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Null Controllability of Nonlinear Infinite Delay Systems with Time Varying Multiple Delays in Control

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Abstract—Sufficient conditions for the null controllability of a nonlinear infinite delay system with time varying multiple delays in the control are developed. Namely, if the uncontrolled system is uniformly asymptotically stable, and if the linear system is controllable, then the nonlinear infinite delay system is null controllable.

Keywords—Controllability, Nonlinear systems, Delay systems.

1. INTRODUCTION

The study of integrodifferential equations with infinite delay has emerged in recent years as an independent branch of modern research because of its connection to many fields such as continuum mechanics, population dynamics, ecology, systems theory, viscoelasticity, biology, epidemics and chemical oscillations [1–4]. For example, in most biological populations the accumulation of metabolic products may seriously inconvenience a population, and one of the consequences can be a fall in the birth rate and an increase in the mortality rate. If it is assumed that the total toxic action on birth and death rates is expressed by an integral term in the logistic equation, then an appropriate model is an integrodifferential equation with infinite delay [5]. The aim of this paper is to study the null controllability of such systems by introducing multiple delays in controls. For motivation of time varying multiple delays in control variables refer to the book by Klamka [6].

Chukwu [7] showed that if the linear delay system

$$\dot{x}(t) = L(t, x_t) \tag{1}$$

is uniformly asymptotically stable and

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) \tag{2}$$

is proper, then

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + f\big(t, x_t, u(t)\big) \tag{3}$$

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is Euclidean null controllable provided f satisfies certain growth and continuity conditions. He also showed in [8] that if (1) is uniformly asymptotically stable, and (2) is function space controllable, then (3) is function space null controllable with constraints. Underwood and Young [9] proved that if the linear approximation (2) to the system

$$\dot{x}(t) = f\big(t, x_t, u(t)\big) \tag{4}$$

is function space controllable, then (4) is function space locally null controllable under certain conditions on f. Chukwu [10] extended this result by assuming that if, in addition, the system

$$\dot{x}(t) = f(t, x_t, 0) \tag{5}$$

is globally uniformly asymptotically stable, then the system (4) is globally null controllable with constraints.

Sinha [11] developed sufficient conditions for the null controllability of the infinite delay system

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(s)x(t+s)\,ds + f\big(t, x(\cdot), u(\cdot)\big), \\ x(t) &= \phi(t) \qquad \text{for } -\infty < t \le 0, \end{aligned} \tag{6}$$

where $L(t, \phi)$ is continuous in t, linear in ϕ and given by

$$L(t, \phi) = \sum_{k=0}^{N} A_k(t)\phi(-h_k). \tag{7}$$

Balachandran and Dauer [12] studied this problem for system (6) with distributed delays in the control. In this paper, nonlinear infinite delay systems of the following form are considered:

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + \sum_{i=0}^{N} B_i(t)u\big(h_i(t)\big) + \int_{-\infty}^{0} A(s)x(t+s)\,ds + f\big(t, x(\cdot), u(\cdot)\big), \\ x(t) &= \phi(t), \qquad t \in (-\infty, t_0]. \end{aligned} \tag{8}$$

2. PRELIMINARIES

In equation (8), each A_k (see (7)) is a continuous $n \times n$ matrix function for $0 \le h_k \le h$, and $A(s)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty, 0]$. The matrix functions $B_i(t)$, $i = 0, 1, \dots, N$ are $n \times p$, continuous in t, and $h = t_0 - \min_i h_i(t_0)$ where $h_i(t)$ are defined below. Here $x \times E^n$ and $u \in E^p$. Let $\gamma \ge h \ge 0$ be given real numbers (γ may be $+\infty$), and E^n be an n-dimensional linear vector space with norm $|\cdot|$. The function $\eta : [-\gamma, 0] \to (0, \infty)$ is Lebesque integrable on $[-\gamma, 0]$, positive and nondecreasing. Let $B([-\gamma, 0], E^n)$ be the Banach space of functions [13] which are continuous and bounded on $[-\gamma, 0]$ and such that

$$|\phi| = \sup_{s \in [-h, 0]} |\phi(s)| + \int_{-\gamma}^{0} \eta(\tau)\,|\phi(\tau)|\,d\tau < \infty.$$

For any $t \in R$, and any $x : [t - \gamma, t] \to E^n$, let $x_t : [-\gamma, 0] \to E^n$ be defined by

$$x_t(s) = x(t+s), \qquad s \in [-\gamma, 0].$$

The functions $h_i : [t_0, t_1] \to R$, $i = 0, 1, \dots, N$, are twice continuously differentiable and strictly increasing in $[t_0, t_1]$. Further

$$h_i(t) \le t \qquad \text{for } t \in [t_0, t_1], \quad i = 0, 1, \dots, N.$$

Let us introduce, as in [14], the following time lead functions r_i with

$$r_i(t) : \big[h_i(t_0), h_i(t_1)\big] \to [t_0, t_1]$$

such that $r_i\big(h_i(t)\big) = t$ for $i = 0, 1, \ldots, N$, $t \in [t_0, t_1]$. Without loss of generality, it can be assumed that $h_0(t) = t$ and the following inequalities hold for $t = t_1$:

$$\begin{aligned} h_N(t_1) \le h_{N-1}(t_1) \le \cdots \le h_{m+1}(t_1) \le t_0 = h_m(t_1) \\ < h_{m-1}(t_1) = \cdots = h_1(t) = h_0(t_1). \end{aligned} \tag{9}$$

Consider the linear homogeneous systems

$$\dot{x}(t) = L(t, x_t) + \sum_{i=0}^{N} B_i(t) u\big(h_i(t)\big), \tag{10}$$

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(s) x(t+s)\, ds. \tag{11}$$

Dafermos [15] reduced the problem of asymptotic stability for viscoelastic materials to the investigation of stability properties for the equation of the form (11). Also, it represents a model of neural networks with infinite delay [16]. Chukwu [17] studied the null controllability of systems of the form (10) with constant and distributed delays in control with respect to the control of global economic growth. Further, Chukwu introduced the solidarity functions in (10) and obtained certain universal principles for the control of economic growth of interconnected systems. Here, the more general case of time varying delays is considered.

In particular, the controllability of (8) is studied when it is assumed that the admissible controls have values in a compact convex subset U of E^p. Hale [13] obtained exponential estimates on the solutions of the linear equation (11).

Let X satisfy the equation

$$\begin{aligned} &\frac{\partial X(t,s)}{\partial t} = L\big(t, X_t(.,s)\big), && t \ge s, \\ &0, && s - h \le t \le s, \\ &X(t,s) = I, && t = s, \end{aligned}$$

where $X_t(.,s)(\theta) = X(t+\theta, s)$, $-h \le \theta \le 0$. Then the solution of (8) is given by

$$\begin{aligned} x(t) = x(t; t_0, \phi) &+ \int_{t_0}^{t} X(t,s) \sum_{i=0}^{N} B_i(s) u\big(h_i(s)\big)\, ds \\ &+ \int_{t_0}^{t} X(t,s) \left(\int_{-\gamma}^{0} A(\theta) x(s+\theta)\, d\theta \right) ds \\ &+ \int_{t_0}^{t} X(t,s) f\big(s, x(\cdot), u(\cdot)\big)\, ds \qquad \text{for } t_0 \le t \le t_1, \\ x(t) = \phi(t) &\qquad \text{for } t \in [-\infty, t_0] \end{aligned} \tag{12}$$

with initial state $z(t_0) = \big(x(t_0); \phi, \eta\big)$ where $u(s) = \eta(s)$ for $s \in [t_0 - h, t_0]$ and $x(t; t_0, \phi)$ is the solution of $\dot{x}(t) = L(t, x_t)$. Using the time lead function and the inequalities (9) we have

$$\begin{aligned} x(t_1) &= x(t_1; t_0, \phi) + \sum_{i=0}^{m} \int_{h_i(t_0)}^{t_0} X\big(t, r_i(s)\big) B_i\big(r_i(s)\big) \dot{r}_i(s) \eta(s)\, ds \\ &+ \sum_{i=m+1}^{N} \int_{h_i(t_0)}^{h_i(t_1)} X\big(t_1, r_i(s)\big) B_i\big(r_i(s)\big) \dot{r}_i(s) \eta(s)\, ds \\ &+ \sum_{i=0}^{m} \int_{t_0}^{t_1} X\big(t_1, r_i(s)\big) B_i\big(r_i(s)\big) \dot{r}_i(s) u(s)\, ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) \left(\int_{-r}^{0} A(\theta) x(s+\theta)\, d\theta \right) ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) f\big(s, x(\cdot), u(\cdot)\big)\, ds. \end{aligned} \tag{13}$$

For brevity, introduce the following notations:

$$\begin{aligned} H(t, \eta) &= \sum_{i=0}^{m} \int_{h_i(t_0)}^{t_0} X\big(t, r_i(s)\big) B_i\big(r_i(s)\big) \dot{r}_i(s) \eta(s)\, ds \\ &= + \sum_{i=m+1}^{N} \int_{h_i(t_0)}^{h_i(t)} X\big(t, r_i(s)\big) B_i\big(r_i(s)\big) \dot{r}_i(s) \eta(s)\, ds, \\ q(t_1, \eta) &= x(t_1; t_0, \phi) + H(t_1, \eta) + \int_{t_0}^{t_1} X(t_1, s) f\big(s, x(\cdot), u(\cdot)\big)\, ds \\ &= + \int_{t_0}^{t_1} X(t_1, s) \left(\int_{-r}^{0} A(\theta) x(s+\theta)\, d\theta \right) ds, \\ G_i(t, s) &= \sum_{j=0}^{i} X\big(t, r_i(s)\big) B_j\big(r_j(s)\big) \dot{r}_j(s). \end{aligned}$$

Define the controllability matrix of (10) at time t as

$$W(t_0, t) = \int_{t_0}^{t} G_m(t, s) G_m^{\top}(t, s)\, ds \tag{14}$$

where $\top$ denotes matrix transpose.

DEFINITION. *The system (8) is said to be null controllable if for each $\phi \in B\big([-\gamma, 0], E^n\big)$, there is a $t_1 \geq t_0, u \in L_2\big([t_0, t_1], U\big)$, U a compact convex subset of E^p, such that the solution $x(t; t_0, \phi, u)$ of (8) satisfies $x_{t_0}(t_1; \phi, u) = \phi$ and $x(t_1; t_0, \phi, u) = 0$.*

3. MAIN RESULTS

THEOREM. *Suppose that the constraint set U is an arbitrary compact subset of E^p and that the following hold:*

(i) *System (11) is uniformly asymptotically stable, so that the solution $x_t(t_0, \phi)$ satisfies*

$$\|x_t(t_0, \phi)\| \leq M e^{-\alpha(t-t_0)} \|\phi\| \qquad \text{for some } \alpha > 0, \quad M > 0.$$

(ii) *The linear control system (10) is controllable.*

(iii) *The continuous function f satisfies*

$$\big|f\big(t, x(\cdot), u(\cdot)\big)\big| \leq \exp(-\beta t) \pi\big(x(\cdot), u(\cdot)\big)$$

for all $(t, x(\cdot), u(\cdot)) \in [t_0, \infty) \times B([-\gamma, 0], E^n) \times L_2([t_0, t_1], U)$ *where*

$$\int_{t_0}^{\infty} \pi(x(\cdot), u(\cdot))\, ds \le K < \infty \quad \text{and} \qquad \beta - \alpha \ge 0.$$

Then (8) is Euclidean null controllable.

PROOF. Since (10) is controllable, $W^{-1}(t_0, t_1)$ exists for each $t_1 > t_0$. Suppose the pair of functions x, u form a solution pair to the following equations:

$$u(t) = -G_m^{\top}(t_1, t) W^{-1}(t_0, t_1) q(t_1, \eta) \tag{15}$$

for some suitably chosen $t_1 \ge t \ge t_0$, $u(t) = \eta(t)$, $t \times [t_0 - h, t_0]$ and

$$\begin{aligned} x(t) &= x(t; t_0, \phi) + H(t, \eta) + \int_{t_0}^{t} G_m(t, s) u(s)\, ds \\ &\quad + \int_{t_0}^{t} \int_{-\gamma}^{0} A(\theta)\, x(t + \theta)\, d\theta\, ds + \int_{t_0}^{t} X(t, s) f(s, x(\cdot), u(\cdot))\, ds, \\ x(t) &= \phi(t), \qquad t \in [t_0 - h, t_0]. \end{aligned} \tag{16}$$

Then u is square integrable on $[t_0 - h, t_1]$ and x is a solution of (8) corresponding to u with initial state $z(t_0) = (x(t_0), \phi, \eta)$. Also, $x(t_1) = 0$. Now it is shown that $u : [t_0, t_1] \to U$ is in a compact constraint subset of E^p; that is, $|u| \le a$ for some constant $a > 0$. Since (11) is uniformly asymptotically stable and B_i are continuous in t, it follows that

$$\begin{aligned} |G_m^{\top}(t_1, t) W^{-1}| &\le C_1 && \text{for some } C_1 > 0, \\ |x_t(t_0, \phi)| &\le C_2 \exp\left[-\alpha(t_1 - t_0)\right] && \text{for some } C_2 > 0, \\ |H(t, \eta)| &\le C_3 \exp\left[-\alpha(t_1 - t_0)\right] && \text{for some } C_3 > 0. \end{aligned}$$

Hence,

$$\begin{aligned} |u(t)| \le C_1 \Bigg[& C_2 \exp\left[-\alpha(t_1 - t_0)\right] + C_3 \\ & + \int_{t_0}^{t_1} M \exp\left[-\alpha(t_1 - s)\right] \exp(-\beta s) \pi(x(\cdot), u(\cdot))\, ds \Bigg], \end{aligned}$$

and therefore

$$|u(t)| \le C_1 [C_2 + C_3] \exp\left[-\alpha(t_1 - t_0)\right] + KM \exp(-\alpha t_1)\Big], \tag{17}$$

since $\beta - \alpha \ge 0$ and $s \ge t_0 \ge 0$. From (17), t_1 can be chosen so large that $|u(t)| \le a$, $t \in [t_0, t_1]$ which proves that u is an admissible control for this choice of t_1.

It remains to prove the existence of a solution pair of the integral equations (15) and (16). Let B be the Banach space of all functions

$$(x, u) : [t_0 - h, t_1] \times [t_0 - h, t_1] \to E^n \times E^p$$

where $x \in B([t_0 - h, t_1], E^n)$ and $u \in L_2([t_0 - h, t_1], E^p)$ with the norm defined by

$$\begin{aligned} \|(x, u)\| &\le \|x\|_2 + \|u\|_2 \quad \text{where} \\ \|x\|_2 &= \left[\int_{t_0 - h}^{t_1} |x(s)|^2\, ds\right]^{1/2}, \\ \|u\|_2 &= \left[\int_{t_0 - h}^{t_1} |u(s)|^2\, ds\right]^{1/2}. \end{aligned}$$

Define the operator $T : B \to B$ by $T(x,u) = (y,v)$ where

$$v(t) = -G_m^{\mathsf{T}}(t_1,t)W^{-1}(t_0,t_1)q(t_1,\eta) \qquad \text{for } t \in J \equiv [t_0,t_1] \tag{18}$$

and

$$\begin{aligned} v(t) &= \eta(t) \qquad \text{for } t \in [t_0-\gamma,t_0], \\ y(t) &= x(t;t_0,\phi) + H(t,\eta) + \int_{t_0}^{t} G_m(t,s)v(s)\,ds \\ &\quad + \int_{t_0}^{t}\int_{-\gamma}^{0} X(t,s)A(\theta)x(t+\theta)\,d\theta\,ds \\ &\quad + \int_{t_0}^{t} X(t,s)f\big(s,x(\cdot),u(\cdot)\big)\,ds \qquad \text{for } t \in J \end{aligned} \tag{19}$$

and $y(t) = \phi(t)$ for $t \in [t_0-\gamma,t_0]$. From equation (17) it is clear that $|v(t)| \le a$, $t \in J$ and also $v : [t_0-h,t_0] \to U$, so $|v(t)| \le a$. Hence

$$\|v\|_2 \le a(t_1+h-t_0)^{1/2} = \beta_0.$$

Next

$$|y(t)| \le C_2 + C_3 \exp\,\big[-\alpha(t-t_0)\big] + C_4\int_{t_0}^{t}|v(s)|\,ds + KM\,\exp\,(-\alpha t_1)$$

where $C_4 = \sup|G_m(t,s)|$. Since $\alpha > 0$, $t \ge t_0 \ge 0$ it follows that

$$\begin{aligned} |y(t)| &\le C_2 + C_3 + C_4\,a(t_1-t_0) + KM \equiv \beta, && t \in J, \\ |y(t)| &\le \sup|\phi(t)| \equiv \delta, && t \in [t_0-h,t_0]. \end{aligned}$$

Hence, if $\lambda = \max\{\beta,\delta\}$, then

$$\|y\|_2 \le \lambda(t_1+h-t_0)^{1/2} \equiv \beta_1.$$

Let $r = \max\{\beta_0,\beta_1\}$. Then letting

$$Q(r) = \Big\{(x,u) \in B : \|x\|_2 \le r, \|u\|_2 \le r\Big\}$$

it follows that $T : Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riesz's theorem [18] it is relatively compact under T. The Schauder theorem implies that T has a fixed point $(x,u) \in Q(r)$. This fixed point (x,u) of T is a solution pair of the set of integral equations (18), (19). Hence the system (8) is Euclidean null controllable. ∎

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