Available online at www.sciencedirect.com

SCIENCE DIRECT®

Theoretical Computer Science 341 (2005) 247–262

Theoretical Computer Science

www.elsevier.com/locate/tcs

Towards a proof of the decidability of the momentary stagnation of the growth function of $D0L$ systems

Blanca Cases[a,1], Manuel Alfonseca[b,*,2]

[a]Departamento de Lenguajes y Sistemas Informáticos, Universidad del País Vasco/Euskal Herriko Unibertsitatea, 20080 Guipúzcoa, Spain

[b]Escuela Politécnica Superior, Department of Computer Engineering, Universidad Autónoma de Madrid, Madrid 28049, Spain

Received 8 June 2004; received in revised form 5 January 2005; accepted 19 May 2005

Communicated by M. Ito

Abstract

This paper proves the decidability of several problems in the theory of $HD0L$, $D0L$ and $PD0L$ systems, some of which that have been proved before but are now proved in a different way. First, the paper tackles the decidability of the nilpotency of $HD0L$ systems and the infinitude of $PD0L$ languages. Then, we prove the decidability of the problem of momentary stagnation of the growth function of $PD0L$ systems. Finally, we suggest a way to solve the decidability of the momentary stagnation of the growth function of $D0L$ systems, proving the decidability of the infinitude of $HD0L$ as a trivial consequence.

Keywords: Formal language theory; $D0L$ systems; $HD0L$ systems

* Corresponding author. Tel.: +34 91 497 2278; fax: +34 91 497 2235.
E-mail address: Manuel.Alfonseca@ii.uam.es (M. Alfonseca).

1 This paper has been partially sponsored by the UPV/EHU project number 9/upv 00003.230-13707/2001.

2 This paper has been partially sponsored by the Spanish Ministry of Science and Technology (MCYT) project number TIC2002-01948.

doi:10.1016/j.tcs.2005.05.002

1. Introduction

The theory of Lindermayer systems, or L systems started with the work by Aristide Lindenmayer [2], whose goal was the development of graphical algorithms to explain the growth of living organisms, specially of plants [5]. The mathematical formalization of the theory of L Systems in terms of Formal Language Theory and Z-rational functions [4,6], has produced a fruitful line of research [7–9,1].

This work proves in a different way several well-known problems related to $HD0L$ and $PD0L$ systems. The triple $\mathbf{V}^* = (V^*, \cdot, \varepsilon)$ represents the free monoid generated by the concatenation operation $x \cdot y = xy$ over the alphabet $V = \{a_1, \dots, a_m\}$ composed of $m > 0$ symbols, with neutral element ε, called the empty word. The set of non-empty words is $V^+ = V^* - \{\varepsilon\}$. The length of a word x is denoted by $|x|$.

A $D0L$ homomorphism in V^* is any function $F : V^* \to V^*$ such that $F(\varepsilon) = \varepsilon$ and $F(xy) = F(x)F(y)$. A $D0L$ sequence is a succession $s(0) = F^0(x) = x$, $s(k+1) = F^{k+1}(x) = F(F^k(x))$ obtained by iterating the homomorphism F from a starting word x. A $D0L$ homomorphism is propagating or $PD0L$ if for all symbols $a_i \in V$, $F(a_i) \neq \varepsilon$ is true. A filtering homomorphism $f : V^* \to V'^*$ together with a $D0L$ sequence $s(k) = F^k(x)$ defines an $HD0L$ sequence $s'(k) = f(s(k)) = f(F^k(x))$.

If the filtering homomorphism is $g : V^* \to \{a\}^*$ in such a way that $g(x) = a^{|x|}$, or equivalently, $g : V^* \to \mathbb{N}$ and $g(x) = |x|$, then $s'(k) = g(s(k)) = g(F^k(x))$ is a growth sequence.

In this paper, we prove the problem of "momentary stagnation" of the growth function of $PD0L$ sequences, which deals with the decidability of the existence of a value k, such that $g(F^k(x)) = g(F^{k+1}(x))$.

We first present a *Theorem of decidability of filtered mononotous systems*, concerning filtered iterated sequences of monotonous functions in finitely founded well behaved quasi-orders. The theorem ensures the decidability of

$$\forall k \;\; \exists z \in Z, \quad z \preccurlyeq' f(F^k(x)) = s'(k),$$

where Z is a finite set, A is a recursive set, $F : A \to A,\ f : A \to A'$ are monotonous functions in recursive finitely founded well behaved quasi-orders $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$. We also prove the existence of an algorithm to decide the infinitude of $s'(\mathbb{N})$ if F and f are strictly monotonous.

As corollaries of this theorem, the decidability of the following problems is proved:

(1) The nilpotency of $HD0L$ homomorphisms, $\exists k,\ s'(k) = f(F^k(x)) = \varepsilon$.
(2) The infinitude of propagating or $PD0L$ languages.
(3) The Parikh momentary stagnation of the growth function of $PD0L$ systems.
(4) The infinitude of $HD0L$ languages.

Sections 2 and 3 describe the basic notions and the notation used in the paper. Basic concepts of algebra are taken from the book [10], while the background on Computability is taken from [3]. Section 4 introduces a few examples of order structures $(\mathbb{N}^m, \leqslant)$ and Parikh's quasi-order $(V^*, \leqslant)$, which are used in the development of this work, proving that they are well behaved quasi-orders.

Original results are presented in Section 5, which proves some propositions on recursive well behaved quasi-orders, and in Section 6, which proves the decidability theorem mentioned above.

Section 7 refers the problem to domain V^* [1], introducing homomorphisms and their relation to matrix theory.

The corollaries of the theorem of decidability of filtered iterated monotonous functions are presented respectively in Sections 8–10. Finally, Section 11 describes an algorithm for the latter problem.

2. Sets, predicates, functions and computability

In the following, we assume that any set A is contained in a universe $\mathcal{A}$, a set which is in correspondence to natural numbers through a computable and bijective succession $s : \mathbb{N} \to \mathcal{A}$. $\#A$ denotes the cardinality of set A and $\emptyset$ the empty set. We define $\neg A = \mathcal{A} - A$ as the set of all the elements of $\mathcal{A}$ which are not in A. Therefore,

$$\neg\mathcal{A} = \emptyset.$$

Consider the cartesian product $A^m = A \times \cdots \times A = \{(x_1, \ldots, x_m) : x_i \in A,\ 1 \leqslant i \leqslant m\}$. To every subset B of A^m we associate a predicate Q of arity m, denoted by $Q(x)$, which is true for $x \in A^m$ if $x \in B$. The complementary predicate is denoted by $\neg Q(x)$, which is true if $x \notin B$. Binary predicates $Q(x_1, x_2)$, with $m = 2$, can be written $x_1 Q x_2$. When the arity is 0, the predicate Q is either a true or a false proposition for all x.

A function $F : A \to A'$ is a predicate $F \subseteq A \times A'$ such that, for all $x \in A$, the image set of x by F, $Im(F, x) = \{y \in A' : (x, y) \in F\}$, has at most one element: $\#Im(F, x) \leqslant 1$. Using the notation of functions, we write $F(x) = y \leftrightarrow F(x, y) \leftrightarrow Im(F, x) = \{y\}$.

The range or image of F is the set $Im(F) = F(A) = \bigcup_{x \in A} Im(F, x)$, while the domain of F is the set $Dom(F) = \{x \in A : \#Im(F, x) = 1\}$. Function F is total if $Dom(F) = A$, surjective if $Im(F) = A'$, and injective if for all pairs (w, x) of different elements of A, $F(w) \neq F(x)$. F is a bijection when it is total, surjective and injective.

A succession in a set A is any total function $s : \mathbb{N} \to A$.

A predicate Q is recursive or decidable when there are algorithms that determine whether $Q(x)$ or $\neg Q(x)$ is true in a finite number of computation steps, for all $x \in A^m$. If the arity of Q is 0, the algorithm determines which is true: Q or $\neg Q$. If an algorithm exists, which always stops if $Q(x)$ is true, but does not stop if $\neg Q(x)$ is true, the predicate is recursively enumerable or semi-decidable. Predicate Q is recursive iff Q and $\neg Q$ are recursively enumerable.

A function $F : A \to A'$ is computable if predicate $F \subseteq A \times A'$ is decidable. Intuitively, a function is computable if there are algorithms that, for each $x \in A$, compute $F(x) \in A'$ in a finite number of computation steps.

A set A is recursively enumerable (recursive) iff predicate $Q(x) \leftrightarrow x \in A$ is recursively enumerable (recursive). Formally, we will apply the following definition of recursive and recursively enumerable sets:

Definition 1. Any set $A \subseteq \mathcal{A}$ is recursively enumerable if it is the empty set, $A = \emptyset$, or if there exists a computable succession $s : \mathbb{N} \to \mathcal{A}$ such that $s(\mathbb{N}) = A$.

We say that A is recursive if A and $\neg A = \mathcal{A} - A$ are recursively enumerable.

It should be noticed that the universe $\mathcal{A}$ is recursive, since it is recursively enumerable, and its complementary set $\mathcal{A} = \emptyset$ is recursively enumerable. We also use the following well known results in computability theory:

- If A and B are recursively enumerable (recursive) sets then their union $A \cup B$ and intersection $A \cap B$ are recursively enumerable (recursive) sets.
- A is recursive iff $\neg A$ is recursive. A set A is co-finite if $\neg A$ is finite. Finite and co-finite sets are recursive.
- If A is recursive, then $A \times A$ is recursive.

3. A background on order relations

3.1. Quasi-orders

A binary predicate $\preccurlyeq\ \subseteq A \times A$ is a quasi-order (or pre-order) if it is reflexive and transitive. The pair $(A, \preccurlyeq)$ is a quasi-order structure. Notation $y \succcurlyeq x$ is equivalent to $x \preccurlyeq y$. The non-reflexive or proper part of the quasi-order is denoted by $\prec$, while $y \succ x$ is equivalent to $x \prec y$.

Let $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$ be partial quasi-orders. A total function $F : A \to A'$ is monotonous if for all $x, y \in A$, $x \preccurlyeq y \to F(x) \preccurlyeq' F(y)$. F is strictly monotonous if $x \prec y \to F(x) \prec F(y)$. $(A, \preccurlyeq)$ is isomorphic to $(A', \preccurlyeq')$ if a bijective and monotonous function exists, $F : A \to A'$, such that $x \preccurlyeq y \leftrightarrow F(x) \preccurlyeq' F(y)$. This is denoted by $(A, \preccurlyeq) =_F (A', \preccurlyeq')$.

Let $(A, \preccurlyeq)$ be a quasi-order. A succession $s : \mathbb{N} \to A$ is increasing if s is monotonous in the order $(\mathbb{N}, \leqslant)$ of natural numbers, which means that, for all k, $s(0) \preccurlyeq s(1) \preccurlyeq \cdots \preccurlyeq s(k) \preccurlyeq \cdots$. The succession is strictly increasing if $s(0) \prec s(1) \prec \cdots \prec s(k) \prec \cdots$. The succession is decreasing if, for all k, $s(0) \succcurlyeq s(1) \succcurlyeq \cdots \succcurlyeq s(k) \succcurlyeq \cdots$, and strictly decreasing if the same holds for predicate $\succ$.

$(A, \preccurlyeq)$ is a well founded quasi-order structure if A does not contain infinite strictly decreasing chains. This happens iff a strictly decreasing succession $s : \mathbb{N} \to A$ does not exist.

$(A, \preccurlyeq)$ is a well behaved quasi-order structure if, for every succession $s : \mathbb{N} \to A$ there are indices i and j such that $i < j$ and $s(i) \preccurlyeq s(j)$. In this case, we can say that the successions are good in this structure. The condition of being well behaved is stronger than that of being well founded. Every well behaved quasi-order is a well founded quasi-order.

3.2. Finitely founded well behaved quasi-orders

Definition 2. Let $(A, \preccurlyeq)$ be a well behaved quasi-order structure. We say that $(A, \preccurlyeq)$ is finitely founded if $\forall x \in A,\ \#\{y \in A : y \preccurlyeq x\} \in \mathbb{N}$ (is finite).

Finitely founded well behaved quasi-orders are a restriction of well behaved quasi-orders that avoids the situations where an element $x \in A$ is greater or equal than an infinite number of different elements $y \in A$.

Example 3. Let $(\mathbb{N}, \leqslant)$ be the order of natural numbers. Take an element $\top \notin \mathbb{N}$. Then $(\mathbb{N} \cup \{\top\}, \sqsubseteq)$, where $\sqsubseteq = \leqslant \cup \{(x, \top) : x \in \mathbb{N} \cup \{\top\}\}$, is trivially a well behaved quasi-order structure which is not finitely founded: $\#\{y \in \mathbb{N} \cup \{\top\} : y \sqsubseteq \top\} = \#\mathbb{N}$ means that element $\top$ is greater or equal than an infinite number of elements.

3.3. Partial orders

If $\preccurlyeq$ is antisymmetric (i.e. $x \preccurlyeq y \wedge y \preccurlyeq x \rightarrow x = y$) then $\preccurlyeq$ is a partial order. We denote a partial order predicate by $\leqslant$, its proper subset by $<$, and a partial order structure by $(A, \leqslant)$. The set A is called a *poset*. The order is total if any two elements x, y are comparable, $x \leqslant y$ or $y \leqslant x$.

Given a quasi-order $(A, \preccurlyeq)$, consider the equivalence relation $x = y \leftrightarrow x \preccurlyeq y \wedge y \preccurlyeq x$. Consider also $(A_{/=}, \preccurlyeq_{/=})$, where $A_{/=}$ is the quotient set of A with respect to the equivalence $=$, i.e. $A_{/=} = \{[x] : x \in A\}$ and $[x] = \{y \in A : x \preccurlyeq y \wedge y \preccurlyeq x\}$. For all $[x], [y] \in A_{/=}$, $[x] \preccurlyeq_{/=} [y] \leftrightarrow x \preccurlyeq y \leftrightarrow \forall x' \in [x], \forall y' \in [y],\ x' \preccurlyeq y'$. Then $(A_{/=}, \preccurlyeq_{/=})$ is a partial order induced by the quasi-order $(A, \preccurlyeq)$.

Let $B \subseteq A$ be a subset of a poset A. An element $x \in B$ is minimal if for all $y \in B$, $\neg\, x > y$. The set of minimal elements in B is denoted by $\min(B)$. A partial order on a set A is a well behaved partial order iff every non-empty subset of $B \subseteq A$ has minimal elements, $\min(B) \neq \emptyset$, but only a finite number of them.

Proposition 4. $(A, \preccurlyeq)$ *is a well behaved quasi-order structure iff* $(A_{/=}), \preccurlyeq_{/=})$ *is a well behaved partial order structure*.

Proof. $s : \mathbb{N} \rightarrow A$ is a succession in A iff $s' : \mathbb{N} \rightarrow A_{/=}$ such that for all k, $s'(k) = [s(k)] \in A_{/=}$ is a succession in $A_{/=}$.

For all indices $i < j$, $s(i) \preccurlyeq s(j)$ iff $s'(i) = [s(i)] \preccurlyeq_{/=} [s(j)] = s'(j)$. In consequence, any sequence s is good in structure $(A, \preccurlyeq)$ iff s' is good in $(A_{/=}, \preccurlyeq_{/=})$. □

Proposition 5. $(A, \preccurlyeq)$ *is a finitely founded well behaved quasi-order structure iff* $(A_{/=}, \preccurlyeq_{/=})$ *is a finitely founded well behaved partial order structure, where each equivalence class* $[x] \in A_{/=}$ *is a finite set*.

Proof. Proposition 4 ensures that $(A, \preccurlyeq)$ is a well behaved quasi-order iff $(A_{/=}, \preccurlyeq_{/=})$ is a well behaved partial order.

We know that, for all $x \in A$, $[x] \in A_{/=}$, $\#\{[y] \in A_{/=} : [y] \preccurlyeq [x]\} \leqslant \#\{y \in A : y \preccurlyeq x\}$. Therefore, the fact that $(A, \preccurlyeq)$ is finitely founded implies that $(A_{/=}, \preccurlyeq_{/=})$ is finitely founded. If $(A, \preccurlyeq)$ is finitely founded, then $[x] = \{y \in A : y \preccurlyeq x \wedge x \preccurlyeq y\} \subseteq \{y \in A : y \preccurlyeq x\}$, which is a finite set. Hence, all classes $[x]$ are finite.

Conversely, if $(A_{/=}, \preccurlyeq_{/=})$ is finitely founded and every class $[y] \subseteq A$ is finite, for all class $[x]$, the set $\{y \in A : y \preccurlyeq x\} = \bigcup_{[y] \preccurlyeq [x]} [y]$ is a finite union of finite sets, and hence finite. In consequence, $\forall x \in A,\ \#\{y \in A : y \preccurlyeq x\} \in \mathbb{N}$ and $(A, \preccurlyeq)$ is finitely founded. $\square$

The following example is useful to see that finitely founded partial orders are a strict sub-class of well behaved partial orders.

Example 6. The well behaved quasi-order $(\mathbb{N} \cup \{\top\}, \sqsubseteq)$ defined in Example 3 is a well behaved partial order which is not finitely founded.

The following example is an application of Proposition 5.

Example 7. Consider the well behaved quasi-order $(\mathbb{N}^2, \lessdot)$, such that $(x, y) \lessdot (x', y') \leftrightarrow x \leqslant x'$. In the quotient partial order, each class $(\mathbb{N}^2_{/=}, \lessdot_{/=})$ is infinite, but $(\mathbb{N}^2_{/=}, \lessdot_{/=})$ is isomorphic to $(\mathbb{N}, \leqslant)$, since $x = [(x, 0)] = \{(x, y) : y \in \mathbb{N}\}$, and hence not finitely founded.

As a consequence of Proposition 5, the structure $(\mathbb{N}^2, \lessdot)$ is not finitely founded.

4. Examples of finitely founded well behaved quasi-ordered sets used in this work

Example 8. The extension to $\mathbb{N}^m$ the order of naturals $(\mathbb{N}, \leqslant)$ is a finitely founded well behaved partial order.

The extension to $(\mathbb{N}^m, \leqslant)$ of the order of naturals $(\mathbb{N}, \leqslant)$ in such a way that $(x_1, \dots, x_m) \leqslant (y_1, \dots, y_m) \leftrightarrow \bigwedge_{i=1}^{m} x_i \leqslant y_i$ is a well behaved partial order whose minimum is $\bar{0} = (0, \dots, 0)$.

Consider any set $B \subseteq \mathbb{N}^m$ of tuples. We prove that $\min(B)$, the set of minimal elements in B is finite and non-empty.

Let $\mu_i = \min\{x_i : x \in B\}$ be the ith coordinate of $\mu \in \mathbb{N}^m$, i.e. the minimum of the ith coordinates of the tuples in B (μ is not necessarily in B). Tuple $\mu = \mu(B)$ is always defined and unique.

Now, for each coordinate μ_i, take any m tuples $x^i \in B$ whose ith coordinate reaches the minimum $x^i_i = \mu_i$ for $1 \leqslant i \leqslant m$. Let $Z = \{x^i : 1 \leqslant i \leqslant m\} \subseteq B$ be the set of such tuples. Z is non-empty iff B is non-empty. Call $B_1 = \{x \in B : \bigvee_{i=1}^{m} x^i \leqslant x\}$. Then $\min(B_1) = Z$.

Consider the tuple $M = (M_1, \dots, M_m)$ (not necessarily in B) where $M_i = \max\{x^1_i, \dots, x^m_i\}$ is the maximum of ith coordinates of the tuples in Z. Then $B_2 = \{x \in B : \mu \leqslant x \leqslant M\}$ is a finite set and $Z \subseteq B_2$ is non-empty.

Since $B = B_1 \cup B_2$, it follows that $\min(B) = \min(\min(B_1) \cup B_2) = \min(Z \cup B_2) = \min(B_2)$ is a finite and non-empty set, and $(\mathbb{N}^m, \leqslant)$ is a well behaved partial order, as defined in Section 3.3.

Consider $B = \{(x, y) : x, y$ are odd numbers and $x \geqslant 3$ or $y \geqslant 5\}$ a subset of $\mathbb{N}^2$ with $\mu = (1, 1) \notin B$. Take $Z = \{(1, 9), (7, 1)\} \subseteq B$, where $M = (7, 9)$.

The set $B_2 = \{(x, y) : x, y \in B,\ \mu = (1, 1) \leqslant (x, y) \leqslant (7, 9) = M\}$ is finite and non-empty. Thus $\min(B) = \{(3, 1), (1, 5)\} = \min(B_2)$ is finite and non-empty.

To see that $(\mathbb{N}^m, \leqslant)$ is finitely founded, consider any tuple $x = (x_1, \dots, x_m)$. If $z = x_i = \max\{x_1, \dots, x_m\}$ is the maximum component of tuple x, then $\{y \in \mathbb{N}^m : y \leqslant x\} \subseteq$

$\{0, \ldots, z\}^m$, which is a finite set. As a consequence, $\#\{y \in \mathbb{N}^m : y \leqslant x\} \in \mathbb{N}$, and $(\mathbb{N}^m, \leqslant)$ is finitely founded.

Example 9. *Parikh's well behaved quasi-order in V^**. Consider the set of words V^* over the alphabet V.

Let $|x|_{a_i}$ count the number of occurrences of symbol $a_i \in V$ in the word $x \in V^*$, and take $\psi : V^* \longrightarrow \mathbb{N}^m$ such that $\psi(x) = (|x|_{a_1}, \ldots, |x|_{a_m})$. Function ψ is called the Parikh's function.

We define $(V^*, \preccurlyeq_\psi)$ thus: $x \preccurlyeq_\psi y \leftrightarrow \psi(x) \leqslant \psi(y)$ in the well behaved partial order $(\mathbb{N}^m, \leqslant)$ defined in Example 8.

Let us illustrate this for the alphabet $V = \{a, b\}$: $aab \preccurlyeq_\psi baaab$, as $\psi(aab) = (2, 1) < (3, 2) = \psi(baaab)$. On the other hand, $baaab \preccurlyeq_\psi bbaaa$ and $bbaaa \preccurlyeq_\psi baaab$, although they are different words. Therefore, $\preccurlyeq_\psi$ is not antisymmetric.

Consider the equivalence $x =_\psi y \leftrightarrow x \preccurlyeq_\psi y \wedge y \preccurlyeq_\psi x \leftrightarrow \psi(x) = \psi(y)$. By Proposition 4, the quotient structure $(V^*_{/=_\psi}, \preccurlyeq_{\psi_{/=_\psi}})$ is a well behaved partial order structure isomorphic to $(\mathbb{N}^m, \leqslant)$ through the bijection $\Psi : V^*_{/=_\psi} \rightarrow \mathbb{N}^m$, such that $\Psi([x]) = [\psi(x)]$.

In Example 8, we proved that $(\mathbb{N}^m, \leqslant)$ is finitely founded. By isomorphism, $(V^*_{/=_\psi}, \preccurlyeq_{\psi_{/=_\psi}})$ is finitely founded. Proposition 5 ensures that $(V^*, \preccurlyeq)$ is finitely founded, since for all $x \in V^*$, $\#[x] \in \mathbb{N}$.

5. Recursive quasi-orders

Definition 10. A quasi-order $(A, \preccurlyeq)$ is recursive if $\preccurlyeq \subseteq A \times A$ is a recursive predicate.

Proposition 11. *Let $(\mathcal{A}, \preccurlyeq)$ be a recursive well behaved quasi-order and $A \subseteq \mathcal{A}$ any recursive subset. Then $(A, \preccurlyeq_A)$, where $\preccurlyeq_A = \preccurlyeq \cap A \times A$, is a recursive well behaved quasi-order structure.*

Proof. The restriction of quasi-order $\preccurlyeq$ to $\preccurlyeq_A$ is trivially a quasi-order. Since any succession $s : \mathbb{N} \rightarrow A$ (computable or not) in the set A is a succession in $\mathcal{A}$, all successions in A are good. Thus, $(A, \preccurlyeq)$ is a well behaved quasi-order structure.

Since $\preccurlyeq_A = \preccurlyeq \cap A \times A$ is the intersection of recursive predicates, the well behaved quasi-order $\preccurlyeq_A$ is recursive. □

Without loss of generality, we can write $(A, \preccurlyeq_A) = (A, \preccurlyeq)$, with the conditions of Proposition 11.

Notice that, being a finitely founded or well-behaved quasi-order, does not imply being recursive, as shown in the following example.

Example 12. Consider the order $(\mathbb{N}, \leqslant)$, a finitely founded and well-behaved total order. Let $\bar{K} \subseteq \mathbb{N}$ be any non-recursively enumerable subset. Following the proof of

Proposition 11, it is easy to see that $(\bar{K}, \leqslant)$ is a finitely founded well behaved quasi-order which is non-recursive.

6. Decidability results in filtered iterated monotonous functions

Given a function $F : A \to A$, we call $F^k(x)$ the kth iteration of F, $k \in \mathbb{N}$, where $F^0(x) = x$ and $F^{k+1}(x) = F(F^k(x))$.

Theorem 13. *Let $s(k) = F^k(x)$ be a succession, where $F : A \to A$, $f : A \to A'$ are total, computable and monotonous functions in recursive finitely founded well behaved quasi-orders $(A, \preccurlyeq)$ and $(A', \preccurlyeq')$. This gives rise to the succession*[3] $s'(k) = f(s(k)) = f(F^k(x))$.

(1) *Predicate $\forall k\, \exists z \in Z,\ z \preccurlyeq' f(F^k(x)) = s'(k)$ is decidable, where Z is a finite subset of A'.*

(2) *If F and f are strictly monotonous, then there exists an algorithm to decide if $s'(\mathbb{N})$ is finite or infinite.*

Proof. Successions s and s' are trivially computable. Since $(A, \preccurlyeq)$ is a recursive well behaved quasi-order, all successions in A are good. Then, succession $s(k) = F^k(x)$ is computable and good.

Consequently, the iteration $s(0), s(1), \ldots, s(i), \ldots, s(j)$ to find terms i, j with $i < j$ and $s(i) \preccurlyeq s(j)$ always halts. Call $p = j - i$.

Function F is monotonous and $s(i) \preccurlyeq s(j) \to \forall n,\ s(i+n) = F^n(s(i)) \preccurlyeq F^n(s(j)) = s(i+p+n)$. Since $n = (n \operatorname{div} p)p + (n \bmod p) = kp + r$ bijectively, we have that for all k and for all $r < p = j - i$:

$$s(i+r) \preccurlyeq \cdots \preccurlyeq s(i+kp+r) \preccurlyeq s(i+(k+1)p+r). \tag{1}$$

(1) The fact that function f in Eq. (1) is monotonous, implies that for all k and for all $r < p$:

$$f(s(i+r)) \preccurlyeq' \cdots \preccurlyeq' f(s(i+kp+r)) \preccurlyeq' f(s(i+(k+1)p+r)) \leftrightarrow \tag{2}$$
$$s'(i+r) \preccurlyeq' \cdots \preccurlyeq' s'(i+kp+r) \preccurlyeq' s'(i+(k+1)p+r). \tag{3}$$

There are two possibilities:

- Observing the $j-1$ first terms in the succession s', we can deduce that $\forall k < j\, \exists z \in Z,\ z \preccurlyeq s'(k) = f(F^k(x))$ is decidable, because Z is a finite set and the number of terms to be compared is finite. The proposition $\forall k\, \exists z \in Z,\ z \preccurlyeq' f(F^k(x)) = s'(k)$ is false if it fails in the first $j-1$ elements.
- Otherwise the proposition is true. Assume that it is true for the first $j-1$ elements. Then, for terms $s'(i) \ldots s'(i+r) \ldots s'(i+(p-1)) = s'(j-1)$, where $0 \leqslant r < j-i = p$,

[3] Actually, the succession depends on the starting value x, and should be written $s(k, x)$, but to simplify the notation we write just $s(k)$.

there are $z_0, \ldots, z_r, \ldots, z_{p-1} \in Z$ such that:

$$z_0 \preccurlyeq' s'(i+0), \ldots, z_r \preccurlyeq' s'(i+r), \ldots, z_{p-1} \preccurlyeq' s'(i+(p-1)). \tag{4}$$

From Eqs. (2) and (4), for all k and for all $r < j - i = p$, $z_r \in Z$ exists, such that:

$$z_r \preccurlyeq' s'(i+r) \preccurlyeq' \cdots \preccurlyeq' s'(i+kp+r) \preccurlyeq' s'(i+(k+1)p+r). \tag{5}$$

In consequence, $\forall k\, \exists z \in Z\; z \preccurlyeq' f(F^k(x)) = s'(k)$ is true.

(2) If F and f are strictly monotonous, there are two possibilities with Eq. (1):

- For all $r < p$, and for all $k \in \mathbb{N}$, $s_r(k) = s(i+kp+r) \succcurlyeq s(i+(k+1)p+r) = s_r(k+1)$ is a (non-strictly) decreasing sub-succession. Since A is finitely founded, there are k and k', such that $s_r(k) = s_r(k')$. Therefore, the range $s(\mathbb{N})$ is a finite set and $s'(\mathbb{N}) = f(s(\mathbb{N}))$ is finite.
- If $r < p$ exists, such that $s(i+r) \prec s(i+p+r)$, then necessarily $s'(i+r) = f(s(i+r)) \prec f(s(i+p+r)) = s'(i+p+r)$. For all k, $f(s(i+r)) \prec' f(s(i+p+r)) \prec' f(s(i+kp+r))$. Consequently, s' contains an infinitely strictly increasing sub-succession and $s'(\mathbb{N})$ is infinite. □

Example 14. Consider the recursive well behaved partial order $(\mathbb{N}^3, \leqslant)$ introduced in Example 8. Let $F(\bar{x}) = \bar{x}L_F$ be the product of vector $\bar{x} \in \mathbb{N}^3$ by a square matrix of natural numbers L_F, with dimension 3×3. It is easy to see that F is monotonous. Let us look at the special case where

$$\bar{x} = (1, 0, 0), \quad L_F = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $i = 0 < j = 2$ such that $s(i) = F^i(\bar{x}) \leqslant s(j) = F^j(\bar{x})$, where $p = j - i = 2$ and for all k:

$$s(0) = s(i+0p+0) = (1,0,0) \preccurlyeq s(2) = s(i+p+0) = (2,0,0),$$
$$s(1) = s(i+0p+1) = (0,2,0) \preccurlyeq s(3) = s(i+p+1) = (0,4,0).$$

In consequence, for all k and for all $r < p$

$$s(i+kp+r) \preccurlyeq s(i+(k+1)p+r).$$

Consider the trivially monotonous function $f : \mathbb{N}^m \to \mathbb{N}$ such that:

$$f(\bar{x}) = \bar{x} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Let the finite set be $Z = \{0\} \subseteq \mathbb{N}$. Now, $s'(0) = f(s(i+0p+0)) = 1 > z_0 = 0 \in Z$ and $s'(1) = f(s(i+0p+1)) = 2 > z_1 = 0 \in Z$. Consequently, for all $k \in \mathbb{N}$ we have $z = 0 \in Z$ such that $s'(k) = f(s(k)) > z$.

7. Characterizing monotonous and strictly monotonous homomorphisms in the Parikh's well behaved quasi-order

In this section, we apply the results developed above to solve a few decidability problems related to homomorphisms in $(V^*, \varepsilon, \cdot)$, introduced in Examples 8 and 9. Consider $(V'^*, \varepsilon, \cdot)$ over alphabet V', where $\#V = m$ and $\#V' = n$.

A homomorphism $f : V^* \to V'^*$ is a total and computable function, such that $f(\varepsilon) = \varepsilon$ and $f(xy) = f(x)f(y)$. Consider the partial quasi-orders $(V^*, \leqslant_\psi)$ and $(V'^*, \leqslant_\psi)$. Remember that, for all $x \in V^*$, $\psi(x) = \bar{x} \in \mathbb{N}^m$ is the Parikh's vector of the word *x*.

$\psi(f(x)) = \bar{x}L_f \in \mathbb{N}^n$, where L_f is a $m \times n$ matrix of natural numbers with the form

$$L_f = \begin{pmatrix} \psi(f(a_1)) \\ \dots \\ \psi(f(a_m)) \end{pmatrix},$$

where $\psi(f(a_i)) \in \mathbb{N}^n$ is the *i*th row of L_f.

Consider the vectors $\bar{i} = \psi(a_i) = (0, \dots, 0, \overbrace{1}^{i}, 0 \dots, 0) \in \mathbb{N}^m$, whose *i*th component is one, while all other components are zero. Each vector $\bar{x} \in \mathbb{N}^m$ is a linear combination $\bar{x} = x_1\bar{1} + \cdots + x_n\bar{m}$, where the coefficients $x_1, \dots, x_m \in \mathbb{N}$ are unique.

(1) Every homomorphism $f : V^* \to V'^*$ from the Parikh's well behaved quasi-order $(V^*, \leqslant_\psi)$ to the partial well behaved quasi-order $(V'^*, \leqslant_\psi)$ is monotonous:

For all $x, y \in V^*$, $x \leqslant_\psi y \leftrightarrow \bar{x} \leqslant \bar{y}$. Thus, $\psi(f(x)) = \bar{x}L_f \leqslant \bar{y}L_f = \psi(f(y))$, as $\psi(f(y)) - \psi(f(x)) = (\bar{y} - \bar{x})L_f$ and $(\bar{y} - \bar{x}) \geqslant \bar{0}$.

(2) A homomorphism $f : V^* \to V'^*$ is strictly monotonous if $\bar{x}L_f > \bar{0}$ is true for all $\bar{x} > 0$. A homomorphism $f : V^* \to V'^*$ is strictly monotonous iff L_f does not have a zero row:

If L_f has a zero row, assume (without loss of generality) that it has the form

$$L_f = \left(\begin{array}{c} A_f \\ \hline 0 \cdots 0 \end{array} \right).$$

Consequently, for any $\bar{x} = (0, \dots, 0, x_m) > \bar{0}$, $\bar{x}L_f = \bar{0}$ is true, and L_f is not strictly monotonous.

Conversely, if all rows in L_f were non-zero, for every vector $\bar{i} = \psi(a_i) = (0, \dots, 0, \overbrace{1}^{i}, 0 \dots, 0) \in \mathbb{N}^m$, $\bar{i}L_f > \bar{0}$ is true. Consequently, for all $\bar{x} = x_1\bar{1} + \cdots + x_n\bar{m} > \bar{0}$, $\bar{x}L_f = x_1\bar{1}L_f + \cdots + x_m\bar{m}L_f > \bar{0}$ is true.

8. Decidability of the nilpotency of *D0L*, *PD0L* and *HD0L* systems

A $D0L$ system is a homomorphism $F : V^* \to V^*$, iterated from an initial condition $x \in V^*$, which can be represented by a succession $s(k) = F^k(x)$. We call F a $D0L$ homomorphism.

A $D0L$ homomorphism is propagating or $PD0L$ if for all symbols $a_i \in V$, $F(a_i) \neq \varepsilon$ is true.

A filtering or $HD0L$ system is a $D0L$ system, together with a filter homomorphism $f : V^* \to V'^*$, that defines a succession $s'(k) = f(F^k(x))$.

Notice that $PD0L \subset D0L \subset HD0L$ as given in [1]. Since $f = I$, the identity function, is a homomorphism, all the definitions and results enunciated for $HD0L$ systems are correspondingly valid for classes $D0L$ and $PD0L$.

Example 15. If $V = \{a, b\}$ and $F(a) = aa$, $F(b) = a$, the corresponding homomorphism in $\mathbb{N}^2$ is $F = \psi(F)$ (we shall use the same name, as a shortcut):

$$F(x_1, x_2) = (x_1, x_2)\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = (x_1, x_2)L_F.$$

This function defines the iteration $s(k) = F^k(x) = xL_F^k$, starting at $s(0) = x = (x_1, x_2) \in \mathbb{N}^2$ in the well behaved partial order $(\mathbb{N}^2, \leqslant)$. Notice that F is a $PD0L$ system.

Consider now the alphabet $V' = \{a\}$ and the homomorphism $g : V^* \to V'^*$, $g(a) = a$, $g(b) = a$. Working with Parikh's images, let us call $g = \psi(g)$, $g : \mathbb{N}^2 \to \mathbb{N}$, such that:

$$g(x_1, x_2) = (x_1, x_2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (x_1, x_2)L_g = x_1 + x_2.$$

Definition 16. The filtering homomorphism $g : V^* \to \{a\}^*$, such that $g(x) = a^{|x|}$ where $|x|$ is the length of word x, is called the growth function.

Another way to express it is:

$$|g(x)| = \bar{x}L_g = (x_1, \ldots, x_m)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = x_1 + \cdots + x_m.$$

An $HD0L$ system is nilpotent if a k exists, such that $s'(k) = f(F^k(x)) = \varepsilon$.

Corollary 17. *Consider an $HD0L$ system, made of a $D0L$ system $F : V^* \to V^*$, iterated from $x \in V^*$, and of a filtering homomorphism $f : V^* \to V'^*$. The nilpotency problem $\exists k,\ s'(k) = f(F^k(x)) = \varepsilon$ is decidable.*

Proof. $(V^*, \leqslant_\psi)$ and $(V'^*, \leqslant_\psi)$ are well behaved partial orders with absolute minimum ε, as was proved in Examples 8 and 9. Trivially, orders $(V^*, \leqslant_\psi)$ and $(V'^*, \leqslant_\psi)$ are recursive and finitely founded.

Let $Z = \{y \in V'^* : |y| = 1\}$ be the words in V'^* of length 1: by Theorem 13, $\forall k\, \exists z \in Z,\ z \preccurlyeq' f(F^k(x)) = s'(k)$ is decidable. Consequently, $\exists k,\ s'(k) = f(F^k(x)) = \varepsilon$ is decidable. $\square$

The previous corollary was well-known [1], and here has been proved by a different method.

The nilpotency problem for $D0L$ and $PD0L$ systems is equally decidable: if $f = I : V^* \to V^*$ is the identity homomorphism, $\exists k,\ \varepsilon = s(k) = F^k(x)$ is a sub-case of $HD0L$ systems.

9. A proof of the decidability of the infinite growth of *PD0L* systems

From the definition given in Section 8, it is easy to see that the matrix L_F, associated to a $PD0L$ homomorphism F, only has non-zero rows.

Using the growth function g introduced in Definition 16, any vector $\bar{x} > \bar{0}$ iff $\bar{x}L_g > 0$. The condition for F being propagating is similar: $L_F L_g \geqslant L_g$, meaning that every row in matrix L_F contains at least a non-zero entry.

Example 18.

$$L_F L_g = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \geqslant \begin{pmatrix} 1 \\ 1 \end{pmatrix} = L_g.$$

From the characterization given in Section 7, every $PD0L$ homomorphism is strictly monotonous in the Parikh's quasi-order, as L_F does not have a zero row.

Corollary 19. *Let $F : V^* \to V^*$, iterated from x, be a PD0L system, $s(k) = F^k(x)$. Let the language derived by the system be $s(\mathbb{N})$. Then, "$s(\mathbb{N})$ is an infinite language" is a decidable problem.*

Proof. Since F, iterated from x, is a $PD0L$ system, and g is its growth function, F and g are strictly monotonous. By Theorem 13, "$s'(\mathbb{N}) = g(s(\mathbb{N}))$ has an infinite cardinality" is decidable. □

The previous corollary was well-known [1], and here has been proved by a different method.

10. Proving the decidability of the problem of Parikh momentary stagnation of the growth functions of *PD0L* systems

Corollary 20. *Consider an HD0L system, made of a PD0L system $F : V^* \to V^*$, iterated from $x \in V^*$, and of the growth function $g : V^* \to \{a\}^*$. Then, $\exists k,\ s'(k) = g(F^k(x)) = g(F^{k+1}(x)) = s'(k+1)$ is decidable.*

Proof. We prove that the problem in Eq. (6) is decidable:

$$\exists k\ \ (\bar{x}L_F^{k+1}L_g = \bar{x}L_F^{k}L_g). \tag{6}$$

For a $PD0L$ system, L_F is strictly monotonous and does not have any zero rows. Thus, Eq. (6) is true if $\exists k\ (\bar{x}L_F^k(L_F - I)L_g) = 0$, where I is the identity matrix of dimension m.

The homomorphism

$$(L_F - I)L_g = L_{f'} = \begin{pmatrix} f'_1 \\ \vdots \\ f'_m \end{pmatrix}$$

is also monotonous: if $L_{f'}$ has a negative component $f'_j < 0$, then row j in matrix L_F is zero, and L_F is not strictly monotonous, as proved in Section 7.

By Corollary 17, the problem in Eq. (6) is decidable. $\square$

11. An algorithm to compute the problem of momentary stagnation of *PD0L* systems

Since matrix L_F is propagating, by Corollary 17 we have an algorithm that tests $\bar{x} \in$ *OUTPUT*:

$$OUTPUT = \{\bar{x} \in \mathbb{N}^m : \forall k(\bar{x}L_F^{k-1}(L_F - I)L_f > 0)\}.$$

For this purpose, we compute the sequence $s(0) = \bar{x}_y,\ s(1) = \bar{x}_y L_F, \ldots, s(h) = \bar{x}_y L_F^h, \ldots, s(h+p)$ of the first $h+p$ mutually incomparable elements, where $s(h+p+1)$ is the first element comparable to $s(h)$:

- If $s(h) \geqslant s(h+p+1)$, then $s(\mathbb{N})$ is finite and the succession trivially converges to a value $s(h+k) = s(h+k')$, in at most $k' \leqslant k \leqslant x_1 + \ldots + x_m$ steps, where $\bar{x} = (x_1, \cdots, x_m)$. Thus, $\bar{x} \in OUTPUT \leftrightarrow\ \forall k \leqslant k'\ (s'(k) = s(k)L_f \neq s'(k+1))$.
- If $s(h) < s(h+p+1)$, since L_F is propagating, $s(\mathbb{N})$ is infinite. Therefore, $(\bar{x} \in OUTPUT \leftrightarrow \bar{x}(L_F - I)L_f > 0)$.

In this way, the algorithm computes:

$$OUTPUT = \{\bar{x} \in \mathbb{N}^m : \forall k(\bar{x}L_F^k L_f \neq \bar{x}L_F^{k+1}L_f)\}.$$

12. Towards a proof of the problem of momentary stagnation of *HD0L*

12.1. Reducing the problem of momentary stagnation of D0L to the momentary stagnation of HD0L

Consider an $HD0L$ system, made of a $D0L$ system $F : V^* \to V^*$, iterated from $x \in V^*$, and the growth function $g : V^* \to V'^*$. Then, the problem of momentary stagnation of $D0L$ systems can be expressed thus: $\exists k,\ s'(k) = g(F^k(x)) = g(F^{k+1}(x)) = s'(k+1)$ is decidable.

If L_F is monotonous, but not strictly monotonous, it has at least a zero row. By ordering conveniently the rows and columns, L_F and L_g may be given the form

$$L_F = \left(\begin{array}{c|c} M_{i\times i} & B_{i\times(m-i)} \\ \hline 0_{(m-i)\times i} & 0_{(m-i)\times(m-i)} \end{array}\right) L_g = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} L_{1_g} \\ \hline L_{2_g} \end{pmatrix},$$

where $(M|B)_{i\times m}$ only has non-zero rows.

In the following, the sub-indices indicating the dimensions of the matrices will be omitted, as they are the same given above. Hence, matrix $L_F^{k+1}L_g$ has the form

$$L_F^{k+1}L_g = \left(\begin{array}{c|c} M^{k+1} & M^k B \\ \hline 0 & 0 \end{array}\right) L_g = \begin{pmatrix} M^k(ML_{1_g} + BL_{2_g}) \\ 0 \end{pmatrix}. \tag{7}$$

For all $\bar{x} \in \mathbb{N}^m$, we represent $\bar{x} = (u_x, v_x)$, where $u_x \in \mathbb{N}^i$ and $v_x \in \mathbb{N}^{m-i}$, since $\mathbb{N}^m$ is isomorphic to $\mathbb{N}^i \times \mathbb{N}^{m-i}$. Therefore

$$\bar{x}L_F^{k+1}L_g = \begin{pmatrix} u_x(M^k(ML_{1_g} + BL_{2_g})) \\ 0 \end{pmatrix} \tag{8}$$

and the following equation proves that the momentary stagnation of a $D0L$ system is reduced to the momentary stagnation of an equivalent $HD0L$ system with less or equal dimensions (this problem is trivially decidable for dimension $m = 1$).

$$\exists\, k \ \ (\bar{x}L_F^{k+1}L_g = \bar{x}L_F^k L_g) \leftrightarrow \tag{9}$$

$$\exists\, k \ \ (u_x M^k(ML_{1_g} + BL_{2_g}) = u_x M^{k-1}(ML_{1_g} + BL_{2_g})) \leftrightarrow \tag{10}$$

$$\exists\, k \ \ (u_x M^k L_{f'} = u_x M^{k-1} L_{f'}), \tag{11}$$

where M has dimension $i \leqslant m-1$ and $L_{f'} = ML_{1_g} + BL_{2_g}$. Now the momentary stagnation of the problem in Eq. (9) is decidable in the following cases:

- If $(M-I)L_{f'} = L_{f''} \geqslant 0_{i\times 1}$; by Corollary 17, the nilpotency is decidable: $\exists k\ \bar{x}M^k L_{f''} = 0$ for all x.
- If $\bar{x}(M - I) \geqslant \bar{x}$; by Corollary 17, the nilpotency is decidable: $\exists k\ \bar{x}M^k L_{f''} = 0$ for all homomorphism $L_{f''}$.

Difficulties to prove the momentary stagnation of $HD0L$ arise when matrices M or B are non-strictly monotonous.

12.2. Reducing the problem of momentary stagnation of $D0L$ systems to the momentary stagnation of $HPD0L$ systems with a strict filter

Let us look a little more at the problem left open in the previous subsection: assume, without loss of generality (through permutation of the coordinates), that matrix L_F has the form

$$L_F = \left(\begin{array}{c|c|c} M_{i\times i} & B1_{i\times j} & C_{i\times(m-i-j)} \\ \hline 0_{j\times i} & D_{j\times j} & B2_{j\times(m-i-j)} \\ \hline 0_{(m-i-j)\times i} & 0_{(m-i-j)\times j} & 0_{(m-i-j)\times(m-i-j)} \end{array}\right). \tag{12}$$

The dimensions of the sub-matrices are indicated by sub-indices, where $0_{i\times j}$ is a null matrix. Matrix M in Eq. (12) is strictly monotonous. Matrix D is such that $D(i, j) = 0$ if $i \geqslant j$ and $D(i, j) \geqslant 0$ if $i < j$. We call D a diagonalized matrix.

It is easy to see that, for a diagonalized matrix $D_{j\times j}$, $D^j_{j\times j} = 0_{j\times j}$. Thus, after $j+1$ iterations, the matrix in Eq. (12) becomes

$$L_F^{j+1} L_g = \left(\begin{array}{c|c|c} M^{j+1} & B1' & C' \\ \hline 0 & D^{j+1} = 0 & B2' = D^j B2 = 0 \\ \hline 0 & 0 & 0 \end{array}\right) L_g \tag{13}$$

$$= \left(\begin{array}{c|c} A = M^{j+1} & B \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} L_{1_g} \\ \hline L_{2_g} \end{array}\right) = AL_{1_g} + BL_{2_g} = L_f. \tag{14}$$

Homomorphism L_f is strictly monotonous, as matrix A is strictly monotonous. Take a vector $y = (u_y, v_y) \in \mathbb{N}^i \times \mathbb{N}^{m-i}$:

$$\exists\, k \ \ (yL_F^{k+1+j+1} L_g = yL_F^{k+j+1} L_g) \leftrightarrow \tag{15}$$

$$\exists\, k \ \ (yL_F^{k+1} L_f = yL_F^{k} L_f) \leftrightarrow \tag{16}$$

$$\exists\, k \ \ (u_y M^{k+1} L_f = u_y M^k L_f). \tag{17}$$

Now, it is clear that the only difficult case in the proof of the problem of momentary stagnation of the growth functions of $D0L$ systems appears when homomorphism BL_{2_g} in Eq. (14) contains zeros, that is, matrix B contains zero rows.

It is clear also that a proof of the decidability of the momentary stagnation of the growth functions of $D0L$ systems which includes this case, will also provide a proof of the decidability of the momentary stagnation of $HD0L$ systems.

12.3. *Proving the decidability of the finitude of HD0L systems*

To complete the results given in this paper we prove the decidability of the infinitude of $HD0L$ systems.

Corollary 21. *Let $F : V^* \to V^*$, iterated from x, be a $D0L$ system, $s(k) = F^k(x)$. Let the language derived by the system be $s(\mathbb{N})$. Then, "$s(\mathbb{N})$ is an infinite language" is a decidable problem.*

Proof. The problem of determining if a $D0L$ system is finite, is a trivial consequence of the procedure described in this section. The nilpotency of $D0L$ systems is decidable by Corollary 17.

Otherwise, by Eq. (17), the $D0L$ system is reduced to an equivalent $HPD0L$ with a strict filtering homomorphism: by Corollary 9, the finitude of the set $\{u_y M^k : k \in \mathbb{N}\}$ is decidable, since L_f is strictly monotonous, and M is the matrix of a $PD0L$ system. □

13. Conclusions

In this paper, we solve the problem of momentary stagnation of the growth function of $PD0L$ systems, by proving the decidability of the existence of $k \in \mathbb{N}$, such that $f(F^k(x)) = f(F^{k+1}(x))$, where F is a $PD0L$ homomorphism and f is a filtering homomorphism (the growth function). Two other well-known, previously solved problems (the infinitude of $PD0L$ languages and the nilpotency of $HD0L$ homomorphisms) have here been solved by a different method.

Still open is the generalization of the problem of momentary stagnation of the growth function for $D0L$ systems, as described in [1]. We will try to tackle this as the next step in our work. For this purpose, this paper analyzes the difficult cases of a possible proof, following the approach given in Theorem 13. This approach is interesting, because it makes clear that the easy cases of the problem of momentary stagnation are due to the property of monotony of the functions, and are not related to the fact of being homomorphisms.

As a further advance towards a proof of the problem of momentary stagnation of $HD0L$ systems, we reduce the problem of momentary stagnation of the growth function of $D0L$ systems to the momentary stagnation of $HD0L$ systems, to show the easy cases. We refine the difficult cases, reducing the problem of momentary stagnation of the growth of $D0L$ systems to the momentary stagnation of the growth of $HPD0L$ systems with a strictly monotonous homomorphism.

References

[1] L. Kari, G. Rozenberg, A. Salomaa, L systems, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Word, Language, Grammar, Vol. 1, Springer, Berlin, 1997.
[2] A. Lindemayer, Mathematical models for cellular interaction in development—I and II, J. Theoret. Biol. (18) (1968) 280–315.
[3] R.N. Moll, M.A. Arbib, A. Kfoury, A Programming Approach to Computability, Springer, Berlin, 1988.
[4] A. Paz, A. Salomaa, Integral sequential growth functions and growth equivalence of Lindenmayer systems, Inform. Control (23) (1973) 313–343.
[5] P. Prusinkiewicz, A. Lindenmayer, The Algorithmic Beauty of Plants, Springer, Berlin, 1990.
[6] G. Rozenberg, Extension of tabled 0L systems and languages, Internat. J. Comput. Inform. Sci. (2) (1973) 311–334.
[7] G. Rozenberg, A. Salomaa, The Mathematical Theory of L Systems, Academic Press, New York, 1980.
[8] G. Rozenberg, A. Salomaa, The Book of L, Springer, Berlin, 1980.
[9] G. Rozenberg, A. Salomaa, Lindenmayer Systems, Springer, Berlin, 1992.
[10] W. Wechler, Universal Algebra for Computer Scientists, Springer, Berlin, 1992.