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Estimation of time-dependent heat flux using temperature distribution at a point in a two layer system

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Good stability.

Abstract In this paper, the conjugate gradient method, coupled with the adjoint problem, is used in order to solve the inverse heat conduction problem and estimation of the time-dependent heat flux, using temperature distribution at a point in a two layer system. Also, the effect of noisy data on the final solution is studied. The numerical solution of the governing equations is obtained by employing a finite-difference technique. For solving this problem, the general coordinate method is used. The irregular region in the physical domain (r, z) is transformed into a rectangle in the computational domain (ξ, η). The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. The obtained results for few selected examples show the good accuracy of the presented method. Also, the solutions have good stability even if the input data includes noise. The problem is solved in an axisymmetric case. Applications of this model are in the thermal protect systems (t.p.s.) and heat shield systems.

1. Introduction

Direct heat conduction problems are concerned with the determination of temperature at interior points of a region, when the initial and boundary conditions, thermophysical properties and heat generation are specified [1]. In contrast to direct problems, Inverse Heat Conduction Problems (IHCP) are defined as the estimation of initial/boundary conditions, properties of the system/material, sources or sink terms, shape and governing equations from transient temperature measurements at one or several interior locations [2]. The solution of inverse problems is much more difficult in comparison with direct problems, due to instability in the solution, where these problems are called mathematically ill-posed. With the improvement of computer capability, inverse techniques have become a popular means of resolving heat transfer problems in the last decade. Important applications for inverse heat conduction problem solutions include, for example, controlled cooling of electronic components, estimation of jet-flow rate of cooling in machining or quenching, determination of conditions at the interface between the mold and metal during metal casting or rolling process [3], heat flux estimation on the surface of a wall subjected to fire or the inside surface of a combustion chamber [4] and, also, on surfaces where ablation takes place or on surfaces going through welding processes [5]. Some other applications of the IHCP are prediction of the inner wall temperature of a reactor, determination of the heat transfer coefficient and outer surface conditions in the re-entry of a space vehicle, modeling of the temperature or heat flux at the tool-work interface of machine cutting [6] and also in the transpiration cooling control [7], estimation of the unknown time-dependent heat flux and temperature distributions for the system composed of a multi-layer composite strip and semi-infinite foundation from the knowledge of temperature measurements taken within the strip [8], the regularization method for determining a moving boundary from Cauchy data in a one-dimensional heat equation with a multilayer domain [9], estimation of the boundary thermal behavior of a furnace with two layer walls [10], an input estimation method to recursively estimate both the time varied heat flux and the inner wall temperature in the chamber [11], and computation of the temperature field in multi-dimensional, multi-layer bodies [12].

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There are many different methods for solving inverse heat conduction problems and some of them will be listed here. For instance, the exact solution technique, the inversion of Duhamel's integral, Laplace transformation techniques, the control volume method, the use of the Helmholtz equation, the finite difference method and approaches, the digital filtering method, the Tikhonov regularization method, the Alifannov iterative regularization, etc. [13–17]. Jiang et al. [18] obtained the time-dependent boundary heat flux applied on a solid bar by using the conjugate gradient method with an adjoint equation and zeroth-order Tikhonov regularization to stabilize the inverse solution. They used the finite difference method to solve their problem. San Guay Chen et al. [19] calculated the heat flux and temperature distribution of the quenching surface, using the inverse method. They applied a conjugate gradient method to improve estimation of the distribution of the surface temperature and heat flux for a two-dimensional cylindrical coordinate problem, and solved the governing equations with the finite element method. Bao Liu [20] used a hybrid method to identify simultaneously the fluid thermal conductivity and heat capacity for a transient inverse heat transfer problem. Their proposed method is a combination of the modified genetic algorithm and the Levenberge Marquardt method.

In this research, we use the conjugate gradient method, coupled with an adjoint equation approach, to solve the inverse heat conduction problem and estimate the time-dependent heat flux using temperature distribution at a point in a two layer system. The problem is solved in an axisymmetric case and the general coordinate method is used. The irregular region in the physical domain (r, z) is transformed into a rectangle in the computational domain (ξ, η). The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. The governing equations are solved by employing the finite difference method. The obtained results show that the applied method causes high stability, even if the input data includes considerable noise. Applications of this model are in thermal protection systems (t.p.s.) and heat shield systems.

2. Problem formulation and solution

2.1. Direct problem

In this paper, we solve the problem in the axisymmetric cylindrical coordinate system (r, θ, z). The symmetry is with respect to the z-axis. Therefore, the derivative of any quantity, with respect to θ, would be zero. The application example could be any type of body nose. Therefore, the energy equation has actually been presented in the cylindrical coordinate system, (r, z), as in Figure 1. As shown, a time-dependent heat flux is applied to the outer surface, while the inner and side surfaces have been insulated. We aim to obtain the unknown heat flux $q(t)$ on the outer surface for the time, $0 \le t \le t_f$, using the temperature field at a point. The input data could include noise. In the numerical solution, the general coordinate method is applied. The calculations have been done in the rectangular coordinate system (ξ, η), initially, and the results are then transferred to the physical coordinate system (r, z). The computational plane and corresponding boundary conditions are shown in Figure 2. The heat conduction equation in the cylindrical coordinate system, in the axisymmetric case, with the initial and boundary conditions are as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)=\rho C_P\frac{\partial T}{\partial t}, \tag{1}$$

$$T_z=\frac{1}{J}(r_\eta T_\xi - r_\xi T_\eta), \tag{2}$$

$$T_r=\frac{1}{J}(-z_\eta T_\xi + z_\xi T_\eta), \tag{3}$$

$$\nabla^2 T = \frac{1}{J^2}\left[\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta}\right] + \left[(\nabla^2\xi)T_\xi + (\nabla^2\eta)T_\eta\right], \tag{4}$$

$$\alpha = z_\eta^2 + r_\eta^2, \tag{5}$$

$$\beta = z_\xi z_\eta + r_\xi r_\eta, \tag{6}$$

$$\gamma = z_\xi^2 + r_\xi^2, \tag{7}$$

$$\nabla^2\xi = \frac{k_1(r_{\xi\xi}z_\eta - z_{\xi\xi}r_\eta) + k_2(r_{\xi\eta}z_\eta - z_{\xi\eta}r_\eta)}{J} + \frac{k_3(r_{\eta\eta}z_\eta - z_{\eta\eta}r_\eta)}{J}, \tag{8}$$

$$\nabla^2\eta = \frac{k_1(z_{\xi\xi}r_\xi - r_{\xi\xi}z_\xi) + k_2(z_{\xi\eta}r_\xi - r_{\xi\eta}z_\xi)}{J} + \frac{k_3(z_{\eta\eta}r_\xi - r_{\eta\eta}z_\xi)}{J}, \tag{9}$$

$$k_1 = \frac{1}{J^2}(z_\eta^2 + r_\eta^2), \tag{10}$$

$$k_2 = \frac{-2}{J^2}(z_\xi z_\eta + r_\xi r_\eta), \tag{11}$$

$$k_3 = \frac{1}{J^2}(z_\xi^2 + r_\xi^2), \tag{12}$$

$$\xi_z = \frac{1}{J}r_\eta, \tag{13}$$

$$\xi_r = -\frac{1}{J}z_\eta, \tag{14}$$

$$\eta_z = -\frac{1}{J}r_\xi, \tag{15}$$

$$\eta_r = \frac{1}{J}z_\xi, \tag{16}$$

$$J = z_\xi r_\eta - r_\xi z_\eta, \tag{17}$$

where the subscripts denote differentiation, with respect to the variable considered:

$$\frac{\partial T}{\partial z} = 0 \qquad \xi = 1, \qquad \xi = nz, \quad t > 0, \tag{18}$$

$$\frac{\partial T}{\partial r} = 0 \qquad \eta = 1, \quad t > 0, \tag{19}$$

$$k\frac{\partial T}{\partial r} = q(t) \qquad \eta = nr, \quad t > 0, \tag{20}$$

$$T(\xi, \eta, 0) = 0 \quad 1 < \xi < nz,\ 1 < \eta < nr,\ t = 0. \tag{21}$$

In the above relations T, t, $q(t)$, ρ, k and C_p are temperature, time, time-dependent heat flux, density, thermal conductivity and specified thermal capacity, respectively. In the interface of materials, the following relations are used:

$$q_{\xi\,\text{in}} + q_{\eta\,\text{in}} = q_{\xi\,\text{out}} + q_{\eta\,\text{out}}, \tag{22}$$

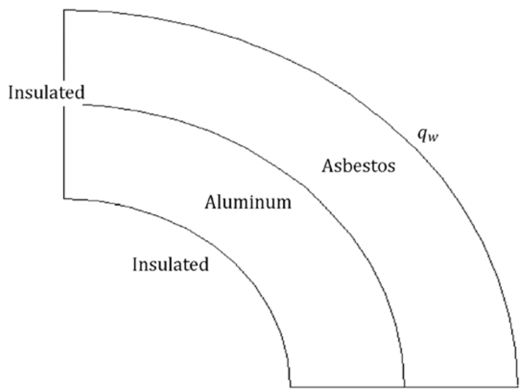

Figure 1: Geometry of the problem and boundary condition.

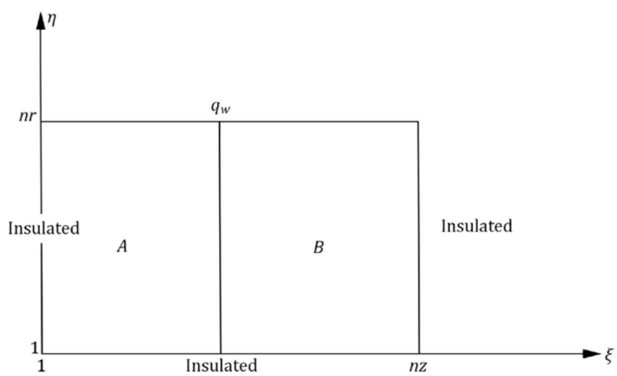

Figure 2: Boundary condition in computational plane.

$$k_A(T_{i,j}-T_{i-1,j})+\frac{2k_Ak_B}{k_A+k_B}(T_{i,j}+T_{i-1,j})$$
$$=k_B(T_{i+1,j}-T_{i,j})+\frac{2k_Ak_B}{k_A+k_B}(T_{i,j+1}+T_{i,j}). \tag{23}$$

As shown in Figure 3, by considering a boundary element in the physical plane and applying the energy equation, the boundary conditions are calculated as follows:

$$k_{i,j}ds_1\frac{T^{n-1}_{nz-1,j}-T^{n-1}_{nz,j}}{ds_2}+k_{i,j}\frac{ds_2}{2}\frac{T^{n-1}_{nz,j-1}-T^{n-1}_{nz,j}}{ds_1}$$
$$+k_{i,j}\frac{ds_2}{2}\frac{T^{n-1}_{nz,j+1}-T^{n-1}_{nz,j}}{ds_1}+q_w ds_1$$
$$=\rho_{i,j}C_{i,j}ds_1\frac{ds_2}{2}\left(\frac{T^n_{nz,j}-T^{n-1}_{nz,j}}{\Delta t}\right), \tag{24}$$

$$T^n_{nz,j}=\frac{1}{F\left(T^{n-1}_{nz,j}+A1+A2+\left(\frac{2q_w\Delta t}{\rho_{i,j}C_{i,j}ds_2}\right)\right)}, \tag{25}$$

where $T^n_{nz,j}$ in the above relation is:

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1nz,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2nz,j}}, \tag{26}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t T^{n-1}_{nz-1,j}}{ds^2_{2nz,j}}, \tag{27}$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(T^{n-1}_{nz,j+1}+T^{n-1}_{nz,j-1}\right)}{ds^2_{1nz,j}}, \tag{28}$$

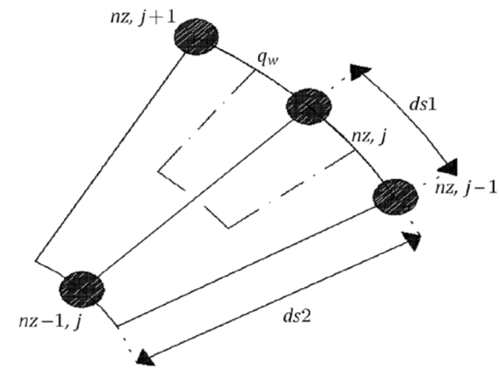

Figure 3: Boundary element in physical plane.

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}. \tag{29}$$

With a similar method for other boundary conditions, we have:

$$T^n_{1,j}=\frac{1}{F\left(T^{n-1}_{1,j}+A1+A2\right)}, \tag{30}$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{11,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{21,j}}, \tag{31}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t T^{n-1}_{2,j}}{ds^2_{21,j}}, \tag{32}$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(T^{n-1}_{1,j+1}+T^{n-1}_{1,j-1}\right)}{ds^2_{11,j}}, \tag{33}$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \tag{34}$$

$$T^n_{i,1}=\frac{1}{F\left(T^{n-1}_{i,1}+A1+A2\right)}, \tag{35}$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1i,1}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2i,1}}, \tag{36}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t T^{n-1}_{i,2}}{ds^2_{1i,1}}, \tag{37}$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(T^{n-1}_{i+1,1}+T^{n-1}_{i-1,1}\right)}{ds^2_{2i,1}}, \tag{38}$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}. \tag{39}$$

2.2. Inverse problem

In inverse problems, the time-dependent heat flux, using measured transient temperatures, is estimated with a sensor positioned at a point. The inverse problem should be solved as the following function is minimized [21]:

$$S[q(t)]=\frac{1}{2}\int_{t=0}^{t_f}\sum_{m=1}^{NS}\left[T(\xi_m,\eta_m,t;q)-Y_m(t)\right]^2\mathrm{d}t. \tag{40}$$

In the above relation, $T(\xi_m,\eta_m,t;q)$, and $Y_m(t)$ are estimated temperatures and measured temperature, respectively. Also, the number of sensors, Ns, is equal to 1.

The above equation will be minimized by using the conjugate gradient method, based on iterative processes. In the conjugate algorithm, the direction of seeking the unknown heat flux is dependent on the gradient of the error function, which will be solved with adjoint equations [15,16,21]:

2.3. Adjoint problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial\lambda}{\partial r}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial\lambda}{\partial z}\right)+\sum_{m=1}^{NS}[T(\xi,\eta,t;q)-Y_m(t)]\,\delta(\eta-\eta_m)\delta(\xi-\xi_m)=\rho C_P\frac{\partial\lambda}{\partial t}, \quad (41)$$

$$\frac{\partial\lambda}{\partial z}=0 \qquad \xi=1, \qquad \xi=nz, \quad t>0, \quad (42)$$

$$\frac{\partial\lambda}{\partial r}=0 \qquad \eta=1, \qquad \eta=nz, \quad t>0, \quad (43)$$

$$\lambda(\xi,\eta,t_f)=0, \quad 1<\xi<nz,\ 1<\eta<nr,\ t=t_f. \quad (44)$$

In the interface of materials, the following relation is used:

$$k_A(\lambda_{i,j}-\lambda_{i-1,j})+\frac{2k_Ak_B}{k_A+k_B}(\lambda_{i,j}+\lambda_{i-1,j})=k_B(\lambda_{i+1,j}-\lambda_{i,j})+\frac{2k_Ak_B}{k_A+k_B}(\lambda_{i,j+1}+\lambda_{i,j}). \quad (45)$$

By considering a boundary element in the physical plane and applying the energy equation, the boundary conditions are calculated as follows:

$$k_{i,j}ds_1\frac{\lambda^{n-1}_{nz-1,j}-\lambda^{n-1}_{nz,j}}{ds_2}+k_{i,j}\frac{ds_2}{2}\frac{\lambda^{n-1}_{nz,j-1}-\lambda^{n-1}_{nz,j}}{ds_1}+k_{i,j}\frac{ds_2}{2}\frac{\lambda^{n-1}_{nz,j+1}-\lambda^{n-1}_{nz,j}}{ds_1}=\rho_{i,j}C_{i,j}ds_1\frac{ds_2}{2}\left(\frac{\lambda^{n}_{nz,j}-\lambda^{n-1}_{nz,j}}{\Delta t}\right), \quad (46)$$

where $\lambda^n_{nz,j}$ in the above relation is as:

$$\lambda^n_{nz,j}=\frac{1}{F\left(\lambda^{n-1}_{nz,j}+A1+A2\right)}, \quad (47)$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1nz,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2nz,j}}, \quad (48)$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\lambda^{n-1}_{nz-1,j}}{ds^2_{2nz,j}}, \quad (49)$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(\lambda^{n-1}_{nz,j+1}+\lambda^{n-1}_{nz,j-1}\right)}{ds^2_{1nz,j}}, \quad (50)$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \quad (51)$$

$$\lambda^n_{1,j}=\frac{1}{F\left(\lambda^{n-1}_{1,j}+A1+A2\right)}, \quad (52)$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{11,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{21,j}}, \quad (53)$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\lambda^{n-1}_{2,j}}{ds^2_{21,j}}, \quad (54)$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(\lambda^{n-1}_{1,j+1}+\lambda^{n-1}_{1,j-1}\right)}{ds^2_{11,j}}, \quad (55)$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}. \quad (56)$$

With a similar method for other boundary conditions, we have:

$$\lambda^n_{i,1}=\frac{1}{F\left(\lambda^{n-1}_{i,1}+A1+A2\right)}, \quad (57)$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1i,1}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2i,1}}, \quad (58)$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\lambda^{n-1}_{i,2}}{ds^2_{1i,1}}, \quad (59)$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(\lambda^{n-1}_{i+1,1}+\lambda^{n-1}_{i-1,1}\right)}{ds^2_{2i,1}}, \quad (60)$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \quad (61)$$

where parameter λ is the adjoint temperature and δ is the dirac delta function. The optimum step size can be obtained based on the sensitivity problem, which is defined as in [16,21].

2.4. Sensitivity problem

To obtain the sensitivity equation, it is assumed that perturbing $q(t)$ by $\Delta q(t)$ would change $T(r,z,t)$ by $\Delta T(r,z,t)$. Thus, in a direct problem, the quantities $[T(r,z,t)+\Delta T(r,z,t)]$ and $[q(t)+\Delta q(t)]$ are replaced by $T(r,z,t)$ and $q(t)$, and the resulting expression is subtracted from the direct problem [21]. In this way, the sensitivity equation is obtained as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial\Delta T}{\partial r}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial\Delta T}{\partial z}\right)=\rho C_P\frac{\partial\Delta T}{\partial t}, \quad (62)$$

$$\frac{\partial\Delta T}{\partial z}=0 \qquad \xi=1, \qquad \xi=nz, \quad t>0, \quad (63)$$

$$\frac{\partial\Delta T}{\partial r}=0 \qquad \eta=1, \quad t>0, \quad (64)$$

$$k\frac{\partial\Delta T}{\partial r}=\Delta q \qquad \eta=nr, \quad t>0, \quad (65)$$

$$\Delta T(\xi,\eta,0)=0, \quad 1<\xi<nz,\ 1<\eta<nr,\ t=0. \quad (66)$$

In the interface of materials, below relation is used:

$$k_A(\Delta T_{i,j}-\Delta T_{i-1,j})+\frac{2k_Ak_B}{k_A+k_B}(\Delta T_{i,j}+\Delta T_{i-1,j})=k_B(\Delta T_{i+1,j}-\Delta T_{i,j})+\frac{2k_Ak_B}{k_A+k_B}(\Delta T_{i,j+1}+\Delta T_{i,j}). \quad (67)$$

As explained before, by considering a boundary element in the physical plane and applying energy balance relations, the boundary conditions are calculated as follows:

$$k_{i,j}ds_1\frac{\Delta T^{n-1}_{nz-1,j}-\Delta T^{n-1}_{nz,j}}{ds_2}+k_{i,j}\frac{ds_2}{2}\frac{\Delta T^{n-1}_{nz,j-1}-\Delta T^{n-1}_{nz,j}}{ds_1}$$

$$+k_{i,j}\frac{ds_2}{2}\frac{\Delta T^{n-1}_{nz,j+1}-\Delta T^{n-1}_{nz,j}}{ds_1}+\Delta q ds_1$$
$$=\rho_{i,j}C_{i,j}ds_1\frac{ds_2}{2}\left(\frac{\Delta T^{n}_{nz,j}-\Delta T^{n-1}_{nz,j}}{\Delta t}\right). \tag{68}$$

From which $\Delta T^n_{nz,j}$ is calculated as below:

$$\Delta T^n_{nz,j}=\frac{1}{F\left(\Delta T^{n-1}_{nz,j}+A1+A2+\left(\frac{2\Delta q\Delta t}{\rho_{i,j}C_{i,j}ds_2}\right)\right)}, \tag{69}$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1nz,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2nz,j}}, \tag{70}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\Delta T^{n-1}_{nz-1,j}}{ds^2_{2nz,j}}, \tag{71}$$

$$A_2=\frac{\alpha_{i,j}\Delta t(\Delta T^{n-1}_{nz,j+1}+\Delta T^{n-1}_{nz,j-1})}{ds^2_{1nz,j}}, \tag{72}$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \tag{73}$$

$$\Delta T^n_{1,j}=\frac{1}{F\left(\Delta T^{n-1}_{1,j}+A1+A2\right)}, \tag{74}$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{11,j}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{21,j}}, \tag{75}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\Delta T^{n-1}_{2,j}}{ds^2_{21,j}}, \tag{76}$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(\Delta T^{n-1}_{1,j+1}+\Delta T^{n-1}_{1,j-1}\right)}{ds^2_{11,j}}. \tag{77}$$

Other boundary conditions are obtained in a similar manner as:

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \tag{78}$$

$$\Delta T^n_{i,1}=\frac{1}{F\left(\Delta T^{n-1}_{i,1}+A1+A2\right)}, \tag{79}$$

$$F=1+\frac{2\alpha_{i,j}\Delta t}{ds^2_{1i,1}}+\frac{2\alpha_{i,j}\Delta t}{ds^2_{2i,1}}, \tag{80}$$

$$A_1=\frac{2\alpha_{i,j}\Delta t\Delta T^{n-1}_{i,2}}{ds^2_{1i,1}}, \tag{81}$$

$$A_2=\frac{\alpha_{i,j}\Delta t\left(\Delta T^{n-1}_{i+1,1}+\Delta T^{n-1}_{i-1,1}\right)}{ds^2_{2i,1}}, \tag{82}$$

$$\alpha_{i,j}=\frac{k_{i,j}}{C_{i,j}}, \tag{83}$$

where ΔT is the sensitivity temperature.

The transient heat flux, $q(t)$, which is an unknown function, can be estimated by minimizing the function, $S[q(t)]$, in Eq. (40). The iterative equation for estimating $q(t)$ is as below [15,17,21]:

$$q^{k+1}(t)=q^k(t)-\beta^k d^k(t). \tag{84}$$

In which k is the number of iterations. The direction of descent, $d^k(t)$, is determined as in [15,17,21]:

$$d^k(t)=\nabla S[q^k(t)]+\gamma^k d^{k-1}(t). \tag{85}$$

Here, γ^k is the conjugate coefficient, as in [16,18,21], which are calculated by:

$$\gamma^k=\frac{\int_{t=0}^{t_f}\left\{\nabla S\left[q^k(t)\right]\right\}^2 \mathrm{d}t}{\int_{t=0}^{t_f}\left\{\nabla S\left[q^{k-1}(t)\right]\right\}^2 \mathrm{d}t}, \tag{86}$$

where γ^0 is assumed zero. To calculate $\nabla S[q^k(t)]$, the following relation is used:

$$\nabla S[q(t)]=\lambda(\xi,nr,t). \tag{87}$$

The above equality depends on the position of the unknown function. The search step-size, β^k, is obtained by minimizing $S[q^{k+1}(t)]$, with respect to β^k, as follows [16,18,21]:

$$\beta^k=\frac{\int_{t=0}^{t_f}\sum_{m=1}^{NS}\left[T\left(\xi_m,\eta_m,t;q^k\right)-Y_S(t)\right]\Delta T\left(\xi_m,\eta_m,t;d^k\right)\mathrm{d}t}{\int_{t=0}^{t_f}\sum_{m=1}^{NS}\left[\Delta T\left(\xi_m,\eta_m,t;d^k\right)\right]^2\mathrm{d}t}, \tag{88}$$

where $\Delta T(\xi_m,\eta_m,t;d^k)$ is obtained from the sensitivity problem by considering $\Delta q^k(t)=d^k(t)$.

By checking Eq. (87), it is determined that the gradient equation in final time (t_f) is equal to zero. Therefore, the initial guess used for $q(t)$ in $t=t_f$ does not change with the iterative process in the conjugate gradient method. When the initial guess is very far from the exact solution, the estimated function in the neighborhood of t_f can deviate from the exact solution. This solution can be eliminated easily by use of a larger value of final time. Thus, the effect of an initial guess on the actual time of the problem is not significant [21]. The iterative procedure mentioned above continues until the stopping criterion is satisfied. The stopping criterion is defined as follows:

$$S[q(t)]\le\varepsilon. \tag{89}$$

In the above relation, $S[q(t)]$ is obtained from Eq. (40). The value of ε should be selected such that if there were errors in the measured data, the accuracy of the results would be satisfactory.

2.5. Computational algorithm

The computational procedure for obtaining the unknown heat flux can be summarized as follows [21]:

1. Choose an initial guess, for example $q^0(t)$ for the function $q(t)$, and set $k=0$.
2. Solve the direct problem to obtain $T(z,r,t)$ based on $q^k(t)$ (Eqs. (1)–(23)).
3. Check the stopping criterion and continue if not satisfied (Eq. (89)).
4. Solve the adjoint equation, and compute $\lambda(\xi,nr,t)$ by knowing $T(\xi_m,\eta_m,t)$ and the measured temperature, $Y_m(t)$ (Eqs. (41)–(45)).
5. Knowing $\lambda(\xi,nr,t)$, compute $\nabla S[q^k(t)]$ from Eq. (87).
6. Knowing $\nabla S[q^k(t)]$, compute γ^k from Eq. (86) and $d^k(t)$ from Eq. (85).
7. Set $\Delta q^k(t)=d^k(t)$ and solve the sensitivity problem to obtain $\Delta T(\xi_m,\eta_m,t;d^k)$ (Eqs. (62)–(66)).
8. Knowing $\Delta T(\xi_m,\eta_m,t;d^k)$, compute β^k from Eq. (88).
9. Knowing β^k and $d^k(t)$, compute $q^{k+1}(t)$ and return to step 2 (Eq. (84)).

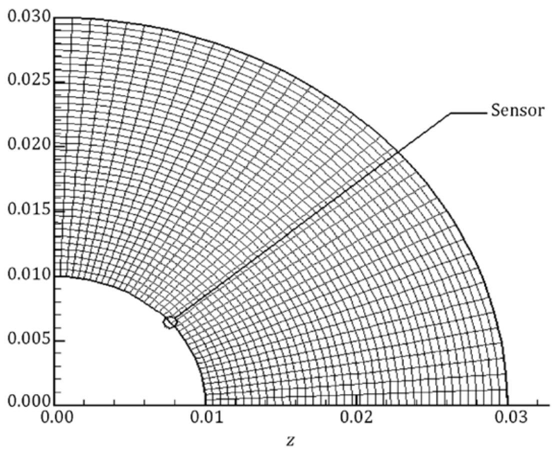

Figure 4: Using grid in solution of problem and sensor position.

3. Results and discussions

We aim to estimate the unknown heat flux in a two layer system, using the conjugate gradient method, when there is no information about the unknown function. It should be noted that in the conjugated gradient method, the initial guess for the unknown function is arbitrary; in other words, the method is independent of an initial guess. Here, initial estimation of the heat flux is assumed zero. The governing equations were discredited by the finite-difference method and the mesh size used in the numerical method is a uniform 35×35, 45×45 and 55×55. This all shows that the problem is independent of mesh size, but by noting the calculation time, we chose the 35×35 mesh size. The final time, $t_f = 10$, and time step, $\Delta t = 0.01$, are considered. In this work, by measuring the temperature at a point only in the inner layer, the heat flux on the outer layer is estimated, and the sensitivity of the problem for noisy data is investigated. In Figure 4, the mesh used and the position of the sensor are shown.

To investigate the accuracy of the presented solution, a step-function is considered as:

$$q(t) = \begin{cases} 10^7 & \text{for } 4 < t < 8 \\ 0 & \text{for } t \leq 4 \text{ and } t \geq 8. \end{cases}$$

One should note that the discontinuous and sharp corner functions are well known for being highly ill-posed (see Figure 5). Therefore, these functions can be used to evaluate the accuracy of the solutions [21].

In the next example, a sinusoidal function is considered for the heat flux as $q(t) = 10^7 \sin(\pi t)$, presented in Figure 6.

In the next example, a combination of sine and cosine functions is considered for the heat flux as:

$$q(t) = 10^7 \sin(0.1t) + 10^7 \cos(2t),$$

with results depicted in Figure 7.

As the last example, a triangle function is considered for the heat flux, with results shown in Figure 8.

In the next part, the inverse solution, with noisy data, is presented. In practice, there are errors in measured data; therefore, noisy data are used to simulate errors by using data with 6% noise. The effect of noisy data can be seen in Figures 9–12 in comparison to noiseless cases presented in Figures 5–8. It is found that despite a noise in data, results have very good stability.

The mesh study has been done for $q(t) = 10^7 \sin(0.1t) + 10^7 \cos(2t)$, using three mesh sizes: 35×35, 45×45 and

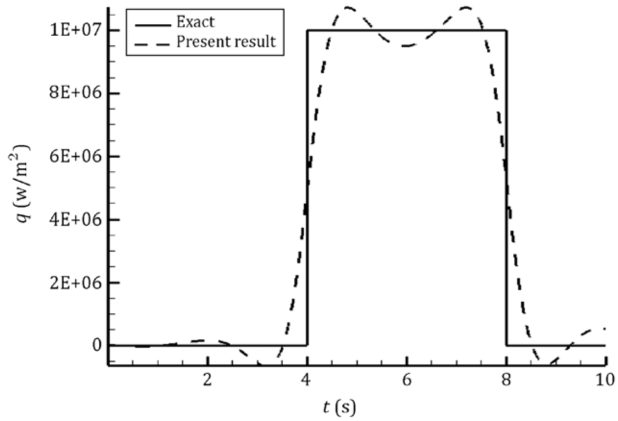

Figure 5: Estimated heat flux in comparison with exact function for step-function.

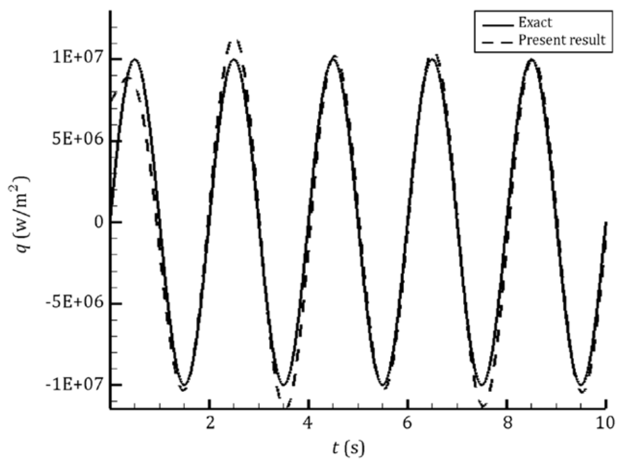

Figure 6: Estimated heat flux in comparison with exact function for sine function.

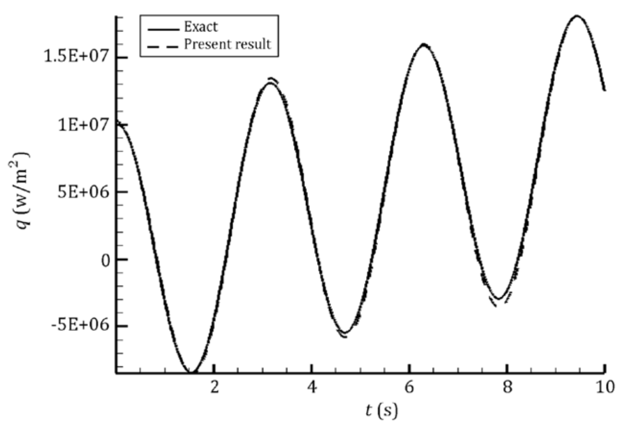

Figure 7: Estimated heat flux in comparison with exact function for a combination of sine and cosine functions.

55×55. As seen in Figure 13, the exact heat flux is recovered by the inverse solution using all mesh sizes, thus the results are independent of mesh size.

4. Conclusions

The conjugate gradient method with an adjoint problem has been successfully applied for the solution of inverse heat conduction to estimate the unknown time-dependent

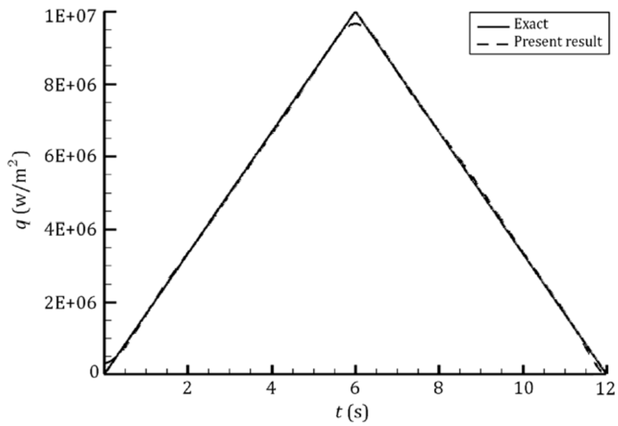

Figure 8: Estimated heat flux in comparison with exact function for triangle function.

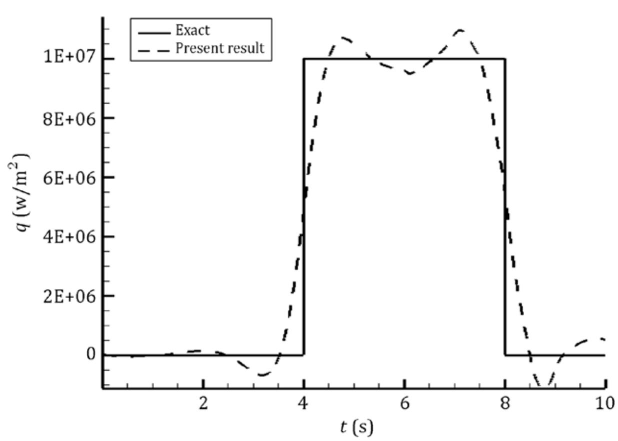

Figure 9: Estimated heat flux with noisy data in comparison with exact function for step-function.

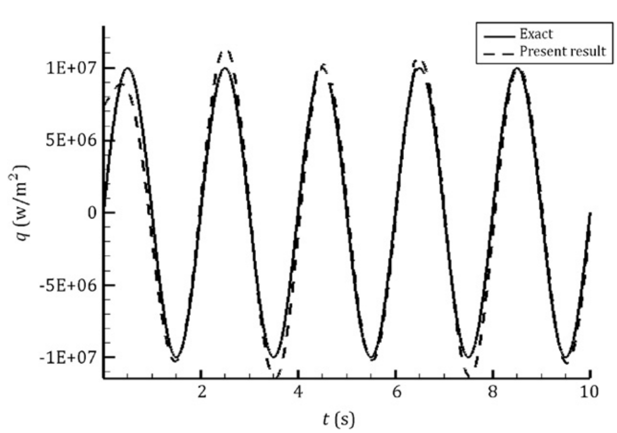

Figure 10: Estimated heat flux with noisy data in comparison with exact function for sine function.

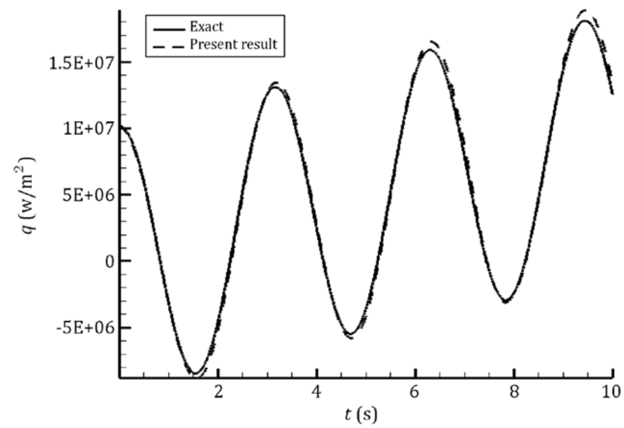

Figure 11: Estimated heat flux with noisy data in comparison with exact function for a combination of sine and cosine functions.

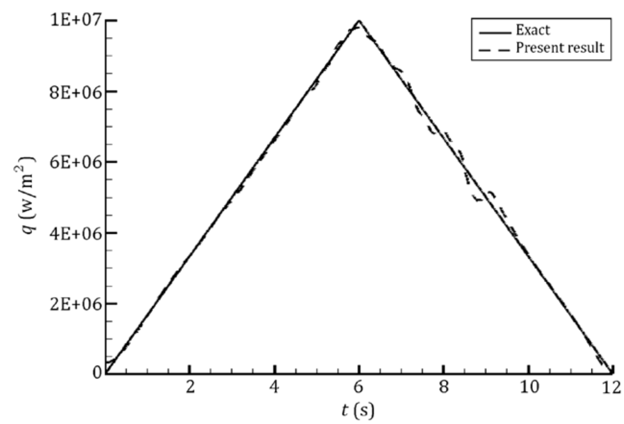

Figure 12: Estimated heat flux with noisy data in comparison with exact function for triangle function.

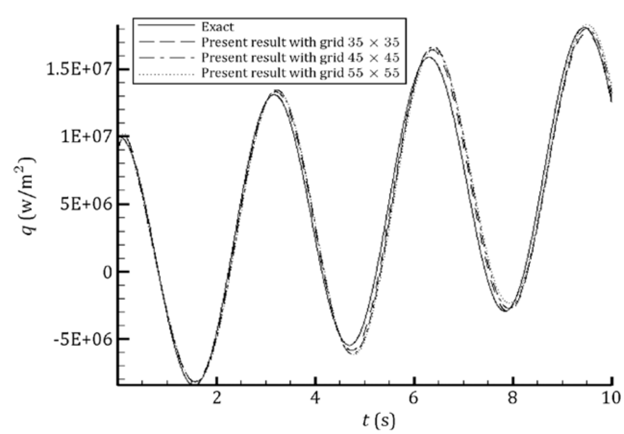

Figure 13: Mesh size study on obtained results.

heat flux using temperature distribution at a point in a two layer system, and the general coordinate method is used. The present formulation is general and can be applied to the solution of boundary inverse heat conduction problems over any region that can be mapped into a rectangle. In this paper, discontinuous and sharp corner functions that are well known for being highly ill-posed were used for illustrating the good accuracy of the presented method. The obtained results show that the presented solution has good stability when there is noise in input data up to 6%. Therefore, the presented method is good for estimating the time-dependent unknown heat flux in multi layer systems.

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