_t\| - t)$. Using this, (4.12) and (4.11), we get

$$f(x) > 1 - 4(\lambda - 1)^{-1}(\delta + 2\alpha). \tag{4.15}$$

By (4.14), there is a functional $h \in X^*$ such that $\|h\| = 1$ and $h(u - x) > \varepsilon$, hence $h(u) > h(x) + \varepsilon$. Denote $g = f + \alpha h$. Then $\|g\| \leqslant 1 + \alpha$ and $\|g\| \geqslant \|f\| - \alpha\, \|h\| = 1 - \alpha$. Particularly, since $\alpha < 1$, we have $g \neq 0$. Define $g_1 = g/\|g\|$. Then $\|g_1 - g\| = |1 - \|g\|\,| \leqslant \alpha$, thus $\|g_1 - f\| \leqslant \|g_1 - g\| + \|g - f\| \leqslant 2\alpha$. From this and from (4.15) we get

$$g_1(x) > 1 - 4(\lambda - 1)^{-1}(\delta + 2\alpha) - 2\alpha. \tag{4.16}$$

Let $v \in U$ be such that $\|v - x\| < \varepsilon$; then

$$\begin{aligned} g(v) &= f(v) + \alpha h(v) \\ &\leqslant 1 + \alpha(h(x) + \|v - x\|) \\ &< 1 + \alpha(h(x) + \varepsilon) \\ &< f(u) + \alpha h(u) = g(u), \end{aligned}$$

thus

$$g_1(v) < g_1(u) \qquad \text{whenever} \quad v \in U,\ \|v - x\| < \varepsilon. \tag{4.17}$$

Applying (4.17) to $v = x$, we obtain

$$g_1(x) < g_1(u). \tag{4.18}$$

Denote $\beta = g_1(u) - g_1(x)$. By (4.17), g_1 maps the ε-neighbourhood of x in U onto a set which does not contain the point $g_1(u)$ of $g_1(U)$ and, by (4.18), the distance of this point from $g_1(x)$ is β. Thus, by Remark 2.1 (applied to $T = g_1$, $x_0 = x$ and $c = 1$), we have $\rho(x, \varepsilon) \leqslant \beta$. Finally, observing that $\beta \leqslant 1 - g_1(x)$ and applying (4.16), we get (4.8). Since α can be arbitrarily small, we obtain (3.3) and, by taking the infimum over $x \in U$, (3.4) follows. ∎

The following lemma is used in the proofs of Theorem 3.4 and Theorem 3.5. We note that for a map T the kernel of which is closed the assertion of the lemma follows immediately from Theorem 3.1 and Remark 2.1.

LEMMA 4.4. *Let T be a linear map defined on X and $c > 0$ be such that for each $y \in T(X)$ of norm $\|y\| < c$ there is $x \in X$ of norm $\|x\| < 1$ such that $T(x) = y$. Then, for each $x \in U$ and $\varepsilon \in (0, 1)$, T maps the ε-neighbourhood of*

x in U onto a set containing the r-neighbourhood of $T(x)$ in $T(U)$ with $r = c\varepsilon\, \delta(x, \varepsilon)/5$.

Proof. Denote $K = T(U)$ and, for a fixed $x \in U$ and $\varepsilon \in (0, 1)$, let $y = T(x)$ and $\delta = \delta(x, \varepsilon)$. We have $\delta \leqslant 1$. Consider two cases.

Case 1. Suppose that $(1+\delta)\, y \notin K$. Let u be an arbitrary element of U such that

$$\|T(u) - y\| < c. \tag{4.19}$$

We show that $\|u - x\| < \varepsilon$. Suppose that this is false. Then, by the definition of $\delta = \delta(x, \varepsilon)$, we have $\|(u+x)/2\| \leqslant 1 - \delta$. Denote $x_1 = 2^{-1}(1+\delta)(u+x)$ and $y_1 = T(x_1) = 2^{-1}(1+\delta)(T(u) + y)$. We have $x_1 \in U$, so $y_1 \in K$. Since K contains the c-neighbourhood of 0 in $T(X)$, (4.19) implies that the point $y_2 = 2^{-1}(1+\delta)(y - T(u))$ is in K. By the convexity of K, $(1+\delta)\, y = (y_1 + y_2)/2$ is in K, which contradicts the assumption at the beginning of Case 1.

Thus T maps the ε-neighbourhood of x in U onto a set containing the c-neighbourhood of $T(x)$ in $T(U)$.

Case 2. Suppose that $(1+\delta)\, y \in K$. Then there is a $u \in U$ such that $T(u) = (1+\delta)\, y$, hence for $x_1 = (1+\delta)^{-1} u$ we have $\|x_1\| \leqslant 1 - \delta/2$ and $T(x_1) = y$. Denote $\alpha = \varepsilon\delta/5$, and, let $t = 2\varepsilon/5$ and $x_t = t x_1 + (1-t)\, x$. Then

$$\begin{aligned} \|x_t - x\| &= t\, \|x_1 - x\| \\ &\leqslant 2t = 4\varepsilon/5 \leqslant \varepsilon - \alpha \end{aligned}$$

and

$$\begin{aligned} \|x_t\| &\leqslant t\, \|x_1\| + (1-t)\, \|x\| \\ &\leqslant t(1 - \delta/2) + 1 - t \\ &= 1 - t\delta/2 = 1 - \alpha. \end{aligned}$$

Let V be the α-neighbourhood of x_t in X. For any $v \in V$, we have

$$\|v - x\| \leqslant \|v - x_t\| + \|x_t - x\| < \alpha + (\varepsilon - \alpha) = \varepsilon$$

and

$$\|v\| \leqslant \|v - x_t\| + \|x_t\| < \alpha + (1 - \alpha) = 1,$$

thus the ε-neighbourhood of x in U contains V. Since $T(x_t) = y$, the assumptions on T yield that $T(V)$ contains the r-neighbourhood of y in $T(X)$ with $r = c\alpha = c\varepsilon\delta/5$, which completes the proof. ∎

Proof of Theorem 3.4 *and Theorem* 3.5. Since the quotient maps in the statements (ii) are equally relatively open on X, (iii) imply (ii) immediately. That (ii) imply (i) is an easy consequence of Theorem 3.3. Indeed, the condition (ii) of Theorem 3.4 yields $\rho(x, \varepsilon) > 0$ for each $x \in U$ and $\varepsilon > 0$, hence, by (3.3), $\delta(x, \varepsilon) > 0$ for each $x \in U$ and $\varepsilon > 0$, thus X is locally uniformly convex. Similarly for Theorem 3.5: (ii) implies $\rho(\varepsilon) > 0$ for each $\varepsilon > 0$, therefore, by (3.4), $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$, hence X is uniformly convex.

Finally, (i) imply (iii) by Lemma 4.4; equal relative openness of quotient maps on X yields that the constant $c > 0$ used in Lemma 4.4 can be chosen so that it is independent on the map. To conclude this implication for Theorem 3.5, use (2.1) and the fact that in a uniformly convex space X we have $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$. ∎

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