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Existence and multiplicity of positive periodic solutions for a class of higher-dimension functional differential equations with impulses ☆

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ABSTRACT

This paper deals with the existence of multiple periodic solutions for n-dimensional functional differential equations with impulses. By employing the Krasnoselskii fixed point theorem, we obtain some easily verifiable sufficient criteria which extend previous results.

1. Introduction

As usual, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{N} = \{1, 2, \dots, \}$ and $\mathbb{R}_+^n = \{(x_1, \dots, x_n)^{\mathrm{T}} : x_i \geq 0, 1 \leq i \leq n\}$, respectively. For each $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, the norm of x is defined as $|x| = \sum_{i=1}^n |x_i|$. Let BC denote the Banach space of bounded continuous functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|\phi\| = \sup_{\theta \in R} \sum_{i=1}^n |\phi_i(\theta)|$, where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^{\mathrm{T}}$. Let $\mathbb{J} \subset \mathbb{R}$ and $PC(\mathbb{J}, \mathbb{R}^n)$ denote the set of operators $\phi : \mathbb{J} \to \mathbb{R}^n$ which are continuous for $t \in \mathbb{J}$, $t \neq \tau_k$ and have discontinuities of the first kind at the points $\tau_k \in \mathbb{J}(k \in \mathbb{N})$, but are continuous from the left at these points.

Recent years have witnessed increasing interest in the existence of positive periodic solutions. By employing the powerful and efficient method of coincidence degree [1–4] and theory in cones [5–10], some verifiable sufficient criteria for the existence of positive periodic solutions have been established. However, compared with advances in the area of studying the existence of periodic solutions of continuous differential equations, less progress has been achieved in the so-called impulsive differential equations, which are subject to short-time perturbation or change very rapidly at certain instants; only a few papers are concerned with this subject: see [5,11,10]. In fact, differential equations involving impulse effects occur in almost every domain of applied science: physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. Therefore, the study of this class of dynamical systems is becoming a rapidly growing field. Motivated by this, in the present paper, by utilizing the fixed point theorem due to Krasnoselskii, we aim to study the existence and multiplicity of periodic solutions of the following differential equation with impulses:

$$\begin{cases} x'(t) = A(t, x(t))x(t) + f(t, x_t), & t \neq \tau_k,\ k \in \mathbb{N}, \\ x(\tau_k^+) = x(\tau_k) + E_k(x(\tau_k)), & t = \tau_k, \end{cases} \tag{1}$$

where $A(t, x(t)) = \operatorname{diag}[a_1(t, x(t)), a_2(t, x(t)), \dots, a_n(t, x(t))]$, $a_i \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is ω-periodic; $f = (f_1, f_2, \dots, f_n)^{\mathrm{T}}$, $f : \mathbb{R} \times BC \to \mathbb{R}^n$ and $f(t, x_t)$ is ω-periodic whenever x is ω-periodic; x_t is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in \mathbb{R}$. By

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this definition, it is easy to see that, if $x \in BC$, then $x_t \in BC$ for any $t \in \mathbb{R}$. $x(\tau_k^+)$ represents the right limit of $x(t)$ at the point τ_k, $E_k = (E_k^1, E_k^2, \ldots, E_k^n) \in C(\mathbb{R}_+^n, \mathbb{R}_-^n)$ (here $\mathbb{R}_-^n = \{(x_1, \ldots, x_n)^{\mathrm{T}} : x_i \le 0, 1 \le i \le n\}$). We assume that there exists an integer $p > 0$ such that $\tau_{k+p} = \tau_k + \omega$, $E_{k+p} = E_k$, where $0 < \tau_1 < \tau_2 < \cdots < \tau_p < \omega$.

The paper is organized as follows. In Section 2, we list some preliminaries involving the famous Krasnoselskii fixed point theory and some assumptions and lemmas. In Section 3, by employing the Krasnoselskii fixed point theory, we investigate the existence of at least one positive periodic solution and we study the existence of multiple periodic solutions for the impulsive functional differential equation (1) in Section 4. In Section 5, we give some conclusions.

2. Preliminaries

In this section, we make some preparations for the following sections. First, we give the following related definition and the famous fixed point theorem that will be needed in our arguments.

Definition. Let X be Banach space and K be a closed, nonempty subset of X; K is said to be a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$
(ii) $u, -u \in K$ imply $u = 0$.

Theorem A (*Krasnoselskii Fixed Point Theorem [12]*). *Let X be a Banach space, and let K be a cone in X. Suppose that Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

is a completely continuous operator and satisfies either

(i) *$\|Tx\| \ge \|x\|$ for any $x \in K \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$ for any $x \in K \cap \partial\Omega_2$; or*
(ii) *$\|Tx\| \le \|x\|$ for any $x \in K \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$ for any $x \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

To obtain our main results, we make the following assumptions throughout this paper.

(H1) There exist continuous ω-periodic functions $a_i^\ell(t), a_i^L(t)$, such that $a_i^\ell(t) \le a_i(t, x) \le a_i^L(t)$ and $\int_0^\omega a_i^\ell(t)\mathrm{d}t > 0$, for $1 \le i \le n$,
(H2) $f_i(t, x_t)$ is a continuous function of t for each $x \in BC(\mathbb{R}, \mathbb{R}_+^n)$.
(H3) $f_i(t, \phi_t) \int_0^\omega a_i(s, x(s))\mathrm{d}s \le 0$ for all $(t, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}_+^n)$, $1 \le i \le n$.
(H4) For any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $[\phi, \psi \in BC, \|\phi\| \le L, \|\psi\| \le L, \|\phi - \psi\| < \delta, 0 \le s \le \omega]$ imply $|f_i(s, \phi_s) - f_i(s, \psi_s)| < \varepsilon$, $1 \le i \le n$.

First, we have the following lemma that will be needed in our arguments.

Lemma 2.1. *The function $x(t)$ is an ω-periodic solution of (1) if and only if $x(t)$ is an ω-periodic solution of the following system:*

$$x(t) = \int_t^{t+\omega} G(t, s) f(s, x_s)\mathrm{d}s + \sum_{j=1}^{p} G(t, \tau_{m_j} + n\omega) E_j(x(\tau_{m_j})), \tag{2}$$

where

$$G(t, s) = \mathrm{diag}[G_1(t, s), G_2(t, s), \ldots, G_n(t, s)], \tag{3}$$

and

$$E_j(x) = \mathrm{diag}\left[E_j^1(x), E_j^2(x), \ldots, E_j^p(x)\right].$$

Proof. The proof of sufficiency is similar to that in [11], so we omit it. Here we only prove the necessity part. If $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} \in X$ is a solution of system (1), then

$$\left[x_i(t) \exp\left\{-\int_0^t a_i(u, x(u))\mathrm{d}u\right\}\right]' = \exp\left\{-\int_0^t a_i(u, x(u))\mathrm{d}u\right\} f_i(t, x_t), \quad t \ne \tau_k,\ i = 1, 2, \ldots, n. \tag{4}$$

Integrating both sides of (4) over $[t, t + \omega]$, we obtain

$$\begin{aligned} & x_i(s) \exp\left\{-\int_0^s a_i(u, x(u))\mathrm{d}u\right\}\Bigg|_t^{\tau_{m_1}+n\omega} + x_i(s) \exp\left\{-\int_0^s a_i(u, x(u))\mathrm{d}u\right\}\Bigg|_{\tau_{m_1}+n\omega}^{\tau_{m_2}+n\omega} \\ & \quad + \cdots + x_i(s) \exp\left\{-\int_0^s a_i(u, x(u))\mathrm{d}u\right\}\Bigg|_{\tau_{m_p}+n\omega}^{t+\omega} \\ & = \int_t^{t+\omega} \exp\left\{-\int_0^s a_i(u, x(u))\mathrm{d}u\right\} f_i(s, x_s)\mathrm{d}s, \end{aligned}$$

where $\tau_{m_j} + n\omega \in (t, t+\omega)$, $m_j \in \{1, 2, \dots, p\}$, $j = 1, 2, \dots, p$, $n \in N$. Then

$$x_i(t)\exp\left\{-\int_0^t a_i(u,x(u))\mathrm{d}u\right\}\left[\exp\left\{-\int_t^{t+\omega} a_i(u,x(u))\mathrm{d}u\right\}-1\right]-\sum_{j=1}^{p}\Delta x_i(\tau_{m_j})\exp\left\{-\int_0^{\tau_{m_j}+n\omega} a_i(u,x(u))\mathrm{d}u\right\}$$
$$=\int_t^{t+\omega}\exp\left\{-\int_0^s a_i(u,x(u))\mathrm{d}u\right\}f_i(s,x_s)\mathrm{d}s,$$

where $\Delta x_i(\tau_k) = x_i(\tau_k^+) - x_i(\tau_k)$. In turn, this expression can be transformed into

$$x_i(t) = \int_t^{t+\omega} G_i(t,s)f_i(s,x_s)\mathrm{d}s + \sum_{j=1}^{p} G_i(t, \tau_{m_j}+n\omega)E_j^i(x(\tau_{m_j})),$$

where

$$G_i(t,s) = \frac{\exp\{-\int_t^s a_i(u,x(u))\mathrm{d}u\}}{\exp\{-\int_0^\omega a_i(u,x(u))\mathrm{d}u\}-1}, \quad 1 \le i \le n. \quad \blacksquare \tag{5}$$

By the definition of G in (3) and (5), it is clear that $G(t,s) = G(t+\omega, s+\omega)$ for all $(t,s) \in \mathbb{R}^2$ and by (H3),

$$G_i(t,s)f_i(u,\phi_u) \ge 0$$

for $(t,s) \in \mathbb{R}^2$ and $(u,\phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}_+^n)$.

Let

$$A_i = \max\{|a_i^\ell(t)|, |a_i^L(t)|\}, \quad t \in [0,\omega],\ 1 \le i \le n.$$

Then by direct calculation and (H1), we get

$$m_i := \frac{\exp\{-\int_0^\omega A_i(u)\mathrm{d}u\}}{|\exp\{-\int_0^\omega a_i^L(u)\mathrm{d}u\}-1|} \le |G_i(t,s)| \le \frac{\exp\{\int_0^\omega A_i(u)\mathrm{d}u\}}{|\exp\{-\int_0^\omega a_i^\ell(u)\mathrm{d}u\}-1|} =: M_i, \tag{6}$$

where a_i^ℓ, a_i^L are defined in (H1).

We define

$$\sigma = \min\left\{\exp\left\{-2\int_0^\omega A_i(u)\mathrm{d}u\right\}\frac{|\exp\{-\int_0^\omega a_i^\ell(u)\mathrm{d}u\}-1|}{|\exp\{-\int_0^\omega a_i^L(u)\mathrm{d}u\}-1|},\ 1 \le i \le n\right\}; \tag{7}$$

then it is clear that $0 < \sigma < 1$.

Define also

$$X = \{x = (x_1(t), \dots, x_n(t))^{\mathrm{T}} \in PC(\mathbb{R}, \mathbb{R}^n) : x(t+\omega) = x(t),\ t \in \mathbb{R}\}$$

with the norm $\|x\| = \sum_{i=1}^n |x_i|_0$, where $|x_i|_0 = \sup_{t\in[0,\omega]} |x_i(t)|$, and

$$K = \{x \in X : x_i(t) \ge \sigma|x_i|_0,\ t \in [0,\omega],\ x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}\}. \tag{8}$$

One may readily verify that X is a Banach space and K is a cone.

Moreover, define, for r a positive number, Ω_r by

$$\Omega_r = \{x \in K : \|x\| < r\}.$$

Note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$.

Let the map $T : K \to X$ be defined by

$$(Tx)(t) = \int_t^{t+\omega} G(t,s)f(s,x_s)\mathrm{d}s + \sum_{j=1}^{p} G(t, \tau_{m_j}+n\omega)E_j(x(\tau_{m_j})), \tag{9}$$

for $x \in K$, $t \in \mathbb{R}$, and let

$$(Tx) = (T_1x, T_2x, \dots, T_nx)^{\mathrm{T}}.$$

Then it can be immediately obtained from the assumptions (H3) and (H4) that the map T is completely continuous. On the other hand, it follows from Lemma 2.1 that $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^{\mathrm{T}}$ is a positive ω-periodic solution of (1) if and only if $x^*(t)$ is a fixed point of the operator T.

Lemma 2.2. *The mapping T maps K into K, i.e., $TK \subset K$.*

Proof. For any $x \in K$, it is easy to see that $Tx \in K$. From (6) and (9), we have

$$|T_i x|_0 \leq M_i \int_0^\omega |f_i(s, x_s)|\mathrm{d}s + M_i \sum_{j=1}^{p} |E_j^i(x(\tau_{m_j}))|.$$

Noting that $G_i(t, s)f_i(u, \phi_u) \geq 0$, we can also obtain

$$\begin{aligned}(T_i x)(t) &\geq m_i \int_0^\omega |f_i(s, x_s)|\mathrm{d}s + m_i \sum_{j=1}^{p} |E_j^i(x(\tau_{m_j}))| \\ &\geq \frac{m_i}{M_i}|T_i x|_0 \geq \sigma |T_i x|_0.\end{aligned}$$

Therefore, $TK \subset K$. The proof is complete. ■

Lemma 2.3. *If there exists $\eta > 0$ such that*

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(\phi(\tau_{m_j}))| \geq \eta\|\phi\|, \quad \text{for } \phi \in K,$$

then

$$\|Tx\| \geq m\eta\|x\|, \quad \text{for } x \in K,$$

where $m = \min_{1\leq i\leq n} m_i$ and m_i is defined in (6).

Proof. If $x \in K$, then

$$\begin{aligned}(T_i x)(t) &\geq m_i \int_t^{t+\omega} |f_i(s, x_s)|\mathrm{d}s + m_i \sum_{j=1}^{p} |E_j^i(x(\tau_{m_j}))| \\ &= m_i \int_0^\omega |f_i(s, x_s)|\mathrm{d}s + m_i \sum_{j=1}^{p} |E_j^i(x(\tau_{m_j}))|.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|Tx\| = \sup_{t\in\mathbb{R}} \sum_{i=1}^{n} |(T_i x)(t)| &\geq \sum_{i=1}^{n} \left[m_i \int_0^\omega |f_i(s, x_s)|\mathrm{d}s + m_i \sum_{j=1}^{p} |E_j^i(x(\tau_{m_j}))| \right] \\ &\geq m \left[\int_0^\omega |f(s, x_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(x(\tau_{m_j}))| \right] \geq m\eta\|x\|. \quad ■\end{aligned}$$

Lemma 2.4. *If there exists a sufficiently small $\varepsilon > 0$ and any number $r > 0$ such that*

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(\phi(\tau_{m_j}))| \leq \varepsilon r, \quad \text{for } \phi \in K \cap \partial\Omega_r,$$

then

$$\|Tx\| \leq M\varepsilon\|x\|, \quad \text{for } x \in K \cap \partial\Omega_r,$$

where $M = \max_{1\leq i\leq n} M_i$ and M_i is defined in (6).

Proof. Following the ideas in the proof of Lemma 2.3, we obtain the assertion. ■

For the sake of convenience, we introduce the following notations:

$$f_\nu = \lim_{\|\phi\|\to\nu} \inf_{\phi\in K} \frac{\int_0^\omega |f(s, \phi_s)|\mathrm{d}s}{\|\phi\|}, \qquad E_\nu = \lim_{\|\phi\|\to\nu} \inf_{\phi\in K} \frac{\sum_{j=1}^{p} |E_j(\phi)|}{\|\phi\|},$$

$$f^\nu = \lim_{\|\phi\|\to\nu} \sup_{\phi\in K} \frac{\int_0^\omega |f(s, \phi_s)|\mathrm{d}s}{\|\phi\|}, \qquad E^\nu = \lim_{\|\phi\|\to\nu} \sup_{\phi\in K} \frac{\sum_{j=1}^{p} |E_j(\phi)|}{\|\phi\|},$$

where ν denotes either 0 or ∞.

3. Existence of at least one periodic solution

Theorem 3.1. *If*

$$\text{(P1)}\quad f_0 = E_0 = \infty \quad \textit{and} \quad \text{(P2)}\quad f^\infty = E^\infty = 0$$

hold, then (1) *has at least one positive ω-periodic solution.*

Proof. Since $f_0 = E_0 = \infty$, one can find an $r_0 > 0$ such that

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s \geq \eta_1\|\phi\| \quad \text{and} \quad \sum_{j=1}^p |E_j(\phi)| \geq \eta_2\|\phi\|, \quad \text{for } \phi \in K,\ 0 < \|\phi\| \leq r_0,$$

where the constants $\eta_1,\ \eta_2$ satisfy $2m\eta_i > 1$, $i = 1, 2$. Choose a constant $\eta > 0$ satisfying $\eta = \min\{2\eta_1, 2\eta_2\}$. Then

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(\phi)| \geq \frac{\eta}{2}\|\phi\| + \frac{\eta}{2}\|\phi\| = \eta\|\phi\|.$$

Therefore, by Lemma 2.3, we obtain

$$\|Tx\| \geq m\eta\|x\| > \|x\|, \quad \text{for } x \in K \cap \partial\Omega_{r_0}.$$

Moreover, using $f^\infty = E^\infty = 0$, we know there exist $N_1 > r_0$ and $\varepsilon_1,\ \varepsilon_2 > 0$ such that

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s \leq \varepsilon_1\|\phi\| \quad \text{and} \quad \sum_{j=1}^p |E_j(\phi)| \leq \varepsilon_2\|\phi\|, \quad \text{for } \phi \in K,\ \|\phi\| \geq N_1.$$

Choose $\varepsilon = \max\{2\varepsilon_1, 2\varepsilon_2\}$ satisfying $0 < \varepsilon \leq \frac{1}{2M}$; then

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(\phi)| \leq \varepsilon\|\phi\|.$$

Take

$$r_1 > N_1 + 1 + 2M \sup_{\substack{\|\phi\| < N_1 \\ \phi \in K}} \left[\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(\phi)|\right].$$

If $x \in K \cap \partial\Omega_{r_1}$, then

$$\begin{aligned} \|Tx\| &\leq M\left[\int_0^\omega |f(s, x_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(x)|\right] \\ &= M[\rho(I_1) + \rho(I_2)] \leq \frac{r_1}{2} + \frac{\|x\|}{2} = \|x\|, \end{aligned}$$

where $\rho(I_i) = [\int_0^\omega |f(s, x_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(x)|]|_{x \in I_i}$, $i = 1, 2$ and $I_1 = \{x \in K, \|x\| < N_1\}$, $I_2 = \{x \in K, \|x\| \geq N_1\}$. This implies that $\|Tx\| \leq \|x\|$ for any $x \in K \cap \partial\Omega_{r_1}$.

Therefore, under the conditions (P1) and (P2), T satisfies all the requirements in Theorem A; then T has a fixed point in $K \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_0})$. We complete the proof. ■

Theorem 3.2. *If*

$$\text{(P3)}\quad f_\infty = E_\infty = \infty \quad \textit{and} \quad \text{(P4)}\quad f^0 = E^0 = 0$$

hold, then (1) *has at least one positive ω-periodic solution.*

Proof. By (P4), there exist $r_2 > 0$ and sufficiently small $\varepsilon_1,\ \varepsilon_2 > 0$, such that

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s \leq \varepsilon_1\|\phi\| \leq \varepsilon_1 r_2 \quad \text{and} \quad \sum_{j=1}^p |E_j(\phi)| \leq \varepsilon_2\|\phi\| \leq \varepsilon_2 r_2, \quad \text{for } \phi \in K,\ 0 < \|\phi\| \leq r_2.$$

Choose $\varepsilon = \max\{2\varepsilon_1, 2\varepsilon_2\}$ satisfying $0 < \varepsilon \leq \frac{1}{M}$, and we have

$$\int_0^\omega |f(s, \phi_s)|\mathrm{d}s + \sum_{j=1}^p |E_j(\phi)| \leq \varepsilon r_2.$$

Then, by Lemma 2.4, we have

$$\|Tx\| \le M\varepsilon\|x\| \le \|x\|, \quad \text{for } x \in K \cap \partial\Omega_{r_2}.$$

Next, by (P3), there exists an $r_3 > r_2 > 0$ such that

$$\int_o^{\omega} |f(s,\phi_s)|\mathrm{d}s \ge \eta_1\|\phi\| \quad \text{and} \quad \sum_{j=1}^{p}|E_j(\phi)| \ge \eta_2\|\phi\|, \quad \text{for } \phi \in K,\ \|\phi\| \ge r_3,$$

where $\eta_i, i = 1, 2$ are chosen so that $2m\eta_i > 1$. Take $\eta = \min\{2\eta_1, 2\eta_2\}$; then

$$\int_o^{\omega} |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(\phi)| \ge \eta\|\phi\|.$$

It follows from Lemma 2.3 that

$$\|Tx\| \ge m\eta\|x\| > \|x\|, \quad \text{for } x \in K \cap \partial\Omega_{r_3}.$$

Thus, by Theorem A, we know that (1) has a positive ω-periodic solution. ■

In order to obtain more results, we introduce two extra assumptions in the following:

(A1) There exists $d_1 > 0$ such that

$$\int_0^{\omega} |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(\phi(\tau_{m_j}))| > \frac{d_1}{m}, \quad \text{for } \sigma d_1 \le \|\phi\| \le d_1,$$

(A2) There exists $d_2 > 0$ such that

$$\int_0^{\omega} |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(\phi(\tau_{m_j}))| < \frac{d_2}{M}, \quad \text{for } \|\phi\| \le d_2,$$

where $\sigma,\ m,\ M$ are defined in (7), Lemma 2.3 and Lemma 2.4, respectively.

Theorem 3.3. *If* (A1) *and* (A2) *hold, then* (1) *has at least one positive ω-periodic solution.*

Proof. Without loss of generality, we may assume that $d_2 < d_1$. If $x \in K \cap \partial\Omega_{d_2}$, then, by (A2), we get

$$\|Tx\| \le M\left[\int_0^{\omega} |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(x)|\right] < M\frac{d_2}{M} = d_2 = \|x\|.$$

In particular, $\|Tx\| < \|x\|$ for all $x \in K \bigcap \partial\Omega_{d_2}$.

On the other hand, by (A1), one has

$$\|Tx\| \ge m\left[\int_0^{\omega} |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(x)|\right] > m\frac{d_1}{m} = d_1 = \|x\|,$$

which produces $\|Tx\| > \|x\|$ for all $x \in K \bigcap \partial\Omega_{d_1}$. Therefore, by Theorem A, we obtain the conclusion, and this completes the proof. ■

Theorem 3.4. *If*

$$\text{(P5)} \quad f^0 = \alpha_1 \in \left[0, \frac{1}{2M}\right), \qquad E^0 = \alpha_2 \in \left[0, \frac{1}{2M}\right),$$

and

$$\text{(P6)} \quad f_\infty = \beta_1 \in \left(\frac{1}{2m\sigma}, \infty\right), \qquad E_\infty = \beta_2 \in \left(\frac{1}{2m\sigma}, \infty\right)$$

hold, then (1) *has at least one positive ω-periodic solution.*

Proof. By (P5), for any $\varepsilon > 0$ there exists a sufficiently small $d_2 > 0$ such that

$$\sup_{x\in K}\frac{\int_0^{\omega}|f(s,x_s)|\mathrm{d}s}{\|x\|} < \alpha_1 + \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{x\in K}\frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|} < \alpha_2 + \frac{\varepsilon}{2}, \quad \text{for } \|x\| \le d_2.$$

Choose $\alpha = \max\{2\alpha_1, 2\alpha_2\}$ and $\varepsilon = \frac{1}{M} - \alpha > 0$. Then

$$\sup_{x\in K}\left[\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|} + \frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|}\right] < \alpha + \varepsilon = \frac{1}{M};$$

that is,

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(x)| < \frac{\|x\|}{M} \le \frac{d_2}{M}, \quad \text{for } \|x\| \le d_2.$$

So, (A2) is satisfied.

By (P6), there exists a sufficiently large $d_1 > 0$ such that

$$\inf_{x\in K}\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|} > \beta_1 - \frac{\varepsilon}{2} \quad \text{and} \quad \inf_{x\in K}\frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|} > \beta_2 - \frac{\varepsilon}{2}, \quad \text{for } \|x\| \ge \sigma d_1.$$

Choose $\beta = \min\{2\beta_1, 2\beta_2\}$ and $\varepsilon = \beta - \frac{1}{m\sigma} > 0$. Then

$$\inf_{x\in K}\left[\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|} + \frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|}\right] > \beta - \varepsilon = \frac{1}{m\sigma};$$

that is,

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(x)| > \frac{\|x\|}{m\sigma} \ge \frac{\sigma d_1}{m\sigma} = \frac{d_1}{m}.$$

Therefore, (A1) holds. Now, the assertion follows from Theorem 3.3. ■

Theorem 3.5. *If*

$$\text{(P7)} \quad f_0 = \alpha_3 \in \left(\frac{1}{2m\sigma}, \infty\right), \qquad E_0 = \alpha_4 \in \left(\frac{1}{2m\sigma}, \infty\right),$$

and

$$\text{(P8)} \quad 0 \le f^\infty = \beta_3 < \frac{1}{2M}, \qquad 0 \le E^\infty = \beta_4 < \frac{1}{2M}$$

hold, then (1) has at least one positive ω-periodic solution.

Proof. By (P7), for any $\varepsilon > 0$ there exists a sufficiently small $d_1 > 0$ such that

$$\inf_{x\in K}\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|} > \alpha_3 - \frac{\varepsilon}{2} \quad \text{and} \quad \inf_{x\in K}\frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|} > \alpha_4 - \frac{\varepsilon}{2}, \quad \text{for } 0 < \|x\| \le d_1.$$

Choose $\alpha = \min\{2\alpha_3, 2\alpha_4\}$ and $\varepsilon = \alpha - \frac{1}{m\sigma} > 0$. Then

$$\inf_{x\in K}\left[\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|}\|x\| + \frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|}\right] > \alpha - \varepsilon = \frac{1}{m\sigma}, \quad \text{for } 0 < \|x\| \le d_1;$$

that is,

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p}|E_j(x)| > \frac{\sigma d_1}{m\sigma} = \frac{d_1}{m}, \quad \text{for } \sigma d_1 \le \|x\| \le d_1,$$

which satisfies (A1).

Again, by (P8), there exists a sufficiently large d such that

$$\sup_{x\in K}\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|}<\beta_3+\frac{\varepsilon}{2}\quad\text{and}\quad\sup_{x\in K}\frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|}<\beta_4+\frac{\varepsilon}{2},\quad\text{for }\|x\|>d.$$

Choose $\beta=\max\{2\beta_3,2\beta_4\}$ and $\varepsilon=\frac{1}{M}-\beta>0$. Then

$$\sup_{x\in K}\left[\frac{\int_0^\omega |f(s,x_s)|\mathrm{d}s}{\|x\|}+\frac{\sum_{j=1}^{p}|E_j(x)|}{\|x\|}\right]<\beta+\varepsilon,\quad\text{for }\|x\|>d.$$

In the following, we consider two cases to prove (A2) to be satisfied:

(i) $\sup_{x\in K}\left[\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|\right]<\infty$,
(ii) $\sup_{x\in K}\left[\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|\right]=\infty$.

The bounded case is clear. If case (ii) is valid, then there exists $y\in BC(\mathbb{R},\mathbb{R}_+^n)$, $\|y\|=d_2>d$ such that

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|\le\int_0^\omega |f(s,y_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(y)|,\quad\text{for }0<\|x\|\le\|y\|=d_2.$$

Since $\|y\|=d_2>d$, then we have

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|\le\int_0^\omega |f(s,y_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(y)|<\frac{\|y\|}{M}=\frac{d_2}{M},\quad\text{for }0<\|x\|\le d_2,$$

which implies that condition (A2) holds. Therefore, by Theorem 3.3 we complete the proof. ■

Theorem 3.6. *If* (P1) *and* (P8) *hold, then* (1) *has at least one positive* ω*-periodic solution.*

Proof. From (P1) and the proof of Theorem 3.1, we know that $\|Tx\|\ge\|x\|$ for all $x\in K\cap\partial\Omega_{r_0}$.

Furthermore, from (P8) and the proof of Theorem 3.5, we know that

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|<\frac{r_1}{M},\quad\text{for }\|x\|\le r_1,$$

and

$$\|Tx\|\le M\left[\int_0^\omega |f(s,x_s)|\mathrm{d}s+\sum_{j=1}^{p}|E_j(x)|\right]<M\frac{r_1}{M}=r_1=\|x\|,$$

which implies that $\|Tx\|<\|x\|$ for all $x\in K\cap\partial\Omega_{r_1}$. This completes the proof. ■

Similarly to Theorem 3.6, one immediately has the following consequences.

Theorem 3.7. *If* (P3) *and* (P5) *hold, then* (1) *has at least one positive* ω*-periodic solution.*

Theorem 3.8. *If* (P2) *and* (P7) *hold, then* (1) *has at least one positive* ω*-periodic solution.*

Theorem 3.9. *If* (P4) *and* (P6) *hold, then* (1) *has at least one positive* ω*-periodic solution.*

Summarizing the above results, one can easily obtain the following result.

Corollary 3.1. *If one of the following pairs*

(P1) *and* (P2); (P3) *and* (P4); (P5) *and* (P6); (P7) *and* (P8); (P1) *and* (P8);
(P3) *and* (P5); (P2) *and* (P7); (P4) *and* (P6); (A1) *and* (A2)

is valid, then system (1) *has at least one positive* ω*-periodic solution.*

4. Existence of multiple periodic solutions

Theorem 4.1. *If* (P2), (P4) *and* (A1) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Proof. By (P4), for any $\varepsilon_1,\ \varepsilon_2 > 0$ there exists $r_4 < d_1$ such that

$$\int_0^\omega |f(s,\phi_s)|\mathrm{d}s \le \varepsilon_1\|\phi\|, \quad \text{and} \quad \sum_{j=1}^{p} |E_j(\phi)| \le \varepsilon_2\|\phi\|, \quad \text{for } \phi \in K,\ 0 < \|\phi\| \le r_4.$$

Choose $\varepsilon = \max\{2\varepsilon_1, 2\varepsilon_2\}$ and $0 < \varepsilon \le \frac{1}{M}$. Then

$$\int_0^\omega |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(\phi)| \le \varepsilon\|\phi\|.$$

Therefore, by Lemma 2.4, we obtain

$$\|Tx\| \le M\varepsilon\|x\| \le \|x\|, \quad \text{for } x \in K \cap \partial\Omega_{r_4}.$$

Moreover, from (P3), there exists an $N_2 > d_1$ such that

$$\int_0^\omega |f(s,\phi_s)|\mathrm{d}s \le \varepsilon_1\|\phi\| \quad \text{and} \quad \sum_{j=1}^{p} |E_j(\phi)| \le \varepsilon_2\|\phi\|, \quad \text{for } \|\phi\| \ge N_2.$$

Choose $\varepsilon = \max\{2\varepsilon_1, 2\varepsilon_2\}$ and $0 < \varepsilon \le \frac{1}{2M}$. Then

$$\int_0^\omega |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(\phi)| \le \varepsilon\|\phi\|, \quad \text{for } \|\phi\| \ge N_2.$$

Take

$$r_5 > N_2 + 1 + 2M \sup_{\substack{\|\phi\|<N_2\\ \phi\in K}} \left[\int_0^\omega |f(s,\phi_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(\phi)|\right].$$

If $x \in K \cap \partial\Omega_{r_5}$, then

$$\begin{aligned}\|Tx\| &\le M\left[\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(x)|\right]\\ &= M[\rho(I_1)+\rho(I_2)] \le \frac{r_5}{2} + \frac{\|x\|}{2} = \|x\|,\end{aligned}$$

where $\rho(I_i) = \left[\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(x)|\right]|_{x\in I_i}$, and $I_1 = \{x \in K, \|x\| < N_2\}$, $I_2 = \{x \in K, \|x\| \ge N_2\}$, which shows that $\|Tx\| \le \|x\|$ for all $x \in K \cap \partial\Omega_{r_5}$.

Denote $\Omega_{d_1} = \{x \in X : \|x\| < d_1\}$. Then, by (A1), for any $x \in K \cap \partial\Omega_{d_1}$, we have

$$\|Tx\| \ge m\left[\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(x)|\right] > m\frac{d_1}{m} = d_1 = \|x\|,$$

which yields $\|Tx\| > \|x\|$ for all $x \in K \cap \partial\Omega_{d_1}$. By Theorem A, T has a fixed point x_1 in $K \cap (\overline{\Omega}_{d_1} \setminus \Omega_{r_4})$ and has a fixed point x_2 in $K \cap (\overline{\Omega}_{r_5} \setminus \Omega_{d_1})$. Moreover, from the above, we know that $r_5 > N_2 > d_1 > r_4$. Therefore, system (1) has two positive ω-periodic solutions satisfying $0 < \|x_1\| < d_1 < \|x_2\|$. This completes the proof. ■

Then next consequence is presented below; the proof parallels that of Theorem 4.1, and is therefore omitted for reasons of space.

Theorem 4.2. *If* (P1), (P3) *and* (A2) *hold, then* (1) *has at least two ω-periodic solutions.*

Theorem 4.3. *If* (P6), (P7) *and* (A2) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Proof. From (P6) and the proof of Theorem 3.4, it follows that there exists a sufficiently large $d_1 > d_2$, such that

$$\int_0^\omega |f(s,x_s)|\mathrm{d}s + \sum_{j=1}^{p} |E_j(x)| > \frac{d_1}{m}, \quad \text{for } \sigma d_1 \le \|x\| \le d_1.$$

That is, (A1) is valid. So, $\|Tx\| > \|x\|$ for all $x \in K \cap \partial\Omega_{d_1}$.

From (P7) and the proof of Theorem 3.5, one can find a sufficiently small $d_1^* \in (0, d_2)$ such that

$$\int_0^{\omega} |f(s, x_s)| \mathrm{d}s + \sum_{j=1}^{p} |E_j(x)| > \frac{d_1^*}{m}, \quad \text{for } \sigma d_1^* \le \|x\| \le d_1^*,$$

which satisfies (A1). So we have $\|Tx\| > \|x\|$ for all $x \in K \cap \partial\Omega_{d_1^*}$.

Incorporating (A2), we know that T has a fixed point x_1 in $K \cap (\overline{\Omega}_{d_2} \setminus \Omega_{d_1^*})$ and has a fixed point x_2 in $K \cap (\overline{\Omega}_{d_1} \setminus \Omega_{d_2})$. That is, system (1) has two positive ω-periodic solutions satisfying $d_1^* < \|x_1\| < d_2 < \|x_2\| < d_1$. ■

From the arguments in the above proof, we have the following consequence.

Theorem 4.4. *If* (P5), (P8) *and* (A1) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Theorem 4.5. *If* (P1), (P6) *and* (A2) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Proof. Let $\Omega_{r_*} = \{x \in X : \|x\| < r_*\}$, where $r_* < d_2$. By assumption (P1) and the proof of Theorem 3.1, we know that $\|Tx\| \ge \|x\|$ for all $x \in K \cap \partial\Omega_{r_*}$.

Let $\Omega_{d_1} = \{x \in X : \|x\| < d_1\}$. By assumption (P6) and the proof of Theorem 3.4, we can see that $\int_0^{\omega} |f(s, x_s)| \mathrm{d}s + \sum_{j=1}^{p} |E_j(x)| > \frac{d_1}{m}$ for $\sigma d_1 \le \|x\| \le d_1$. Incorporating (A2) and the proof of Theorem 3.3, we know that there exist two positive ω-periodic solutions. ■

The following statements are immediately obtained by applying similar arguments as used in the proof of Theorem 4.5.

Theorem 4.6. *If* (P3), (P7) *and* (A2) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Theorem 4.7. *If* (P2), (P5) *and* (A1) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Theorem 4.8. *If* (P4), (P8) *and* (A1) *hold, then* (1) *has at least two positive ω-periodic solutions.*

Corollary 4.1. *If one of the following pairs*

$$\begin{array}{llll} (\text{P1}),\ (\text{P3}) \textit{ and } (\text{A2}); & (\text{P1}),\ (\text{P6}) \textit{ and } (\text{A2}); & (\text{P3}),\ (\text{P7}) \textit{ and } (\text{A2}); & (\text{P2}),\ (\text{P4}) \textit{ and } (\text{A1}); \\ (\text{P2}),\ (\text{P5}) \textit{ and } (\text{A1}); & (\text{P4}),\ (\text{P8}) \textit{ and } (\text{A1}); & (\text{P5}),\ (\text{P8}) \textit{ and } (\text{A1}); & (\text{P6}),\ (\text{P7}) \textit{ and } (\text{A2}) \end{array}$$

is valid, then system (1) *has at least two positive ω-periodic solution.*

5. Conclusions

In this paper, by using the famous Krasnoselskii fixed point theorem, we have investigated the existence and multiplicity of positive periodic solutions for n-dimensional functional differential equations with impulses and have obtained some easily verifiable sufficient criteria which extend previous results. The methodology which we employed in studying the functional differential equations without impulses in [9] can be modified to establish similar sufficient criteria for impulsive functional differential equations. It is worth mentioning that there are still many problems that remain open in this vital field except for the results obtained in this paper: for example, whether or not the combination of (P1) $f_0 = E_0 = \infty$ and (P4) $f^0 = E^0 = 0$ can ensure the existence of a periodic solution, and whether or not our concise criteria can guarantee the stability of positive periodic solutions. More efforts are still needed in the future.

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