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Hyperbolic Trigonometry in the Einstein Relativistic Velocity Model of Hyperbolic Geometry

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Abstract—Hyperbolic geometry is a fundamental aspect of modern physics. We explore in this paper the use of Einstein's velocity addition as a model of vector addition in hyperbolic geometry. Guided by analogies with ordinary vector addition, we develop hyperbolic vector spaces, called gyrovector spaces, which provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry. The resulting gyrovector spaces enable Euclidean trigonometry to be extended to hyperbolic trigonometry. In particular, we present the hyperbolic law of cosines and sines and the Hyperbolic Pythagorean Theorem emerges when the common vector addition is replaced by the Einstein velocity addition. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Vector addition in abstract spaces is a commutative and associative binary operation in a group of vectors, which supports scalar multiplication giving rise to vector spaces. These, in turn, provide the setting for Euclidean geometry. In full analogy, Einstein's addition of relativistically admissible velocities is abstracted into a *gyrocommutative* and *gyroassociative* binary operation in a *gyrogroup* of *gyrovectors*, which supports scalar multiplication giving rise to gyrovector spaces. These, in turn, provide the setting for hyperbolic geometry.

The power and elegance of the capability of gyrovector spaces to unify Euclidean and hyperbolic geometry is demonstrated in this article by exposing the truly hyperbolic law of cosines and sines as well as the truly Pythagorean Theorem in the Einstein gyrovector space model of hyperbolic geometry. In two dimensions, this model of hyperbolic geometry is coincident with the Beltrami (or, Klein) disc model, which is a the well-known model of hyperbolic geometry [1].

Following the presentation of brief history, motivation, and definition of the gyrogroup notion, we will present Einstein's addition as a gyrogroup operation in an abstract gyrocommutative gyrogroup that is turned into a gyrovector space by the incorporation of scalar multiplication. We will show that analogies that Einstein's addition shares with vector addition enables hyperbolic

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and Euclidean geometry to be unified. The unified theory allows, in particular, the introduction of hyperbolic trigonometry that shares remarkable analogies with Euclidean trigonometry.

2. GYROGROUPS AND K-LOOPS

Since coined in 1989 [2], the term *K-loop* is becoming increasingly popular, as evidenced from the number of articles in which it appears in their titles.

The author realized in 1989 [2] that the structure of Einstein's velocity addition that he exposed in 1988 [3] provides the first concrete example of a mathematical structure that illuminates part of a structure studied by Karzel since the 1960s with no known specimen. Therefore, the author named the structure a *K-loop* after Karzel [2]. Later, however, the author realized that the mathematical structure that he uncovered in 1988 [3] from the soil of relativity physics illustrates Kikkawa's important work as well. Therefore, the author dedicated in 1998 [4, fn. 16] the term *K-loop* that he coined in 1989 [2] to both Karzel and Kikkawa.

In subsequent development of K-loops, the author has shown that the K-loop structure can be further generalized into a structure governed by analogies with groups and vector spaces, for which terminology that has evolved and tested by generations of mathematicians already exists. Noting that all the analogies stem from the abstraction of the relativistic effect known as the Thomas precession, the author called the abstract Thomas precession a *Thomas gyration* suggesting the use of *"gyro"* as a prefix that emphasizes analogies shared with classical notations. Thus, for instance, groups are generalized into gyrogroups and vector spaces are generalized into gyrovector spaces. In particular, we have the following.

- The associative group operation becomes a gyroassociative gyrogroup operation, and gyrogroups are classified into gyrocommutative and nongyrocommutative gyrogroups in full analogy with groups, which are classified into commutative and noncommutative groups. K-loops are the same as gyrocommutative gyrogroups. Nongyrocommutative gyrogroups have been discovered in [5,6] long after the term K-loop had been coined in 1989 [2].
- Some commutative groups allow the introduction of scalar multiplication, turning them into vector spaces. Similarly, some gyrocommutative gyrogroups allow the introduction of scalar multiplication, turning them into gyrovector spaces. The latter, in turn, form the setting for non-Euclidean geometry in the same way that vector spaces form the setting for Euclidean geometry.

Historically, the first gyrogroup structure was discovered in the study of Einstein's velocity addition [3]. However, the best way to introduce the gyrogroup notion is provided by the study of the Möbius transformation group of the complex unit disc.

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z : |z| < 1\} \tag{2.1}$$

in the complex z-plane [7],

$$z \mapsto e^{i\theta}\frac{a+z}{1+\overline{a}z} = e^{i\theta}(a \oplus z) \tag{2.2}$$

induces the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus z = \frac{a+z}{1+\overline{a}z} \tag{2.3}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a. Möbius addition $\oplus$ is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$\text{gyr} : \mathbb{D} \times \mathbb{D} \to \text{Aut}(\mathbb{D}, \oplus) \tag{2.4}$$

given by the equation

$$\mathrm{gyr}[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}, \tag{2.5}$$

where $\mathrm{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. We recall that a groupoid $(G, +)$ is a nonempty set G with a binary operation $+$, and an automorphism of a groupoid $(G, +)$ is a bijective self-map of the groupoid (G) which respects its binary operation $+$. The set of all automorphisms of a groupoid $(G, +)$ forms a group, denoted $\mathrm{Aut}(G, +)$, under bijection composition.

The *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyr in (2.5),

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a). \tag{2.6}$$

Surprisingly, the gyration $\mathrm{gyr}[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2.6) repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the *left and right gyroassociative laws*

$$\begin{aligned} a \oplus (b \oplus z) &= (a \oplus b) \oplus \mathrm{gyr}[a,b]z, \\ (a \oplus b) \oplus z &= a \oplus (b \oplus \mathrm{gyr}[b,a]z), \end{aligned} \tag{2.7}$$

for all $a, b, z \in \mathbb{D}$.

Guided by analogies with groups, we take the (gyrocommutative and the) gyroassociative laws and other key features of the Möbius addition as a model of a (gyrocommutative) gyrogroup, obtaining the following.

Definition 2.1. Gyrogroups. *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms and properties. In G, there exists a unique element, 0, called the identity, satisfying*

(g1)	$0 \oplus a = a \oplus 0 = a$	*Identity*

for all $a \in G$. For each a in G, there exists a unique inverse $\ominus a$ in G, satisfying

(g2)	$\ominus a \oplus a = a \ominus a = 0$	*Inverse*

where we use the notation $a \ominus b = a \oplus (\ominus b)$, $a, b \in G$. Moreover, if for any $a, b \in G$ the self-map $\mathrm{gyr}[a,b]$ of G is given by the equation

$$\mathrm{gyr}[a,b]z = -(a \oplus b) \oplus (a \oplus (b \oplus z)), \tag{2.8}$$

for all $z \in G$, then the following hold for all $a, b, c \in G$:

(g3)	$\mathrm{gyr}[a,b] \in \mathrm{Aut}(G, \oplus)$	*Gyroautomorphism Property*
(g4a)	$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c$	*Left Gyroassociative Law*
(g4b)	$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b,a]c)$	*Right Gyroassociative Law*
(g5a)	$\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$	*Left Loop Property*
(g5b)	$\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b \oplus a]$	*Right Loop Property*
(g6)	$\ominus (a \oplus b) = \mathrm{gyr}[a,b](\ominus b \ominus a)$	*Gyrosum Inversion Law*
(g7)	$\mathrm{gyr}^{-1}[a,b] = \mathrm{gyr}[b,a]$	*Gyroautomorphism Inversion.*

A gyrogroup is gyrocommutative if it satisfies

(g8)	$a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$	*Gyrocommutative Law.*

Definition 2.2. The Gyrogroup Dual Operations. *Let $(G, \oplus)$ be a gyrogroup. A secondary binary operation $\boxplus$ in G is defined by the equation*

(g9)	$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b$	*Secondary Operation.*

The primary operation $\oplus$ and the secondary operation $\boxplus$ of a gyrogroup $(G, \oplus)$ are called the dual operations of the gyrogroup. The secondary operation is also called the dual operation.

Theorem 2.3. *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if*

(g10)	$\ominus(a \oplus b) = \ominus a \ominus b$	*Automorphic Inverse.*

THEOREM 2.4. *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if its associated dual groupoid $(G, \oplus)$ is commutative,*

(g11) $a \boxplus b = b \boxplus a$ *Commutative Dual Operation.*

Gyrogroup theory is presented in [8]. The dual binary operations in a gyrogroup expose useful duality symmetries in gyrogroups and in gyrovector spaces, as well as in hyperbolic geometry. In particular, they give rise to the *left cancellation law*

$$a \oplus (\ominus a \oplus b) = b \tag{2.9}$$

and to the two *dual right cancellation laws*

$$\begin{aligned} (b \ominus a) \boxplus a &= a, \\ (b \boxminus a) \oplus a &= a. \end{aligned} \tag{2.10}$$

The left and the right cancellation laws indicate that in order to capture analogies that gyrogroups share with group, the two dual binary operations in a gyrogroup are needed.

3. AN EXAMPLE OF A FINITE GYROGROUP

Gyrogroups abound in group theory, where they laid dormant till the discovery of the relativity gyrocommutative gyrogroup in 1988 [3], whose gyrogroup operation is given by Einstein's addition. In order to demonstrate that there are plenty of gyrogroups around us, we present in this section the multiplication table of a nongyrocommutative gyrogroup of order 16, that was generated by the software package MAGMA and its library [9], using a method developed in [10]. Other finite gyrogroups can be generated in various ways. We denote this gyrogroup of order 16 by K_{16}, and its elements k_i are denoted by their subscripts i, $1 \le i \le 16$.

The gyroautomorphisms of K_{16} are calculated by means of (2.8), and are shown in Table 2. There is only one nonidentity gyroautomorphism, A, whose transformation table is given in (3.1) below:

$$\begin{array}{llll} 1 \to 1 & 5 \to 5 & 9 \to 10 & 13 \to 14 \\ 2 \to 2 & 6 \to 6 & 10 \to 9 & 14 \to 13 \\ 3 \to 3 & 7 \to 7 & 11 \to 12 & 15 \to 16 \\ 4 \to 4 & 8 \to 8 & 12 \to 11 & 16 \to 15. \end{array} \tag{3.1}$$

The gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in K_{16}$ is either A or the identity automorphism I. The gyroautomorphism table for $\mathrm{gyr}[a, b]$ is presented in Table 2.

The set $\{I, A\}$ of all gyroautomorphisms of K_{16} forms a group of order 2. In general, however, the set of all gyroautomorphisms of a gyrogroup need not form a group. Thus, for instance, the gyroautomorphisms of the Einstein two-dimensional gyrogroup $(\mathbb{R}^2_c, \oplus)$ are rotations of the Euclidean plane $\mathbb{R}^2$ about its origin, but there is no gyroautomorphism that rotates the plane about its origin by π radians as we show graphically in [11].

As an example illustrating the use of Tables 1 and 2 and the transformation rule (3.1) of the gyroautomorphism A of K_{16}, let us corroborate the left gyroassociative law

$$a \odot (b \odot c) = (a \odot b) \odot \mathrm{gyr}[a, b]c,$$

for the special case when $a = 6$, $b = 12$, and $c = 9$ in K_{16}. On one hand,

$$a \odot (b \odot c) = 6 \odot (12 \odot 9) = 6 \odot 3 = 8,$$

and on the other

$$\begin{aligned} (a \odot b) \odot \mathrm{gyr}[a, b]c &= (6 \odot 12) \odot \mathrm{gyr}[6, 12]9 \\ &= 13 \odot A(9) \\ &= 13 \odot 10 \\ &= 8 \end{aligned}$$

as expected. In the same way, readers may corroborate any gyrogroup identity, plenty of which have been studied in the literature on gyrogroups; see [2–6,8,10–36].

Table 1. Multiplication table of the nongyrocommutative gyrogroup K_{16} of order 16. The elements of the upper-left 8×8 corner form a subgroup. Accordingly, the entries of that corner in the gyration table of K_{16} are I.

$\odot$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	2	1	7	8	6	5	12	11	9	10	16	15	13	14
4	4	3	1	2	8	7	5	6	11	12	10	9	15	16	14	13
5	5	6	7	8	4	3	1	2	16	15	13	14	10	9	12	11
6	6	5	8	7	3	4	2	1	15	16	14	13	9	10	11	12
7	7	8	6	5	1	2	3	4	14	13	16	15	11	12	10	9
8	8	7	5	6	2	1	4	3	13	14	15	16	12	11	9	10
9	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	11	12	10	9	15	16	14	13	4	3	1	2	8	7	5	6
12	12	11	9	10	16	15	13	14	3	4	2	1	7	8	6	5
13	13	14	15	16	12	11	9	10	7	8	6	5	1	2	3	4
14	14	13	16	15	11	12	10	9	8	7	5	6	2	1	4	3
15	15	16	14	13	9	10	11	12	5	6	7	8	4	3	1	2
16	16	15	13	14	10	9	12	11	6	5	8	7	3	4	2	1

Table 2. Gyration Table of K_{16}. In K_{16}, there are two gyroautomorphisms. These are (i) the identity automorphism I of K_{16}, and (ii) the automorphism A of K_{16} whose transformation table is given in (3.1).

gyr	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
2	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
3	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
4	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
5	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
6	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
7	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
8	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
9	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
10	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
11	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
12	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
13	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
14	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
15	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
16	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I

4. THE ABSTRACT EINSTEIN ADDITION AND SCALAR MULTIPLICATION

Einstein's velocity addition $\oplus$ of relativistically admissible velocities turns out to be a gyrocommutative gyrogroup operation in the c-ball $\mathbb{R}^3_c$,

$$\mathbb{R}^3_c = \left\{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\right\} \tag{4.1}$$

of the Euclidean 3-space $\mathbb{R}^3$, giving rise to the relativity gyrocommutative gyrogroup $(\mathbb{R}^3_c, \oplus)$. It is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + (\mathbf{u} \cdot \mathbf{v})/c^2} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}, \tag{4.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, where $\gamma_{\mathbf{u}}$ is the *Lorentz factor* given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \|\mathbf{u}\|^2/c^2}}. \tag{4.3}$$

Einstein's addition $\oplus$ is thus a binary operation in the open c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$. Texts on relativity present Einstein's addition only for parallel velocities, for the sake of simplicity. Two outstanding exceptions to this are the books by Fock [37], and by Sexl and Urbantke [38], where the Einstein velocity addition law is presented for the general case in which velocities need not be collinear, as in (4.2). Unlike its simple special case, when all velocities are parallel, in its full generality, the Einstein relativistic velocity addition $\oplus$ is neither commutative nor associative.

The groupoid of classical velocities $(\mathbb{R}^3, +)$ under ordinary vector addition forms a commutative group as opposed to Einstein groupoid of relativistically admissible velocities $(\mathbb{R}^3_c, \oplus)$ which, under Einstein's velocity addition, does not form a group. Is the breakdown of associativity in Einstein's velocity addition associated with loss of mathematical regularity? The answer is "no": in full analogy with the commutative and associative vector addition in the group $(\mathbb{R}^3, +)$ of classical velocities, Einstein's addition is a gyrocommutative and gyroassociative gyrovector addition in the gyrogroup $(\mathbb{R}^3_c, \oplus)$ of relativistically admissible velocities. Furthermore, like vector addition, Einstein's addition remains valid in any real inner product space. We, therefore, extend the Einstein addition (4.2) by abstraction, and consider it as a binary operation $\oplus$ in the open c-ball $\mathbb{V}_c$,

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V}_\infty : \|\mathbf{v}\| < c\} \tag{4.4}$$

of an abstract real inner product space $\mathbb{V}_\infty$.

Being a gyrogroup operation in the relativity gyrocommutative gyrogroup $(\mathbb{V}_c, \oplus)$, Einstein's addition $\oplus$ has a dual addition $\boxplus$, Definition 2.2, which turns out to be a commutative binary operation in the c-ball $\mathbb{V}_c$, given by the equation

$$\mathbf{u} \boxplus \mathbf{v} = 2 \otimes \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}, \tag{4.5}$$

where Einstein's scalar multiplication $\otimes$ by 2 is defined by $2 \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v}$. The self-map $2\otimes$ of the c-ball $\mathbb{V}_c$ is bijective, the inverse of which is $(1/2)\otimes$ given by

$$\frac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} \tag{4.6}$$

so that, indeed,

$$2 \otimes \left(\frac{1}{2} \otimes \mathbf{v}\right) = 2 \otimes \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} \oplus \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}\mathbf{v} = \mathbf{v} \tag{4.7}$$

as it should be.

The Einstein scalar multiplications $2\otimes$ and $(1/2)\otimes$ are special cases of the general Einstein scalar multiplication, given by the equation

$$\begin{aligned} r \otimes \mathbf{v} &= c\frac{(1 + \|\mathbf{v}\|/c)^r - (1 - \|\mathbf{v}\|/c)^r}{(1 + \|\mathbf{v}\|/c)^r + (1 - \|\mathbf{v}\|/c)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c \tanh\left(r \tanh^{-1} \frac{\|\mathbf{v}\|}{c^2}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{4.8}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$, and $r \otimes \mathbf{0} = \mathbf{0}$. We use the notation $r \otimes \mathbf{v} = \mathbf{v} \otimes r$.

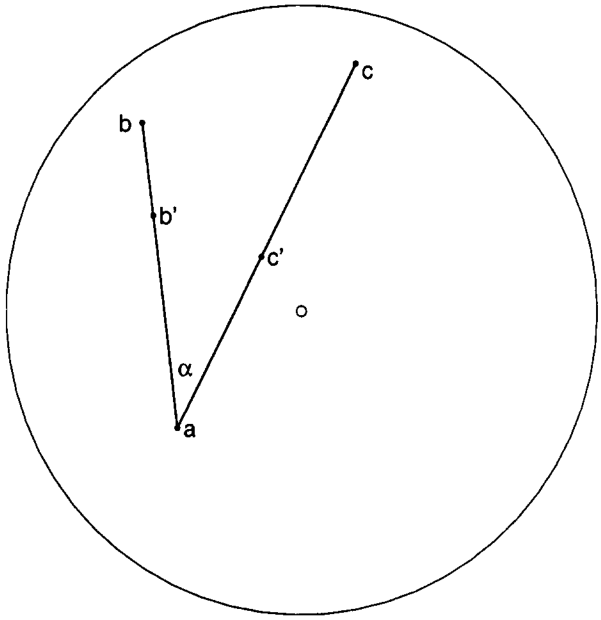

Figure 1. The hyperbolic angle α, (5.1), between two geodesic rays that emanate from a point $\mathbf{a}$ in a gyrovector space and contain, respectively, points $\mathbf{b}$ and $\mathbf{c}$, is independent of the choice of the points $\mathbf{b}$ and $\mathbf{c}$ of the geodesic rays.

The Einstein scalar multiplication possesses the following properties. For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_c$,

$$
\begin{aligned}
n \otimes \mathbf{v} &= \mathbf{v} \oplus \cdots \oplus \mathbf{v} && n \text{ terms} \\
(r_1 + r_2) \otimes \mathbf{v} &= r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v} && \text{Additive Scalar Law} \\
r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) && \text{Special Gyrodistributive Law} \\
(r_1 r_2) \otimes \mathbf{v} &= r_1 \otimes (r_2 \otimes \mathbf{v}) && \text{Multiplicative Scalar Law} \\
\|r \otimes \mathbf{v}\| &= |r| \otimes \|\mathbf{v}\| && \text{Scalar-Norm Product Law} \\
\frac{|r| \otimes \mathbf{v}}{\|r \otimes \mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} && \text{Scaling Property.}
\end{aligned}
\tag{4.9}
$$

5. EINSTEIN'S ADDITION AND THE HYPERBOLIC ANGLE

A vector in physics is determined by its length and orientation relative to other vectors. By analogy, we wish that also a geometric gyrovector be determined by its length and orientation relative to other gyrovectors. Being guided by analogies, to accomplish this task we have to define angles. We define the cosine of the angle α between the two geometric gyrovectors $\ominus \mathbf{a} \oplus \mathbf{b}$ and $\ominus \mathbf{a} \oplus \mathbf{c}$ that emanate from a common point $\mathbf{a}$, by the inner product of corresponding unit gyrovectors,

$$
\cos \alpha = \frac{\ominus \mathbf{a} \oplus \mathbf{b}}{\| \ominus \mathbf{a} \oplus \mathbf{b}\|} \cdot \frac{\ominus \mathbf{a} \oplus \mathbf{c}}{\| \ominus \mathbf{a} \oplus \mathbf{c}\|}. \tag{5.1}
$$

Equation (5.1) determines the angle between two rays $L_{\mathbf{ab}}$ and $L_{\mathbf{ac}}$ which emanate from a common point $\mathbf{a}$, and which contain, respectively, the points $\mathbf{b}$ and $\mathbf{c}$, as shown in Figure 1. The angle that $\cos \alpha$ determines is either $\pm\alpha$ or $\pi \pm \alpha$, $0 \le \alpha \le \pi/2$, depending on the orientation of the rays, in full analogy with angles between directed rays in Euclidean geometry.

The sine of the hyperbolic angle α is defined by the equation

$$
\sin \alpha = \pm\sqrt{1 - \cos^2 \alpha}, \tag{5.2}
$$

where the ambiguous sign is determined as in Euclidean geometry.

To justify calling α an angle between intersecting geodesic rays, we have to show that α is a property of the intersecting geodesic rays rather than a property of points on these geodesic rays. This has already been done in [4] for the Poincaré disc model of hyperbolic geometry.

6. THE HYPERBOLIC LAW OF COSINES AND SINES

The present state of hyperbolic trigonometry, described in [39], is plagued by the lack of hyperbolic trigonometric laws of cosines and sines which are fully analogous to their Euclidean counterparts. We show in this section that these can be exposed by gyrovector space theoretic techniques. Specifically, we show in this section that Einstein's addition captures the law of cosines and the law of sines in a form similar to the one that we know from Euclidean trigonometry, as well as the Hyperbolic Pythagorean Theorem itself, and that the Einstein half (4.6) proves useful in this study.

THEOREM 6.1. *Let* $(\mathbb{V}_c, \oplus, \otimes)$ *be an Einstein gyrovector space. Then, for all* $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$,

$$\frac{\|(1/2) \otimes (\mathbf{a} \oplus \mathbf{b})\|^2}{c} = \frac{\|(1/2) \otimes \mathbf{a}\|^2}{c} \oplus \frac{\|(1/2) \otimes \mathbf{b}\|^2}{c} \oplus \frac{1}{2c} \frac{\mathbf{a} \cdot \mathbf{b}}{1 + (\mathbf{a} \cdot \mathbf{b})/(2c^2)}. \tag{6.1}$$

In particular, if $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal, then*

$$\frac{\|(1/2) \otimes (\mathbf{a} \oplus \mathbf{b})\|^2}{c} = \frac{\|(1/2) \otimes \mathbf{a}\|^2}{c} \oplus \frac{\|(1/2) \otimes \mathbf{b}\|^2}{c}. \tag{6.2}$$

PROOF. The proof is by straightforward computer algebra, which can be simplified by employing identities like (4.6). ∎

THEOREM 6.2. POLARIZATION IDENTITY IN EINSTEIN GYROVECTOR SPACES. *Let* $(\mathbb{V}_c, \oplus, \otimes)$ *be an Einstein gyrovector space. Then, for all* $\mathbf{a}, \mathbf{b} \in \mathbb{V}_c$,

$$\frac{\|(1/2) \otimes (\mathbf{a} \oplus \mathbf{b})\|^2}{c} \ominus \frac{\|(1/2) \otimes (\mathbf{a} \ominus \mathbf{b})\|^2}{c} = \frac{\mathbf{a} \cdot \mathbf{b}}{c}. \tag{6.3}$$

PROOF. The proof is by straightforward computer algebra. ∎

In the limit of large c, $c \to \infty$, the polarization identity (6.3) reduces to the standard polarization identity in a real inner product space,

$$\left\| \frac{\mathbf{a} + \mathbf{b}}{2} \right\|^2 - \left\| \frac{\mathbf{a} - \mathbf{b}}{2} \right\|^2 = \mathbf{a} \cdot \mathbf{b}. \tag{6.4}$$

We will now relate the identities in Theorem 6.1 to hyperbolic triangles, Figure 2, thereby obtaining the Hyperbolic Law of Cosines and the Hyperbolic Pythagorean Theorem in Einstein gyrovector spaces. Let $\Delta\mathbf{abc}$ be a triangle in an Einstein gyrovector space $\mathbb{V}_c$ whose vertices are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. The special case of $\mathbb{V}_c = \mathbb{R}_c^2$,

$$\mathbb{R}_c^2 = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < c\} \tag{6.5}$$

is presented graphically in Figure 2.

The sides of the triangle $\Delta\mathbf{abc}$ are formed by the three geometric gyrovectors $A = \ominus\mathbf{c} \oplus \mathbf{b}$, $B = \ominus\mathbf{c} \oplus \mathbf{a}$, and $C = \ominus\mathbf{a} \oplus \mathbf{b}$. By [8], we have the gyrogroup identity

$$(\ominus\mathbf{c} \oplus \mathbf{b}) \ominus (\ominus\mathbf{c} \oplus \mathbf{a}) = \operatorname{gyr}[\ominus\mathbf{c}, \mathbf{b}](\mathbf{b} \ominus \mathbf{a}), \tag{6.6}$$

which, by the gyrocommutative law, can be written as

$$A \ominus B = \operatorname{gyr}[\ominus\mathbf{c}, \mathbf{b}]\operatorname{gyr}[\mathbf{b}, \ominus\mathbf{a}]C. \tag{6.7}$$

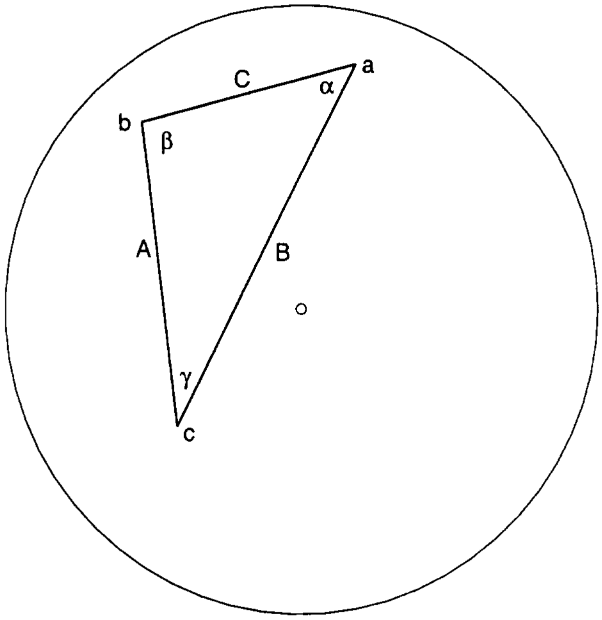

Figure 2. The hyperbolic triangle $\Delta\mathbf{abc}$ in the Einstein disc gyrovector space or, equivalently, in the Beltrami disc model of hyperbolic geometry. Its vertices are the points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (\mathbb{R}_c^2, \oplus, \otimes)$ of the Einstein disc gyrovector space, and its sides, if directed counterclockwise, are $A = \ominus\mathbf{b} \oplus \mathbf{c}$, $B = \ominus\mathbf{c} \oplus \mathbf{a}$, and $C = \ominus\mathbf{a} \oplus \mathbf{b}$. The sum of its three hyperbolic angles is less than π.

Hence, since gyrations are isometries,

$$\|C\|^2 = \|A \ominus B\|^2. \tag{6.8}$$

Noting that $A \ominus B = A \oplus (\ominus B)$, we have by (6.1) and (5.1),

$$\begin{aligned} \frac{\|(1/2) \otimes C\|^2}{c} &= \frac{\|(1/2) \otimes (A \ominus B)\|^2}{c} \\ &= \frac{\|(1/2) \otimes A\|^2}{c} \oplus \frac{\|(1/2) \otimes B\|^2}{c} \ominus \frac{1}{2c} \frac{A \cdot B}{1 - (A \cdot B)/(2c^2)} \\ &= \frac{\|(1/2) \otimes A\|^2}{c} \oplus \frac{\|(1/2) \otimes B\|^2}{c} \ominus \frac{1}{2c} \frac{\|A\|\|B\| \cos\gamma}{1 - (\|A\|\|B\| \cos\gamma)/(2c^2)}, \end{aligned} \tag{6.9}$$

thus obtaining the following.

THEOREM 6.3. HYPERBOLIC LAW OF COSINES IN EINSTEIN GYROVECTOR SPACES. *Let $\Delta\mathbf{abc}$ be a triangle in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ with vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and with sides A, B, and C given by*

$$\begin{aligned} A &= \ominus\mathbf{b} \oplus \mathbf{c}, \\ B &= \ominus\mathbf{c} \oplus \mathbf{a}, \\ C &= \ominus\mathbf{a} \oplus \mathbf{b}, \end{aligned} \tag{6.10}$$

and let γ be the angle between the sides A and B. Then,

$$\frac{\|(1/2) \otimes C\|^2}{c} = \frac{\|(1/2) \otimes A\|^2}{c} \oplus \frac{\|(1/2) \otimes B\|^2}{c} \ominus \frac{1}{2c} \frac{\|A\|\|B\| \cos\gamma}{1 - (\|A\|\|B\| \cos\gamma)/(2c^2)}. \tag{6.11}$$

In the limit of large c, $c \to \infty$, the Einstein Law of Cosines reduces to the standard law of cosines in trigonometry. In Section 7, we will present the Hyperbolic Pythagorean Theorem as a special case of the Hyperbolic Law of Cosines in Theorem 6.3 corresponding to $\gamma = \pi/2$.

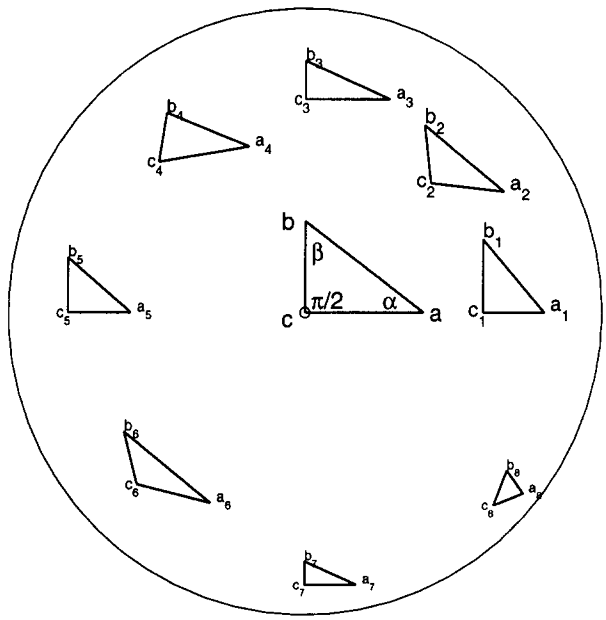

Figure 3. Shown is a hyperbolic right-angled triangle $\Delta\mathbf{abc}$ in the Einstein disc gyrovector spaces $(\mathbb{R}^2_c, \oplus, \otimes)$ with its right angle located at the origin of the disc, and several of its left gyrotranslations in various directions and distances. The hyperbolic right angle $\pi/2$ as well as other hyperbolic angles and hyperbolic lengths seem to Euclidean eyes to be distorted by left gyrotranslations. They are, however, preserved in hyperbolic geometry.

To complete the introduction of trigonometry into Einstein gyrovector spaces, we present without a proof the following theorem [32].

THEOREM 6.4. HYPERBOLIC LAW OF SINES IN EINSTEIN GYROVECTOR SPACES. *Let* $\Delta\mathbf{abc}$ *be a triangle in an Einstein gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with vertices* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *and with sides* A, B, *and* C *given by*

$$\begin{aligned} A &= \ominus\mathbf{b} \oplus \mathbf{c}, \\ B &= \ominus\mathbf{c} \oplus \mathbf{a}, \\ C &= \ominus\mathbf{a} \oplus \mathbf{b}, \end{aligned} \tag{6.12}$$

and let α, β, *and* γ *be the respective angles opposite to these sides. Then,*

$$\frac{\gamma_A \|A\|}{\sin\alpha} = \frac{\gamma_B \|B\|}{\sin\beta} = \frac{\gamma_C \|C\|}{\sin\gamma}. \tag{6.13}$$

The introduction of hyperbolic trigonometry by Theorems 6.3 and 6.4 allows one to solve hyperbolic triangle problems in a way similar to the solution of analogous trigonometric problems in Euclidean trigonometry.

7. THE HYPERBOLIC PYTHAGOREAN THEOREM

A hyperbolic right angle located at the origin of the Beltrami disc model of hyperbolic geometry looks like a Euclidean right angle. This feature is, however, in general distorted to the Euclidean eye when the right angle is left gyrotranslated away from the center of the disc, as shown in Figure 3.

Sometime in the Sixth Century B.C., Pythagoras of Samos discovered the theorem that now bears his name in Euclidean geometry. The extension of the Euclidean Pythagorean Theorem to hyperbolic geometry, which is commonly known as the Hyperbolic Pythagorean Theorem, is restricted to the plane. It asserts that in the hyperbolic plane, the sides of a triangle ABC with angle A equals $\pi/2$ satisfy $\cosh a = \cosh b \cosh c$ (for details, see, for instance, [40]). The Hyperbolic Pythagorean Theorem as we know from the literature prior to [4] is thus

(i) restricted to two dimensions, and
(ii) does not have a form analogous to the Euclidean Pythagorean Theorem.

Following Piel's exploration of the Hyperbolic Pythagorean Theorem [41], and following Calapso's several attempts to give the Hyperbolic Pythagorean Theorem a form analogous to its Euclidean plane counterpart [42–44], it seemed that a truly Hyperbolic Pythagorean Theorem does not exist. Thus, for instance, Wallace and West assert [45] that "the Pythagorean Theorem is strictly Euclidean" since "in the hyperbolic model the Pythagorean Theorem is not valid!"

We realize, however, that the hyperbolic law of cosines (6.11) includes as a special case, a natural formulation of the Hyperbolic Pythagorean Theorem, expressing in a dimension-free form the square of the hyperbolic half-length of the hypotenuse of a hyperbolic right-angled triangle in any real inner product space as an Einstein sum of the squares of the hyperbolic half-lengths of the other two sides, Figure 4. The Hyperbolic Pythagorean Theorem in the Poincaré disc model of hyperbolic geometry, where geodesics are circular arcs that intersect the boundary of the disc orthogonally, is presented in [4,33]. We will now present the same Hyperbolic Pythagorean Theorem, but in the Einstein gyrovector space model of hyperbolic geometry.

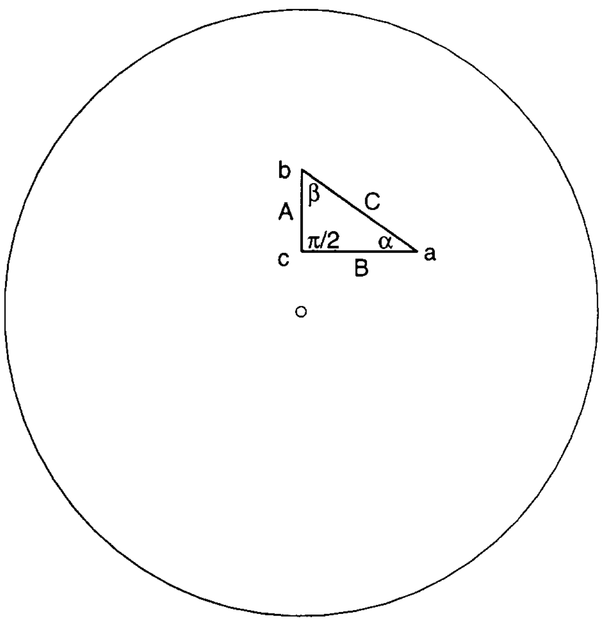

Figure 4. The Hyperbolic Pythagorean Theorem $\|(1/2) \otimes A\|^2 \oplus \|(1/2) \otimes B\|^2 = \|(1/2) \otimes C\|^2$ in the Beltrami disc model of hyperbolic geometry, and in Einstein Gyrovector Spaces. A right-angled triangle in the Beltrami disc model of hyperbolic geometry and the Einstein gyrovector space $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ with vertices **a**, **b**, **c**, and sides A, B, C, is shown, satisfying a Pythagorean identity.

THEOREM 7.1. A HYPERBOLIC PYTHAGOREAN THEOREM. *Let* $\Delta\mathbf{abc}$ *be a right-angled triangle in an Einstein gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with vertices* **a**, **b**, *and* **c**, *whose orthogonal sides* A *and* B *and hypotenuse* C *are*

$$A = \ominus \mathbf{c} \oplus \mathbf{b},$$
$$B = \ominus \mathbf{c} \oplus \mathbf{a},$$
$$C = \ominus \mathbf{a} \oplus \mathbf{b}.$$

Then,

$$\frac{\|(1/2) \otimes A\|^2}{c} \oplus \frac{\|(1/2) \otimes B\|^2}{c} = \frac{\|(1/2) \otimes C\|^2}{c}.$$

PROOF. The proof follows from Theorem 6.3 with $\gamma = \pi/2$. ∎

The special case of the hyperbolic law of sines (6.13) when $\gamma = \pi/2$ is important, resulting in

$$\begin{aligned} \sin\alpha &= \frac{\gamma_A\|A\|}{\gamma_C\|C\|}, \\ \sin\beta &= \frac{\gamma_B\|B\|}{\gamma_C\|C\|}, \end{aligned} \tag{7.1}$$

for the hyperbolic right-angled triangle in Figure 4. The analogy with Euclidean geometry is obvious. Unlike Euclidean geometry, however, $\sin^2\alpha + \sin^2\beta < 1$ since $\alpha + \beta < \pi/2$.

8. RELATIVISTIC UNIFORM ACCELERATION

The hyperbolic geodesic

$$\mathbf{v}(t) = \mathbf{v}_0 \oplus \mathbf{a} \otimes t, \tag{8.1}$$

$\mathbf{v}_0$, $\mathbf{a} \in \mathbb{R}^3_c$, $t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ represents the time dependent velocity of uniform acceleration, where $\mathbf{v}_0$ is an initial velocity and $\mathbf{a}$ is a constant acceleration. The analogies that the relativistic uniform acceleration (8.1) shares with its Newtonian counterpart,

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \tag{8.2}$$

are clear, but incomplete. The noncommutativity of Einstein's addition $\oplus$ raises the question as to whether the right relativistic counterpart of (8.2) is the gyroline (8.1), shown in Figure 5, or, perhaps, the dual gyroline (8.3),

$$\mathbf{v}(t) = \mathbf{a} \otimes t \oplus \mathbf{v}_0 \tag{8.3}$$

shown in Figure 6.

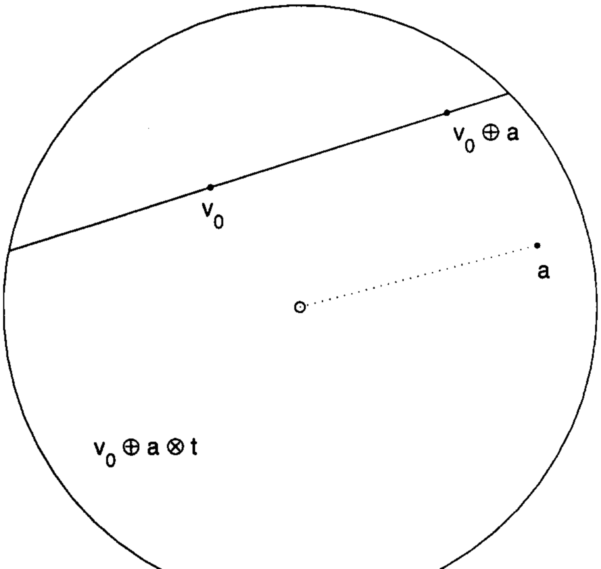

Figure 5. The gyroline $\mathbf{v}_0 \oplus \mathbf{a} \otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is a Euclidean straight line which is a geodesic relative to the Einstein metric (8.7). Unlike the analogous case in Euclidean geometry, this gyroline is not Euclidean parallel to the geometric gyrovector $\mathbf{a}$, in agreement with the fact that the parallel postulate is denied in hyperbolic geometry.

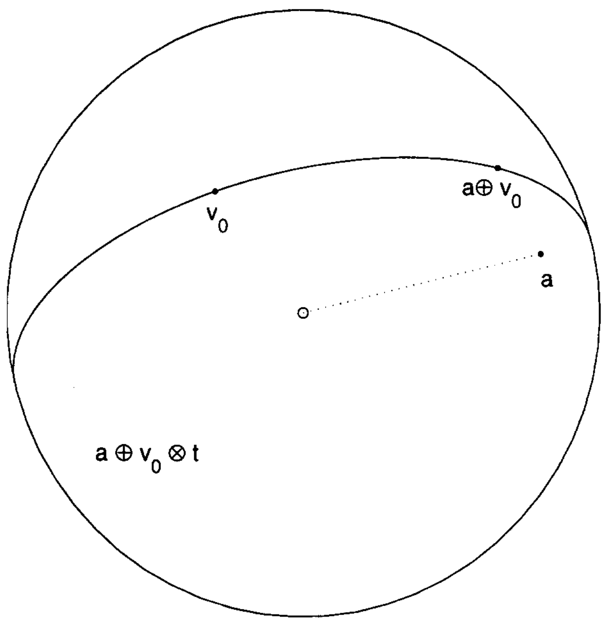

Figure 6. The dual gyroline $\mathbf{a} \otimes t \oplus \mathbf{v}_0$, $t \in \mathbb{R}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is a Euclidean elliptic arc which is also a dual geodesic relative to the Einstein dual metric (9.1). It is "supported" by a diameter of the ball. The supporting diameter is Euclidean parallel to the Euclidean geometric vector **a**. In fact, the Euclidean geometric vector **a** lies on the supporting diameter.

Various definitions of uniform acceleration are possible in relativity theory [46]. However, a widely accepted one states that uniformly accelerated velocities are those which traverse along geodesics in velocity spaces [47]. One, therefore, may argue that (8.1) is superior over (8.3) as a candidate for the hyperbolic counterpart of Newtonian uniform acceleration (8.2) since unlike (8.3), the relativistic uniform acceleration representation (8.1) is a geodesic in the ball $\mathbb{R}^3_c$ relative to Einstein's metric (8.7) of the ball. But, what if (8.3) is also a geodesic relative to some as yet unknown metric of the ball? And what if, moreover, the yet to be discovered metric is also intimately connected with Einstein's addition? To answer that question, we explore the geometric significance of the gyrolines (8.1), shown in Figure 5, and the dual gyrolines (8.3), shown in Figure 6, in terms of analogies that they share with their Euclidean counterpart (8.2).

It is convenient to write the Euclidean geodesic (8.2) in the form

$$\mathbf{a} + (-\mathbf{a} + \mathbf{b})t \tag{8.4}$$

to display the property of being the unique geodesic passing through two given points **a** (when $t = 0$) and **b** (when $t = 1$).

The two hyperbolic counterparts, (8.1) and (8.3), of (8.2) can be written in a form analogous to (8.4) as

$$\mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t \tag{8.5}$$

and

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}. \tag{8.6}$$

The hyperbolic gyroline (8.5) is a geodesic in the ball $\mathbb{V}_c$ of a real inner product space $\mathbb{V}_\infty$ relative to the Einstein metric,

$$d_\ominus(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus \mathbf{v}\| \tag{8.7}$$

of the Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. It is the unique geodesic passing through the points **a** (when $t = 0$) and **b** (when $t = 1$). We should notice that it is due to the left cancellation law in (2.9), that when $t = 1$, the geodesic (8.5) passes through the point **b**.

Similarly, the gyroline (8.6) passes through the point $\mathbf{a}$ when $t = 0$ and, due to the right cancellation law (2.10), it passes through the point $\mathbf{b}$ when $t = 1$. It is this need to employ the right cancellation law that dictates that the term $\ominus \mathbf{a} \oplus \mathbf{b}$ in (8.5) be replaced by the term $\boxminus \mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a}$ in (8.6).

9. EINSTEIN'S DUAL GEODESICS

Being guided by analogies, we conjecture that in full analogy with the gyrolines (8.5), which are geodesics relative to the Einstein metric $d_\ominus$ in (8.7), the "dual" gyrolines (8.6) are geodesics relative to the Einstein dual metric that we naturally define as

$$d_\boxminus(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \boxminus \mathbf{v}\|. \tag{9.1}$$

The binary operation $\boxminus$ in (9.1) is the Einstein dual subtraction given by (4.5) using the notation $\mathbf{u} \boxminus \mathbf{v} = \mathbf{u} \boxplus (\ominus \mathbf{v})$.

The dual metric (9.1) does not possess a triangle inequality. Remarkably, however, it provides a ruler for dual gyrolines in the sense that we show graphically in Figure 7 and explain below. Furthermore, it satisfies an inequality that we call the dual gyrotriangle inequality,

$$\|\mathbf{u} \boxminus \mathbf{v}\| \boxplus \|\mathbf{v} \boxminus \mathbf{w}\| \geq \|\mathbf{u} \ominus \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathrm{gyr}[\mathbf{v}, \mathbf{w}]\mathbf{w}\|, \tag{9.2}$$

which reduces to the equality

$$\begin{aligned} \|\mathbf{u} \boxminus \mathbf{v}\| \boxplus \|\mathbf{v} \boxminus \mathbf{w}\| &= \|\mathbf{u} \ominus \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathrm{gyr}[\mathbf{v}, \mathbf{w}]\mathbf{w}\| \\ &= \|\mathbf{u} \ominus \mathrm{gyr}[\mathbf{u}, \mathbf{w}]\mathbf{w}\| \\ &= \|\mathbf{u} \boxminus \mathbf{w}\|, \end{aligned} \tag{9.3}$$

if and only if the three involved points, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, are dual gyrocollinear, that is, they lie on the same dual gyroline.

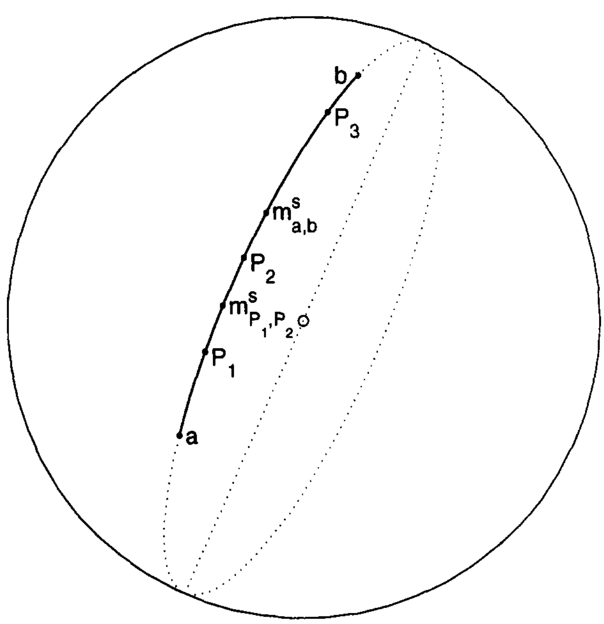

Figure 7. A dual gyroline in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is a semiellipse whose major axis coincides with a diameter of the ball $\mathbb{V}_c$, which is shown here for the Beltrami disc model $\mathbb{V}_c = \mathbb{R}^2_{c=1}$ of hyperbolic geometry. A dual gyroline and the segment connecting two of its points, $\mathbf{a}$ and $\mathbf{b}$, in the Einstein disc gyrogroup $\mathbb{R}^2_{c=1}$ is shown with three points, P_1, P_2, and P_3. On dual gyrolines, and only on dual gyrolines, the ruler equality holds relative to the dual metric (9.1): $d_\boxminus(P_1, P_2) \boxplus d_\boxminus(P_2, P_3) = d_\boxminus(P_1, P_3)$. However, a corresponding gyrotriangle inequality (9.2) is distorted by Thomas gyrations.

The unique two dual gyrolines

$$
\begin{aligned}
L^p_{\mathbf{ab}} &= \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \otimes t, \\
L^s_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a},
\end{aligned}
\tag{9.4}
$$

$t \in \mathbb{R}$, that pass through two given points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ are called collectively the dual geodesics through $\mathbf{a}$ and $\mathbf{b}$. We also call $L^p_{\mathbf{ab}}$ ($L^s_{\mathbf{ab}}$) the geodesic (the dual geodesic), or the primary geodesic (the secondary geodesic), or the gyroline (the dual gyroline) through $\mathbf{a}$ and $\mathbf{b}$.

Figure 7 shows a dual geodesic segment $L^s_{\mathbf{ab}}$ with endpoints $\mathbf{a}$ and $\mathbf{b}$, and their dual (or, secondary) midpoint $m^s_{\mathbf{a},\mathbf{b}}$, and three of its points, P_1, P_2, and P_3, as well as the midpoint $m_{P_1P_2}$ of P_1 and P_2. They are given analytically, in terms of a gyroline parameter $t \in \mathbb{R}$, by the equations

$$
\begin{aligned}
L^s_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}, \\
m^s_{\mathbf{a},\mathbf{b}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes \frac{1}{2} \oplus \mathbf{a}, \\
P_1 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}, \\
P_2 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_2 \oplus \mathbf{a}, \\
m_{P_1P_2} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes \frac{t_1 + t_2}{2} \oplus \mathbf{a}, \\
P_3 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_3 \oplus \mathbf{a},
\end{aligned}
\tag{9.5}
$$

$t_1 < t_2 < t_3$, satisfying

$$
m^s_{\mathbf{a},\mathbf{b}} = m^s_{\mathbf{b},\mathbf{a}}, \tag{9.6}
$$

and, by (9.3),

$$
d_{\boxminus}(P_1, P_2) \boxplus d_{\boxminus}(P_2, P_3) = d_{\boxminus}(P_1, P_3). \tag{9.7}
$$

The dual geodesic segment $L^s_{\mathbf{ab}}$ in Figure 7 lies on a dual gyroline which is a semiellipse whose major axis is a diameter of the disc $\mathbb{R}^2_{c=1}$ (and of the ball $\mathbb{V}_c$ for a general Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$). The diameter is Euclidean parallel to the vector $\mathbf{b} \boxminus \mathbf{a} \in \mathbb{V}_c \subset \mathbb{V}_\infty$. Hence, guided by analogies, we define the hyperbolic orientation of the dual gyroline $L^s_{\mathbf{ab}}$ in (9.5) to be the Euclidean orientation of $\mathbf{b} \boxminus \mathbf{a}$. This, in turn, introduces parallelism into the novel, dual part of hyperbolic geometry.

DEFINITION 9.1. PARALLELISM IN HYPERBOLIC DUAL GEOMETRY. *The two dual geodesics*

$$
\begin{aligned}
L^s_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}, \\
L^s_{\mathbf{cd}} &= (\mathbf{d} \boxminus \mathbf{c}) \otimes t \oplus \mathbf{c},
\end{aligned}
\tag{9.8}
$$

$t \in \mathbb{R}$, *in an Einstein gyrovector space model* $(\mathbb{V}_c, \oplus, \otimes)$ *of hyperbolic geometry are parallel if and only if the two vectors* $\mathbf{b} \boxminus \mathbf{a}$ *and* $\mathbf{d} \boxminus \mathbf{c}$ *in the ball* $\mathbb{V}_c$ *of the real inner product space* $\mathbb{V}_\infty$ *are Euclidean parallel.*

Parallelism, thus, reappears in hyperbolic geometry if we incorporate the duality symmetries that the Thomas precession suggest. The power and elegance of the abstract Thomas precession, that is, the Thomas gyration, in allowing the study of standard hyperbolic geometry by analogies shared with Euclidean geometry is now further evidenced by the exposition of duality symmetries in hyperbolic geometry. Hyperbolic geometry as we presently know from the literature is, in fact, only half of the full theory, the other half being the dual part. The dual part of hyperbolic geometry, as seen in the study of the Einstein gyrovector space model, involves dual geodesics which support parallelism. In Section 10, we explore the dual angles between dual geodesics to see if our duality symmetries are capable of exposing novel geometrically significant results in hyperbolic geometry.

10. EINSTEIN'S DUAL HYPERBOLIC ANGLES

The duality symmetries that the Einstein dual additions $\oplus$ and $\boxplus$ introduce into hyperbolic geometry suggest that we view an Einstein gyrovector space as a bimetric space possessing the dual metrics (8.7) and (9.1), and the dual families of geodesics (9.4). To extend the exploration of duality symmetries in hyperbolic geometry by means of the Einstein gyrovector space model $(\mathbb{V}_c, \oplus, \otimes)$, we associate the geometric dual gyrovectors

$$\boxminus \mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a} \tag{10.1}$$

to ordered pairs $(\mathbf{a}, \mathbf{b})$ of distinct points $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{V}_c$. In dimension $n \leq 3$, we view the geometric dual gyrovector $\mathbf{b} \boxminus \mathbf{a}$ as an elliptic arc from $\mathbf{a}$ to $\mathbf{b}$ with a major axis that coincides with a diameter of $\mathbb{V}_c$, as shown in Figure 8 for $\mathbb{V}_c = \mathbb{R}^2_{c=1}$.

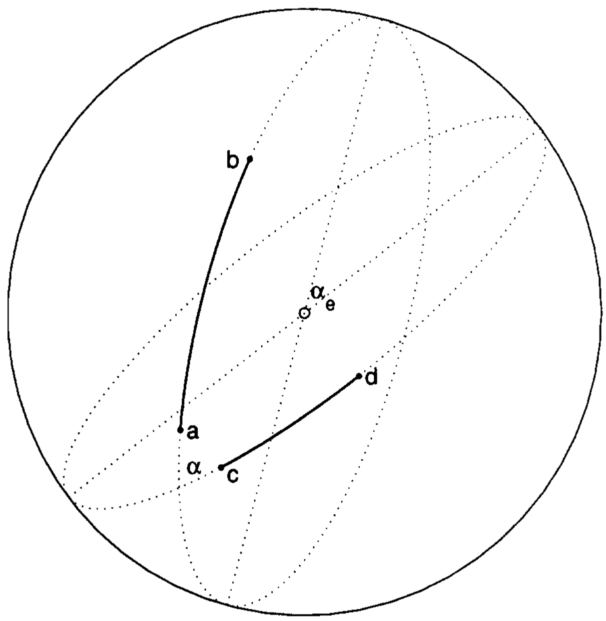

Figure 8. The dual angle α generated by the two geometric dual gyrovectors $\mathbf{b} \boxminus \mathbf{a}$ and $\mathbf{d} \boxminus \mathbf{c}$ in the Einstein disc gyrovector space $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is shown. Its cosine is given by (10.4), and is numerically equal to the Euclidean angle α_e generated by the corresponding supporting diameters.

Two geometric dual gyrovectors $\mathbf{b} \boxminus \mathbf{a}$ and $\mathbf{d} \boxminus \mathbf{c}$, lying on the two dual geodesics

$$\begin{aligned} L^s_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}, \\ L^s_{\mathbf{cd}} &= (\mathbf{d} \boxminus \mathbf{c}) \otimes t \oplus \mathbf{c}, \end{aligned} \tag{10.2}$$

$t \in \mathbb{R}$, as well as their supporting diameters are shown in Figure 8. As we have found following the inspection of Figure 6, the vectors $\mathbf{b} \boxminus \mathbf{a}$ and $\mathbf{d} \boxminus \mathbf{c}$ are Euclidean parallel to their respective supporting diameters. Hence, the cosine of the Euclidean angle α_e between the supporting diameters in Figure 8 is given by

$$\cos \alpha_e = \frac{\boxminus \mathbf{a} \boxplus \mathbf{b}}{\| \boxminus \mathbf{a} \boxplus \mathbf{b} \|} \cdot \frac{\boxminus \mathbf{c} \boxplus \mathbf{d}}{\| \boxminus \mathbf{c} \boxplus \mathbf{d} \|}. \tag{10.3}$$

But, (10.3) is just the dual of (5.1), suggesting the following definition of the measure of the hyperbolic dual angle.

DEFINITION 10.1. THE HYPERBOLIC DUAL ANGLE. *The measure of the hyperbolic dual angle α between two geometric dual gyrovectors $\mathbf{b} \boxminus \mathbf{a}$ and $\mathbf{d} \boxminus \mathbf{c}$ is given by the equation*

$$\cos \alpha = \frac{\boxminus \mathbf{a} \boxplus \mathbf{b}}{\| \boxminus \mathbf{a} \boxplus \mathbf{b} \|} \cdot \frac{\boxminus \mathbf{c} \boxplus \mathbf{d}}{\| \boxminus \mathbf{c} \boxplus \mathbf{d} \|}. \tag{10.4}$$

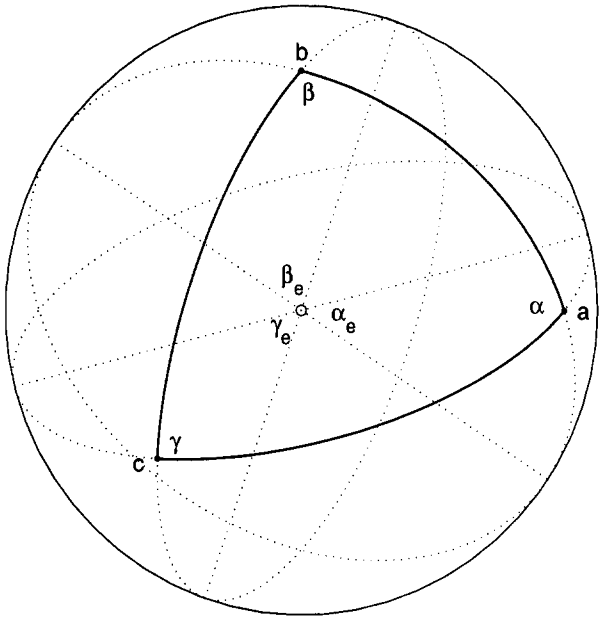

Figure 9. The dual hyperbolic angle between dual geodesics and its associated Euclidean angle between corresponding supporting diameters are equal. Hence, by inspection, the sum of the hyperbolic dual angles of a hyperbolic dual triangle is π.

Dual angles are invariant under rotations, but are not invariant under left gyrotranslations. However, they carry an important Euclidean property that we state in the following.

THEOREM 10.2. THE HYPERBOLIC π THEOREM. *Let* $\Delta\mathbf{abc}$ *be the hyperbolic dual triangle in an Einstein gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *whose three vertices are the points* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. *Then, its three hyperbolic dual angles* α, β, *and* γ, *given by*

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{a}}{\|\mathbf{c} \boxminus \mathbf{a}\|},\\ \cos\beta &= \frac{\mathbf{a} \boxminus \mathbf{b}}{\|\mathbf{a} \boxminus \mathbf{b}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{b}}{\|\mathbf{c} \boxminus \mathbf{b}\|},\\ \cos\gamma &= \frac{\mathbf{a} \boxminus \mathbf{c}}{\|\mathbf{a} \boxminus \mathbf{c}\|} \cdot \frac{\mathbf{b} \boxminus \mathbf{c}}{\|\mathbf{b} \boxminus \mathbf{c}\|},\end{aligned} \tag{10.5}$$

satisfy the identity

$$\alpha + \beta + \gamma = \pi. \tag{10.6}$$

PROOF. It follows from (4.5) that the Einstein dual addition $\boxplus_E$ is given by the equation

$$\mathbf{u} \boxplus_E \mathbf{v} = 2 \otimes_E \frac{\gamma_\mathbf{u}\mathbf{u} + \gamma_\mathbf{v}\mathbf{v}}{\gamma_\mathbf{u} + \gamma_\mathbf{v}}, \tag{10.7}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$ are elements of the ball $\mathbb{V}_c$ of a real inner product space $(\mathbb{V}_\infty, +)$. Furthermore, it follows from the scaling property in (4.9) that

$$\frac{\mathbf{a} \boxplus_E \mathbf{b}}{\|\mathbf{a} \boxplus_E \mathbf{b}\|} = \frac{\gamma_\mathbf{a}\mathbf{a} + \gamma_\mathbf{b}\mathbf{b}}{\|\gamma_\mathbf{a}\mathbf{a} + \gamma_\mathbf{b}\mathbf{b}\|}. \tag{10.8}$$

Hence, the three hyperbolic angles in (10.5) of the hyperbolic triangle with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can be written as

$$\begin{aligned}\cos\alpha &= \frac{\gamma_\mathbf{b}\mathbf{b} - \gamma_\mathbf{a}\mathbf{a}}{\|\gamma_\mathbf{b}\mathbf{b} - \gamma_\mathbf{a}\mathbf{a}\|} \cdot \frac{\gamma_\mathbf{c}\mathbf{c} - \gamma_\mathbf{a}\mathbf{a}}{\|\gamma_\mathbf{c}\mathbf{c} - \gamma_\mathbf{a}\mathbf{a}\|},\\ \cos\beta &= \frac{\gamma_\mathbf{a}\mathbf{a} - \gamma_\mathbf{b}\mathbf{b}}{\|\gamma_\mathbf{a}\mathbf{a} - \gamma_\mathbf{b}\mathbf{b}\|} \cdot \frac{\gamma_\mathbf{c}\mathbf{c} - \gamma_\mathbf{b}\mathbf{b}}{\|\gamma_\mathbf{c}\mathbf{c} - \gamma_\mathbf{b}\mathbf{b}\|},\\ \cos\gamma &= \frac{\gamma_\mathbf{a}\mathbf{a} - \gamma_\mathbf{c}\mathbf{c}}{\|\gamma_\mathbf{a}\mathbf{a} - \gamma_\mathbf{c}\mathbf{c}\|} \cdot \frac{\gamma_\mathbf{b}\mathbf{b} - \gamma_\mathbf{c}\mathbf{c}}{\|\gamma_\mathbf{b}\mathbf{b} - \gamma_\mathbf{c}\mathbf{c}\|}.\end{aligned} \tag{10.9}$$

These, however, are recognized as the three Euclidean angles of the Euclidean triangle in $\mathbb{V}_\infty$ with vertices $\gamma_{\mathbf{a}}\mathbf{a}$, $\gamma_{\mathbf{b}}\mathbf{b}$, and $\gamma_{\mathbf{c}}\mathbf{c}$. As such, their sum must be π. ∎

The duality symmetries emerge naturally when hyperbolic geometry is studied by its underlying gyrovector space structure. They bring elegance and clarity in terms of analogies shared with Euclidean geometry. Understanding the duality idea with its implied Hyperbolic π-Theorem in terms of a physical Principle of Reciprocity has recently been gained by Chen [48].

11. PARALLELISM IN HYPERBOLIC DUAL GEOMETRY

Hyperbolic geometry was discovered as a consequence of questions about the parallel postulate. Appearing in Euclid's original treatise, the parallel postulate provoked two millenia of mathematical investigation about the nature of logic, proof, and geometry. Eventually, hyperbolic geometry emerged from the denial of the parallel postulate. Ironically, as we will see in this section, the parallel postulate did not disappear from hyperbolic geometry. Rather, it reappears in the part of hyperbolic geometry, the hyperbolic dual geometry, that had gone unnoticed prior to its exposition by gyrogroup formalism.

The two dual geodesics L_1 and L_2 in Figure 10 are parallel, Definition 9.1, since they share a common supporting diameter. We see in Figure 10 a dual geodesic intersecting the two parallel dual geodesics L_1 and L_2 in the Beltrami disc model of hyperbolic geometry which is at the same time the Einstein disc gyrovector space $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$. The intersecting dual geodesic generates two alternate interior dual angles α, which, as in Euclidean geometry, are equal.

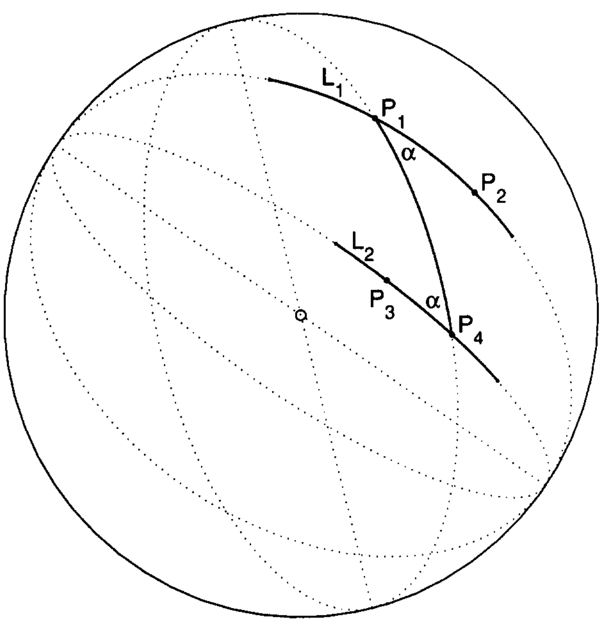

Figure 10. Two hyperbolic alternate interior dual angles α generated by a hyperbolic dual geodesic intersecting two parallel hyperbolic dual geodesics are equal.

THEOREM 11.1. *Let*

$$L^s_{\mathbf{ab}} = (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a}, \qquad L^s_{\mathbf{cd}} = (\mathbf{d} \boxminus \mathbf{c}) \otimes t \oplus \mathbf{c}, \tag{11.1}$$

$t \in \mathbb{R}$, be two parallel dual geodesics in an Einstein gyrovector space model $(\mathbb{V}_c, \oplus, \otimes)$ of hyperbolic geometry, that are intersected by a dual geodesic at the points P_1 and P_4 of L_1 and L_2, respectively. Furthermore, let P_2 and P_3 be points on L_1 and L_2, respectively, which are located on opposite sides of the intersecting dual geodesic, Figure 10. Then the two alternate interior dual angles $\angle P_2P_1P_4$ and $\angle P_3P_4P_2$ are equal.

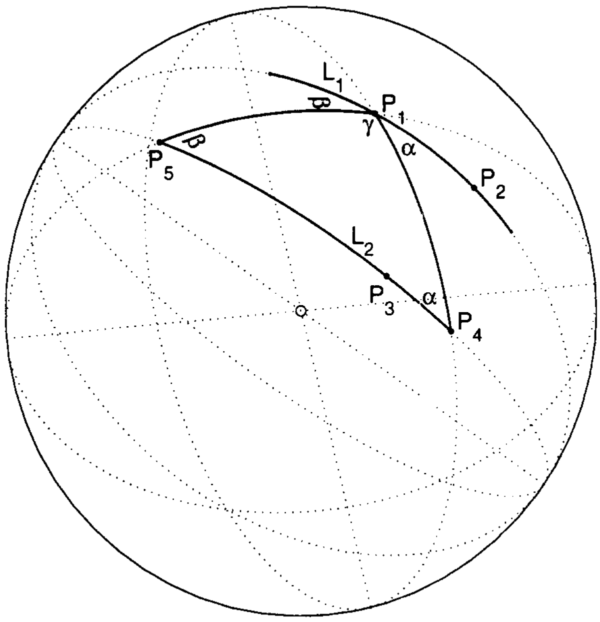

Figure 11. The sum of the dual angles of a hyperbolic dual triangle is π. The proof follows from the equality of hyperbolic alternate interior dual angles, as in the Euclidean case.

Theorem 11.1 is presented in [34], and an alternative proof of Theorem 10.2 is indicated in Figure 11.

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