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# Computing left Kan extensions

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**Abstract**

We describe a new extension of the Todd–Coxeter algorithm adapted to computing left Kan extensions. The algorithm is a much simplified version of that introduced by Carmody and Walters (Category Theory, Proceedings of the International Conference Held in Como, Italy, 22–28 July 1990. Springer) in 1991. The simplification allows us to give a straightforward proof of its correctness and termination when the extension is finite. © 2003 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

Many algorithms have a natural description in the following form. The algorithm concerns a finite presentation of a possibly-infinite algebraic structure. A finite number of ways of modifying presentations are given, called the *actions* of the algorithm. These actions leave invariant the algebra presented by the presentation. A run of the algorithm consists in applying a sequence of these actions, in a particular order, to a given presentation with the idea of simplifying it to the point that the answer to certain questions about the presented algebra become apparent. The correctness of the algorithm follows from the invariance of the presented algebra under the actions. Its termination is a consequence of the particular sequence of actions chosen. Having algorithms in this form separates the questions of correctness, termination, and efficiency. The crucial steps in describing and proving algorithms in this way are finding the appropriate notion of finite presentation, and finding the simplest possible actions which leave invariant the algebra presented.

Two classical examples of algorithms which may be considered in this form are Euclid's algorithm (which concerns the presentation of an ideal in $\mathbf{Z}$) and Gaussian elimination (which concerns the presentation of a linear transformation between two vector spaces). In this paper we describe another example, namely an algorithm for computing the left

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Kan extension of a functor $X : \mathbf{A} \to \mathbf{Sets}$ along a functor $F : \mathbf{A} \to \mathbf{B}$. The state on which the algorithm acts is a finite presentation of a functor $L : \mathbf{B} \to \mathbf{Sets}$ and a natural transformation $\mu : X \to LF$.

The original Todd–Coxeter algorithm, on which the present algorithm is based, concerned a finite presentation of the cosets of a subgroup H in a group G, in terms of certain *tables*. Though this notion of finite generation has been clarified in subsequent works (see, for example, Sims, 1994) the algorithm has never been described and proved in the form outlined in the first paragraph. The reason is that the particular form of presentation taken has always necessitated a recursive subprocedure called *dealing with coincidences* which is perhaps the most obscure part of the algorithm. The essential novelty of this paper, apart from its greater generality in dealing with left Kan extensions rather than the enumeration of cosets, is that we introduce a new notion of finite presentation which removes the need for the subprocedure for dealing with coincidences thus clarifying the algorithm substantially.

The history of our algorithm is as follows. Todd and Coxeter described their coset enumeration algorithm in 1936 (Todd and Coxeter, 1936; see also Coxeter and Moser, 1957). It was perhaps the first abstract algebra algorithm actually implemented on an electronic computer by Haselgrove in 1953 on EDSAC1 in Cambridge (Leech, 1963). The most encyclopaedic reference to later developments in coset enumeration is Sims (1994). The algorithm was extended to the computation of left Kan extension by Carmody and Walters (1991) and Walters (1991). The paper (Carmody and Walters, 1991) contains a proof of the "dealing with coincidences" subprocedure. The algorithm was extended further to left Kan extensions of product-preserving functors in Leeming and Walters (1995).

In trying to find a simple enough presentation of the last algorithm to give a proof of completeness we were led to the flat version in this paper, avoiding the "dealing with coincidences" subprocedure. We were strongly influenced by a conversation with Rod Burstall about a concurrent garbage collection algorithm, and by ideas in Chandy and Misra (1988).

## 2. Congruences and quotients

Throughout this section let $\mathbf{B}$ be an arbitrary category.

**Definition 2.1.** Given a functor $L : \mathbf{B} \longrightarrow \mathbf{Sets}$, a *family of relations on* $L$ is a family $R = \{R_B \subseteq LB \times LB\}_{B \in \mathbf{B}}$. A family $E = \{\sim_B\}_{B\in\mathbf{B}}$ *is called a* congruence on $L$ *if it satisfies*

(i) $\sim_B$ is an equivalence relation for each $B \in \mathbf{B}$.

(ii) If $g : B \to B'$ is a morphism in $\mathbf{B}$ then $x \sim_B y \Rightarrow Lg(x) \sim_{B'} Lg(y)$.

**Definition 2.2.** If $R = \{R_B\}_{B\in\mathbf{B}}$ and $S = \{S_B\}_{B\in\mathbf{B}}$ are two families of relations on $L$ then we say $R \subseteq S$ iff $R_B \subseteq S_B$ for each $B \in \mathbf{B}$.

**Lemma 2.1.** *Let* $L : \mathbf{B} \longrightarrow \mathbf{Sets}$ *be a functor and* $\{R_B\}_{B\in\mathbf{B}}$ *a family of relations on* $L$. *Suppose for each* $i \in I$ *that* $E_i = \{\sim^i_B\}_{B\in\mathbf{B}}$ *is a congruence on* $L$ *containing* $\{R_B\}_{B\in\mathbf{B}}$. *Define for each* $B \in \mathbf{B}$

$$\sim_B = \bigcap_{i\in I} \sim^i_B .$$

*Then* $E = \{\sim_B\}_{B\in\mathbf{B}}$ *is a congruence on* $L$ *containing* $\{R_B\}_{B\in\mathbf{B}}$.

**Proof.** Given $B \in \mathbf{B}$ we have that $R_B \subseteq \sim^i_B$ for each $i \in I$, and so $R_B \subseteq \sim_B$. Also, $\sim_B$ is the intersection of an indexed family of equivalence relations hence $\sim_B$ is an equivalence relation. Finally, if $g : B \to B'$ is a morphism in $\mathbf{B}$ then

$$\begin{aligned} x \sim_B y &\Rightarrow x \sim^i_B y \qquad \text{for each } i \in I \\ &\Rightarrow Lg(x) \sim^i_{B'} Lg(y) \qquad \text{for each } i \in I \\ &\Rightarrow Lg(x) \sim_{B'} Lg(y). \end{aligned}$$

Thus $E = \{\sim_B\}_{B\in\mathbf{B}}$ is a congruence containing $\{R_B\}_{B\in\mathbf{B}}$. □

By Lemma 2.1 we can now define the *minimal congruence on* $L$ *containing* $\{R_B\}_{B\in\mathbf{B}}$ as the family $E = \{\sim_B\}_{B\in\mathbf{B}}$ with $\sim_B$ defined as above where $I$ indexes the collection of *all* congruences on $L$ containing $\{R_B\}_{B\in\mathbf{B}}$. We now give an alternative characterization of this minimal congruence.

**Proposition 2.1.** *Let* $\{\sim_B\}_{B\in\mathbf{B}}$ *be the minimal congruence on* $L$ *containing* $\{R_B\}_{B\in\mathbf{B}}$, *then* $m \sim_B n$ *iff*

1. $m = n$ *or.*
2. *There exist* $u_1, \ldots, u_s \in LB$ *with* $u_1 = m$ *and* $u_s = n$ *such that for each* $i = 1, \ldots, s-1$ *there exists a morphism* $h_i : B_i \to B$ *in* $\mathbf{B}$ *with* $d_i R_{B_i} e_i$ *or* $e_i R_{B_i} d_i$ *where* $Lh_i(d_i) = u_i$ *and* $Lh_i(e_i) = u_{i+1}$.

**Proof.** Define a family of relations $\{T_B\}_{B\in\mathbf{B}}$ by $mT_Bn$ if and only if condition 1 or 2 in the statement of the proposition holds.

We check that $\{T_B\}_{B\in\mathbf{B}}$ is a congruence on $L$ containing $\{R_B\}_{B\in\mathbf{B}}$. If $mR_Bn$ then $mT_Bn$ by condition 2 above, thus $R_B \subseteq T_B$. If $m = n$ then $mT_Bn$ by condition 1 above, thus $T_B$ is reflexive. It is also clear from the nature of condition 2 that each $T_B$ is symmetric and transitive, hence $T_B$ is an equivalence relation for each $B \in \mathbf{B}$. Next, suppose $g : B \to B'$ is a morphism in $\mathbf{B}$ with $x = Lg(m)$ and $y = Lg(n)$. If $mT_Bn$ then either condition 1 or 2 of the construction above must hold. In the first case $m = n$, but then $x = Lg(m) = Lg(n) = y$ and so $xT_{B'}y$. In the second case we have elements $u_1, \ldots, u_s \in LB$ with $u_1 = m$ and $u_s = n$ such that for each $i = 1, \ldots, s-1$ there exists a morphism $h_i : B_i \to B$ in $\mathbf{B}$ with $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i$ where $Lh_i(d_i) = u_i$ and $Lh_i(e_i) = u_{i+1}$. Define $w_i = Lg(u_i)$ for each $i = 1, \ldots, s-1$. It is clear that for each $i$ we have $d_i R_B e_i$ or $e_i R_{B_i} d_i$ where $L(gh_i)(d_i) = w_i$ and $L(gh_i)(e_i) = w_{i+1}$, but then by condition 2 we have $xT_{B'}y$. Thus $\{T_B\}_{B\in\mathbf{B}}$ is a congruence on $L$ containing $\{R_B\}_{B\in\mathbf{B}}$ and so by the minimality of $\{\sim_B\}_{B\in\mathbf{B}}$ it follows that $\sim_B \subseteq T_B$ for each $B \in \mathbf{B}$.

Now we check that $T_B \subseteq \sim_B$ for each $B \in \mathbf{B}$. Suppose that $mT_Bn$, if $m = n$ then $m \sim_B n$ since $\sim_B$ is an equivalence relation. If $m \neq n$ then by definition there must exist $u_1, \ldots, u_s \in L_B$ with $u_1 = m$ and $u_s = n$ satisfying condition 2 above. But $R_B \subseteq \sim_B$ and $\{\sim_B\}_{B\in\mathbf{B}}$ is a congruence thus $u_1 \sim_B u_{i+1}$ for each $i$, which implies $m \sim_B n$. Hence $T_B \subseteq \sim_B$ and so $T_B = \sim_B$ for each $B \in \mathbf{B}$. □

**Remark 2.1.** As a special case, if **B** is a monoid with only one morphism (the identity) then Proposition 2.1 is the usual characterization of the minimal equivalence relation containing a given relation.

**Proposition 2.2.** *Given a functor* $L : \mathbf{B} \longrightarrow \mathbf{Sets}$ *and a congruence* $\{\sim_B\}_{B\in\mathbf{B}}$, *the following defines a new functor* $\hat{L} : \mathbf{B} \longrightarrow \mathbf{Sets}$. *For each* $B \in \mathbf{B}$

$$\hat{L}B = LB/\sim_B$$

*and for each morphism* $g : B \to B'$ *in* **B**

$$(\hat{L}g)[x] = [Lg(x)]$$

*where* $[x]$ *is the equivalence class with respect to* $\sim_B$ *of* $x \in LB$. *The functor* $\hat{L}$ *will be called the quotient of* $L$ *by* $\{\sim_B\}_{B\in\mathbf{B}}$.

**Proof.** Straightforward. □

## 3. Presentations and functors

We consider two finitely presented categories **A** and **B**. Let **B** have a presentation consisting of a finite graph $G_B$ and equations $U_i = V_i$ for $1 \leq i \leq n$ where each $U_i$ and $V_i$ is a word in the morphisms of $G_B$. Let **A** have a presentation consisting of a finite graph $G_\mathbf{A}$ and equations which we do not specify. Suppose also that we are given two functors $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $X : \mathbf{A} \longrightarrow \mathbf{Sets}$. We next define the notion of a *presentation P of a functor from* **B** *to* **Sets** *and a natural transformation from X to the composite of this functor with F*, which for brevity shall just be called a *presentation P*.

**Definition 3.1.** A *presentation P* consists of:

(i) A set $PB$ for each $B \in G_B$.
(ii) A relation $Pg : PB_1 \to PB_2$ for each morphism $g : B_1 \to B_2$ in $G_\mathbf{B}$.
(iii) A symmetric relation $S_B \subseteq PB \times PB$ for each $B \in G_\mathbf{B}$. The elements of each $S_B$ will be called *coincidences*.
(iv) A function $\mu_A : XA \to PFA$ for each object $A \in \mathbf{A}$.

$P$ is said to be a *finite presentation* if the sets $PB$ and $XA$ are finite for all objects $A \in \mathbf{A}$ and $B \in \mathbf{B}$.

Although $Pg$ is defined for morphisms in $G_\mathbf{B}$, we will extend the notation and define $Pg = Pg_1 \ldots Pg_n$ where $g$ is the morphism $g_1 \ldots g_n \in \mathcal{F}G_\mathbf{B}$ (the free category on the graph $G_\mathbf{B}$), and the relations are composed in the usual manner.

Given a presentation $P$ there is an associated functor $\overline{P}$ from **B** to **Sets** and a natural transformation from $X$ to $\overline{P}F$. To construct these first define a functor $Q : \mathbf{B} \longrightarrow \mathbf{Sets}$ on each $B \in \mathbf{B}$ by

$$QB = \sum_{B' \in \mathbf{B}} \mathbf{B}(B', B) \times PB'$$

and on each morphism $g : B_1 \to B_2$ in **B** by

$$Qg(f, x) = (gf, x).$$

It is straightforward to check that $Q$ is indeed a functor and we leave this to the reader. Next define a relation $R_B$ for each object $B \in \mathbf{B}$ by $R_B = K_B \cup L_B \cup N_B$ where $K_B$, $L_B$ and $N_B$ are relations on $QB$ for each $B \in \mathbf{B}$ defined as follows

(i) If $x \in PB_1$ and $g : B_1 \to B_2$ is a morphism in $G_{\mathbf{B}}$ with $y \in Pg(x)$ then $(g, x)K_{B_2}(1_{B_2}, y)$.
(ii) If $m, n \in PB$ and $m \sim_{S_B} n$ then $(1_B, m)L_B(1_B, n)$.
(iii) If $x \in XA_1$ and $f : A_1 \to A_2$ is a morphism in $G_A$ then

$$(Ff, \mu_{A_1}(x))N_{FA_2}(1_{FA_2}, \mu_{A_2}(Xf(x))).$$

Let $\{\sim_B\}_{B\in\mathbf{B}}$ be the minimal congruence on $Q$ containing $\{R_B\}_{B\in\mathbf{B}}$. Using Proposition 2.2 define $\overline{P}$ to be the quotient functor $\hat{Q}$ of the functor $Q$ by the congruence $\sim$.

**Proposition 3.1.** *Define* $\overline{\mu}_A : XA \to \overline{P}FA$ *for each object* $A \in \mathbf{A}$ *by*

$$\overline{\mu}_A(x) = [1_{FA}, \mu_A(x)]$$

*then* $\overline{\mu}$ *is a natural transformation from* $X$ *to* $\overline{P}F$.

**Proof.** Observe that for any element $x \in XA_r$ and morphism $f = f_1 \dots f_r$ in **A** where each $f_i : A_i \to A_{i-1}$ is a morphism in $G_{\mathbf{A}}$, then

$$\begin{aligned}\overline{\mu}_{A_0}(Xf(x)) &= [1_{FA_0}, \mu_{A_0}(Xf_1 \dots f_r(x))]\\ &= [Ff_1, \mu_{A_1}(Xf_2 \dots f_r(x))]\\ &\ \vdots\\ &= [Ff_1 \dots f_r, \mu_{A_r}(x)]\\ &= \overline{P}Ff(\overline{\mu}_{A_r}(x)).\end{aligned}$$

The middle steps in the calculation above follow from repeated usage of condition (iii) in the definition of the relations $\{R_B\}_{B\in\mathbf{B}}$ and also from the closure property of congruences. □

**Definition 3.2.** A pair $(L, \phi)$ consisting of a functor $L : \mathbf{B} \longrightarrow \mathbf{Sets}$ and a natural transformation $\phi : X \stackrel{\cdot}{\longrightarrow} LF$ is said to be finitely presented if there exists a finite presentation $P$ and isomorphism $\psi : \overline{P} \stackrel{\cdot}{\longrightarrow} L$ such that $\psi F \circ \overline{\mu} = \phi$.

**Proposition 3.2.** *Let* $(P, \mu)$ *be a pair consisting of a functor* $P : \mathbf{B} \to \mathbf{Sets}$ *and a natural transformation* $\mu : X \stackrel{\cdot}{\longrightarrow} PF$. *Regard* $P$ *as a presentation by restricting its domain to the* *graph $G_{\mathbf{B}}$, defining $S_B = \emptyset$ for each $B \in \mathbf{B}$ and leaving the maps $\mu_A$ unchanged for each $A \in \mathbf{A}$. Then there exists a natural isomorphism $\psi : P \stackrel{\cdot}{\longrightarrow} \overline{P}$ such that $\psi F \circ \mu = \overline{\mu}$.*

**Proof.** Define $\psi_B : PB \to \overline{P}B$ for each $B \in \mathbf{B}$ by $x \mapsto [1_B, x]$. Immediately it should be clear that $\psi F \circ \mu = \overline{\mu}$. We now show that given any morphism $g = g_1 \dots g_r$ in $\mathbf{B}$ where each $g_i : B_i \to B_{i-1}$ is a morphism in $G_B$, then $[g, x] = [1_{B_0}, Pg(x)]$. First, observe that $[g_r, x] = [1_{B_{r-1}}, Pg_r(x)]$. Now suppose $[g_i \dots g_r, x] = [1_{B_{i-1}}, Pg_i \dots g_r(x)]$ then

$$\begin{aligned}[g_{i-1}g_i \dots g_r, x] &= [g_{i-1}, Pg_i \dots g_r(x)]\\ &= [1_{B_{i-2}}, Pg_{i-1}Pg_i \dots g_r(x)]\\ &= [1_{B_{i-2}}, Pg_{i-1}g_i \dots g_r(x)]\end{aligned}$$

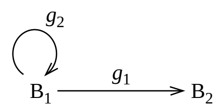
and so by induction $[g, x] = [1_{B_0}, Pg(x)]$. From this result it is clear that the maps $\psi_B$ are natural and surjective.

Showing these maps are injective is equivalent to proving $[1_B, x] = [1_B, y] \Rightarrow x = y$. Suppose $[1_B, x] = [1_B, y]$, then by Proposition 2.1 either $x = y$ or there exist $u_1, \dots, u_s \in QB$ with $u_1 = (1_B, x)$ and $u_s = (1_B, y)$ such that for each $i = 1, \dots, s-1$ there exists a morphism $h_i : B_i \to B$ in $\mathbf{B}$ with $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i$ where $Qh_i(d_i) = u_i$ and $Qh_i(e_i) = u_{i+1}$. Now let $d_i = (f_i, z_i)$ and $e_i = (g_i, w_i)$ for each $i$. Since $R_{B_i} = K_{B_i} \cup L_{B_i} \cup N_{B_i}$ (note that $L_{B_i} = \emptyset$) it follows that if $d_i R_{B_i} e_i$ then we have two cases to consider, either $d_i K_{B_i} e_i$ or $d_i N_{B_i} e_i$ (similarly for $e_i R_{B_i} d_i$). In each of these cases it is not hard to see that $Pf_i(z_i) = Pg_i(w_i)$ and so $Ph_i f_i(z_i) = Ph_i g_i(w_i)$. If we also note that $(h_{i-1}g_{i-1}, w_{i-1}) = Qh_{i-1}(e_{i-1}) = u_i = Qh_i(d_i) = (h_i, f_i, z_i)$ which implies that $w_{i-1} = z_i$ and $h_{i-1}g_{i-1} = h_i f_i$ for any $i$, then moving along $u_1, \dots, u_s$ we have the chain of equalities.

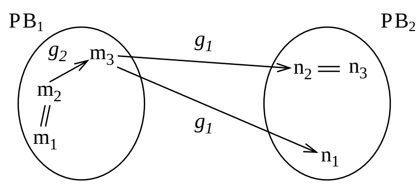
$$\begin{aligned}x = P1_B(x) &= \dots = Ph_{i-1}g_{i-1}(w_{i-1})\\ &= Ph_i f_i(z_i) = Ph_i g_i(w_i) = \dots = P1_B(y) = y. \quad \square\end{aligned}$$

## 4. Modifying presentations

A presentation $P$ can be pictured as a collection of sets with arrows between the elements representing the relations $Pg$ for each morphism $g$ in $G_{\mathbf{B}}$. For example, if $\mathbf{B}$ is the category with finite presentation consisting of the graph




and with no equations, then the following is a typical presentation.



The coincidences given by $S_A$ and $S_B$ on the sets $PA$ and $PB$ have been denoted with pairs of parallel lines between the elements. Note also that by a slight abuse of notation the arrows are labelled with $g_1$ and $g_2$ instead of $Pg_1$ and $Pg_2$. Notice that we have not included the functions $\mu_A$ in the picture above. In general we shall omit these, since they are only referred to infrequently. We now describe a list of permitted actions or modifications of a presentation $P$. In each case we start with $P$ and then construct a new presentation $P'$.

### 4.1. *Action α: add an element*

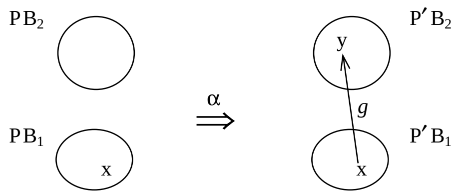
If $x \in PB_1$ and $g : B_1 \to B_2$ is a morphism in $G_{\mathbf{B}}$, then if there does not exist $y \in PB_2$ with $y \in Pg(x)$ we define a new presentation $P'$ on the objects of $G_{\mathbf{B}}$

$$P'B_2 = PB_2 \cup \{y\}$$
$$P'B = PB \qquad \text{where } B \neq B_2$$

and on the morphism of $G_{\mathbf{B}}$

$$P'g = Pg \cup \{(x, y)\}$$
$$P'h = Ph \qquad \text{where } h \neq g$$

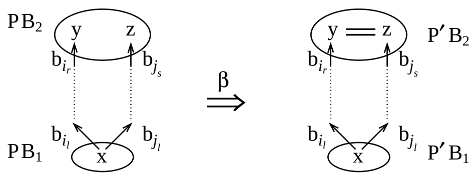
and with the same coincidences $S'_B = S_B$ and functions $\mu'_A = \mu_A$ for each $B \in G_{\mathbf{B}}$ and $A \in G_{\mathbf{A}}$.




### 4.2. *Action β: add a coincidence*

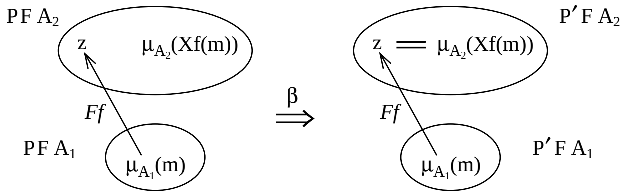
(i) Suppose we have an equation $b_{i_r} \ldots b_{i_1} = b_{j_s} \ldots b_{j_1}$ in the presentation of $\mathbf{B}$. Suppose also that $x \in PB_1$ and $y, z \in PB_2$ with $y \in Pb_{i_r} \ldots Pb_{i_1}(x)$ and $z \in Pb_{j_s} \ldots Pb_{j_1}(x)$. Then if $y \neq z$ we can construct a new presentation $P'$ which is identical to $P$ in relation to objects, morphisms and the functions $\mu_A$, but whose coincidences are defined as follows.

$$S'_{B_2} = S_{B_2} \cup \{(y, z), (z, y)\}$$
$$S'_B = S_B \qquad \text{where } B \neq B_2$$




(ii) if $m \in XA_1$ and $f : A_1 \to A_2$ is a morphism in $G_{\mathbf{A}}$ then if $z \neq y$ where $z \in PFf(\mu_{A_1}(m))$ and $y = \mu_{A_2}(Xf(m))$, construct a new presentation $P'$ which is identical to $P$ in relation to objects, morphisms and the functions $\mu_A$, but whose coincidences are defined as follows

$$S'_{FA_2} = S_{FA_2} \cup \{(y,z),(z,y)\}$$
$$S'_B = S_B \qquad \text{where } B \neq FA_2.$$




### 4.3. Action $\gamma$: delete coincidences

Given a coincidence $(x,y) \in S_{B_1}$ we construct a new presentation $P'$ without this coincidence as follows. If $y = x$ then we define $P'$ identically to $P$ on objects and morphisms of $G_{\mathbf{B}}$. On coincidences we define

$$S'_{B_1} = S_{B_1} \backslash \{(x,x)\}$$
$$S'_B = S_B \qquad \text{where } B \neq B_1.$$

If $y \neq x$ then we define $P'$ on the objects of $G_{\mathbf{B}}$ by

$$P'B_1 = PB_1 \backslash \{y\}$$
$$P'B = PB \qquad \text{where } B \neq B_1.$$

Define $P'$ on each morphism $g$ of $G_{\mathbf{B}}$ with the following list of conditions

1. if $\mathrm{dom}(g) = \mathrm{cod}(g) = B_1$ and $y \in Pg(y)$ then
   $P'g = \{Pg \cup (x \times Pg(y)) \cup ((Pg)^{-1}(y) \times x) \cup (x,x)\} \cap (P'B_1 \times P'B_1)$
2. if $\mathrm{dom}(g) = \mathrm{cod}(g) = B_1$ and $y \notin Pg(y)$ then
   $P'g = \{Pg \cup (x \times Pg(y)) \cup ((Pg)^{-1}(y) \times x)\} \cap (P'B_1 \times P'B_1)$
3. if $\mathrm{dom}(g) = B_1$ but $\mathrm{cod}(g) = B \neq B_1$ then
   $P'g = \{Pg \cup (x \times Pg(y))\} \cap (P'B_1 \times P'B)$
4. if $\mathrm{cod}(g) = B_1$ but $\mathrm{dom}(g) = B \neq B_1$ then
   $P'g = \{Pg \cup ((Pg)^{-1}(y) \times x)\} \cap (P'B \times P'B_1)$
5. if $\mathrm{dom}(g) \neq B_1$ and $\mathrm{cod}(g) \neq B_1$ then $P'_g = Pg$.

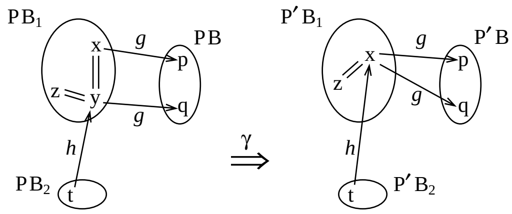
Define the coincidences of $P'$ by

$$S'_B = S_B \qquad \text{where } B \neq B_1$$
$$S'_{B_1} = \{S_{B_1} \cup (x \times S_{B_1}(y)) \cup (S_{B_1}(y) \times x)\} \cap (P'B_1 \times P'B_1).$$

Define the family of functions $\{\mu'_A\}_{A \in \mathbf{A}}$ by

$$\mu'_A(m) = x \qquad \text{where } \mu_A(m) = y$$
$$\mu'_A(n) = \mu_A(n) \qquad \text{where } \mu_A(m) \neq y$$

where $m, n \in XA$.



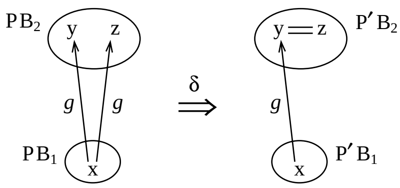

### 4.4. Action $\delta$: delete non-determinism

If $x \in PB_1$ and $y, z \in Pg(x)$ where $g : B_1 \to B_2$ is in $G_{\mathbf{B}}$, then if $y \neq z$ define $P'$ identically to $P$ in relation to the objects of $G_{\mathbf{B}}$ and the functions $\mu_A$. On the morphisms of $G_{\mathbf{B}}$ define

$$P'g = Pg \backslash \{(x, z)\}$$
$$P'h = Ph \qquad \text{where } h \neq g.$$

Define coincidences

$$S'_{B_2} = S_{B_2} \cup \{(y, z), (z, y)\}$$
$$S'_B = S_B \qquad \text{where } B \neq B_2.$$




**Definition 4.1.** Given a presentation $P$ and an action we say that $P$ is *invariant under this action* if it is not applicable to $P$ in any way.

**Proposition 4.1.** *If $P$ is a presentation invariant under the actions $\alpha$, $\beta$, $\gamma$ and $\delta$, and the definition of $P$ is extended to the morphisms of* $\mathbf{B}$ *by defining $Pg = Pg_1 \dots Pg_n$ where $g = g_1 \dots g_n$ in* $\mathbf{B}$ *and $P1_B = 1_{PB}$ for each $B \in \mathbf{B}$. Then $P$ is a functor and $\mu$ is a natural transformation from $X$ to $PF$.*

**Proof.** First observe that the invariance of $P$ under $\alpha$ implies that given any morphism $g : B_1 \to B_2$ in $G_{\mathbf{B}}$ then $Pg(x) \neq \emptyset$ for all $x \in PB_1$. The invariance of $P$ under $\delta$ implies that $Pg$ is a function for each morphism $g : B_1 \to B_2$ in $G_{\mathbf{B}}$. The invariance of $P$ under Action $\beta$(i) implies that given any equation $U_i = V_i$ in the presentation of $\mathbf{B}$ then $PU_i = PV_i$. Thus using the definition of the equivalence relation on the morphisms of $\mathcal{F}G_{\mathbf{B}}$ (used to define morphisms of $\mathbf{B}$) it follows that $Pg$ is well defined for all morphisms $g$ in $\mathbf{B}$. The invariance of $P$ under $\gamma$ implies that $S_{\mathbf{A}} = \emptyset$ for all $A \in \mathbf{B}$. The functorial properties of $P$ follow directly from our definition of $P$ on the morphisms of $\mathbf{B}$. The naturality of $\mu$ follows from the invariance of $P$ under Action $\beta$(ii). □

## 5. Invariance of $\overline{P}$ under modifications

**Theorem 5.1.** *If $P$ is a presentation and $P'$ is the presentation obtained from $P$ by applying one of the actions $\alpha$, $\beta$, $\gamma$ or $\delta$, then there exists a natural isomorphism $\psi : \overline{P} \to \overline{P'}$ satisfying $\psi F \circ \overline{\mu} = \overline{\mu'}$.*

**Proof.** In each case a map $\psi_B : \overline{P}B \to \overline{P'}B$ for each $B \in \mathbf{B}$ is given. These maps are then shown to be

1. Well defined.
2. Injective.
3. Surjective.
4. Natural.

This will show that $\overline{P'}$ is naturally isomorphic to $\overline{P}$ as required. The equation $\psi F \circ \overline{\mu} = \overline{\mu'}$ will also follow in a straightforward manner from the definition. In the course of the proofs $[g, x]$ and $[[g, x]]$ will denote the equivalence classes of $(g, x)$ with respect to the congruences associated with $P$ and $P'$, respectively. In constructing the functor $\overline{P}$ from the presentation $P$ we made use of a functor $Q$. We will let $Q'$ denote the corresponding functor in the construction of $\overline{P'}$ from $P'$. The proof of invariance for each of the actions will now be discussed in turn.

### *5.1. Action $\alpha$*

For each $B \in \mathbf{B}$ define $\psi_B : \overline{P}B \to \overline{P'}B$ by $[f, z] \mapsto [[f, z]]$. Showing this map is well defined is equivalent to proving $[f_1, z_1] = [f_2, z_2] \Rightarrow [[f_1, z_1]] = [[f_2, z_2]]$. So suppose $[f_1, z_1] = [f_2, z_2]$. By Proposition 2.1 either $f_1 = f_2$ and $z_1 = z_2$ or there exist $u_1, \ldots, u_s \in QB$ with $u_1 = (f_1, z_1)$ and $u_s = (f_2, z_2)$ such that for each $i = 1, \ldots, s-1$ there exists a morphism $h_i : B_i \to B$ in $\mathbf{B}$ with $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i$ where $Qh_i(d_i) = u_i$ and $Qh_i(e_i) = u_{i+1}$. Now by definition $R_B = K_B \cup L_B \cup N_B$. Denote the corresponding relations of $P'$ by $R'_B$, $K'_B$, $L'_B$ and $N'_B$. Observe from the definition of $P'$ that $PB \subseteq P'B \quad (\forall B)$, $Pg \subseteq P'g \quad (\forall g)$, $S'_B = S_B \quad (\forall B)$ and $\mu'_A = \mu_A \quad (\forall A)$. Thus $K_B \subseteq K'_B$, $L_B \subseteq L'_B$ and $N_{FA} \subseteq N'_{FA}$, hence $R_B \subseteq R'_B$ for each $B \in \mathbf{B}$. From this it is clear that $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i \Rightarrow d_i R'_{B_i} e_i$ or $e_i R'_{B_i} d_i$ and so by Proposition 2.1 $[[f_1, z_1]] = [[f_2, z_2]]$.

Proving the maps $\psi_B$ are injective is equivalent to showing $[[f_1, z_1]] = [[f_2, z_2]] \Rightarrow [f_1, z_1] = [f_2, z_2]$ where $z_1, z_2 \neq y$. Suppose $[[f_1, z_1]] = [[f_2, z_2]]$. By Proposition 2.1 there exists $B \in \mathbf{B}$ and $u'_1, \ldots, u'_s \in Q'B$ with $u'_1 = (f_1, z_1)$ and $u'_s = (f_2, z_2)$ such that for each $i = 1, \ldots, s-1$ there exists a morphism $h_i : B_i \to B$ in $\mathbf{B}$ with $d'_i R'_{B_i} e'_i$ or $e'_i R'_{B_i} d'_i$ where $Q'h_i(d'_i) = u'_i$ and $Q'h_i(e'_i) = u'_{i+1}$. Define a map $\nu_B : Q'B \to QB$ by $\nu(f, z) = (f, z)$ if $z \neq y$ and $\nu(f, y) = (fg, x)$. In general if $c'_i \in Q'B$ we will let $c_i$ denote $\nu(c'_i)$. Observe that $d'_i \in (Q'h_i)^{-1}(u'_i) \Rightarrow d_i \in (Qh_i)^{-1}(u_i)$ and similarly for $e'_i$. We will now show that $d'_i R'_{B_i} e'_i \Rightarrow d_i = e_i$ or $d_i R_{B_i} e_i$. So suppose $d'_i R'_{B_i} e'_i$, there are three cases to consider

1. If $d'_i K'_{B_i} e'_i$ and neither $d'_i$ or $e'_i$ have $y$ as their second component, then $d'_i = d_i$, $e'_i = e_i$ and $d_i K_{B_i} e_i$. If either $d'_i$ or $e'_i$ contains $y$ then the relation must have the form

$(g, x)K'_{B_i}(1_{B_i}, y)$ since this is the only relation involving $y$ in $K'_{B_i}$. If $d'_i = (1_{B_i}, y)$ then $e'_i = e_i = (g, x)$ and so $d_i = (g, x) = e_i$.

2. If $d'_i L'_{B_i} e'_i$ then $d'_i = d_i$, $e'_i = e_i$ and $d_i L_{B_i} e_i$ since applying action $\alpha$ does not effect coincidences.
3. If $d'_i N'_{B_i} e'_i$ then $d'_i = d_i$ and $e'_i = e_i$ since $y \notin \mu_A = \mu'_A$ for any $A \in \mathbf{A}$, and so $d_i N_{B_i} e_i$.

Thus it can now be seen that $u_1$ and $u_s$ are equivalent with respect to the congruence on $Q$ (i.e. $[f_1, z_1] = [f_2, z_2]$).

It now remains to prove that the maps $\psi_B$ for each $B \in \mathbf{B}$ are surjective and natural. Suppose $[[f, z]] \in Q'B$ for some $B \in \mathbf{B}$. If $z \neq y$ then $\psi_B([f, z]) = [[f, z]]$. If $z = y$ then we observe $(g, x)K'_B(1, y)$ and so $\psi_B([fg, x]) = [[fg, x]] = [[f, y]]$. Thus the maps are surjective. Naturality follows from the following calculation

$$
\begin{aligned}
(\overline{P'}[g] \circ \psi_{B_1})[f, z] &= \overline{P'}[g][[f, z]] \\
&= [[Q'g(f, z)]] \\
&= [[gf, z]] \\
&= \psi_{B_2}[gf, z] \\
&= \psi_{B_2}[Qg(f, z)] \\
&= (\psi_{B_2} \circ \overline{P}[g])[f, z].
\end{aligned}
$$

### 5.2. *Action β(i)*

For each $B \in \mathbf{B}$ define $\psi_B : \overline{P}B \to \overline{P'}B$ by $[f, z] \mapsto [[f, z]]$. Showing these maps are well defined is equivalent to showing $[f_1, z_1] = [f_2, z_2] \Rightarrow [[f_1, z_1]] = [[f_2, z_2]]$. Applying the same reasoning as in the case for Action $\alpha$, this amounts to showing $R_B \subseteq R'_B$ for each $B \in \mathbf{B}$. From the definition of $P'$ it follows that $P'B = PB \quad (\forall B)$, $P'g = Pg \quad (\forall g)$, $S_B \subseteq S'_B \quad (\forall B)$ and $\mu'_A = \mu_A \quad (\forall A)$. Thus $K_B = K'_B$, $L_B \subseteq L'_B$ and $N_{FA} = N'_{FA}$ and hence $R_B \subseteq R'_B$ for each $B \in \mathbf{B}$.

Showing $\psi_B$ is injective is equivalent to showing $[[f_1, z_1]] = [[f_2, z_2]] \Rightarrow [f_1, z_1] = [f_2, z_2]$. We proceed as we did in the case for action $\alpha$ except that we do not need to define the maps $\nu_B$ since $QB = Q'B$ for each $B \in \mathbf{B}$. We show that $d'_i R'_{B_i} e'_i \Rightarrow \exists c_1, \ldots, c_t \in QB_i$ with $c_1 = d'_i$, $c_t = e'_i$ and satisfying $[c_j] = [c_{j+1}]$ for each $j = 1, \ldots, t-1$. There are three cases to consider

1. If $d'_i K'_{B_i} e'_i$ then $d_i K_{B_i} e_i$ since $K'_{B_i} = K_{B_i}$.
2. $d'_i L'_{B_i} e'_i$ then either $d'_i L_{B_i} e'_i$ or the relation must have the form $(1_{B_2}, y)L'_{B_2}(1_{B_2}, z)$ or $(1_{B_2}, z)L'_{B_2}(1_{B_2}, y)$ where $y$ and $z$ are given in the definition of Action $\beta$(i). Let $y_t \in Pb_{i_t} \ldots b_{i_1}(x)$ for $1 \le t \le r$ and $z_t \in Pb_{j_t} \ldots b_{j_1}(x)$ for $1 \le t \le s$ such that $y_t \in Pb_{i_t}(y_{t-1})$, $z_t \in Pb_{j_t}(z_{t-1})$, $y_r = y$ and $z_s = z$. Then we have the chain of equalities

$$[1_{B_2}, y] = [b_{i_r}, y_{r-1}] = \cdots = [b_{i_r} \cdots b_{i_1}, x]$$

and also

$$[b_{j_s} \cdots b_{j_1}, x] = \cdots = [b_{j_s}, z_{s-1}] = [1_{B_2}, z].$$

We note that $[b_{j_s} \dots b_{j_1}, x] = [b_{i_r} \dots b_{i_1}, x]$ since $b_{j_s} \dots b_{j_1} = b_{i_r} \dots b_{i_1}$ is an equation of **B**.

3. If $d'_i N'_{B_i} e'_i$ then $d'_i N_{B_i} e'_i$ since $\mu_A = \mu'_A$ for any $A \in \mathbf{A}$.

Thus it can now be seen that $(f_1, z_1)$ and $(f_2, z_2)$ are equivalent with respect to the congruence on $Q$ (i.e. $[f_1, z_1] = [f_2, z_2]$).

The proofs of surjectivity and naturality are straightforward and we leave them to the reader.

### 5.3. Action β(ii)

For each $B \in \mathbf{B}$ define $\psi_B : \overline{P}B \to \overline{P'}B$ by $[f, z] \mapsto [[f, z]]$. Showing these maps are well defined is equivalent to showing $[f_1, z_1] = [f_2, z_2] \Rightarrow [[f_1, z_1]] = [[f_1, z_1]]$. Applying the same reasoning as in the case for Action $\beta$(i), this amounts to showing $R_B \subseteq R'_B$ for each $B \in \mathbf{B}$. From the definition of $P'$ it follows that $P'B = PB$ $(\forall B)$, $P'g = Pg$ $(\forall g)$, $S_B \subseteq S'_B$ $(\forall B)$ and $\mu'_A = \mu_A$ $(\forall A)$. Thus $K_B = K'_B$, $L_B \subseteq L'_B$ and $N_{FA} = N'_{FA}$ and hence $R_B \subseteq R'_B$ for each $B \in \mathbf{B}$.

Showing $\psi_B$ is injective is equivalent to showing $[[f_1, z_1]] = [[f_2, z_2]] \Rightarrow [f_1, z_1] = [f_2, z_2]$. We proceed as we did in the case for action $\alpha$ except that we do not need to define the maps $\nu_B$ since $QB = Q'B$ for each $B \in \mathbf{B}$. We show that $d'_i R'_{B_i} e'_i \Rightarrow \exists c_1, \dots, c_t \in QB_i$ with $c_1 = d'_i$, $c_t = e'_i$ and satisfying $[c_j] = [c_{j+1}]$ for each $j = 1, \dots, t-1$. There are three cases to consider

1. If $d'_i K'_{B_i} e'_i$ then $d'_i K_{B_i} e'_i$ since $K'_{B_i} = K_{B_i}$.
2. If $d'_i L'_{B_i} e'_i$ then either $d'_i L_{B_i} e'_i$ or the relation must have the form $(1_{FA_2}, y) L'_{FA_2} (1_{FA_2}, z)$ or $(1_{FA_2}, z) L'_{FA_2} (1_{FA_2}, y)$ where $y$ and $z$ are given in the definition of Action $\beta$(ii). It then follows that $(1_{FA_2}, y) N_{FA_2} (1_{FA_2}, z)$.
3. If $d'_i N'_{B_i} e'_i$ then $d'_i N_{B_i} e'_i$ since $\mu_A = \mu'_A$ for any $A \in \mathbf{A}$.

Thus it can now be see that $(f_1, z_1)$ and $(f_2, z_2)$ are equivalent with respect to the congruence on $Q$ (i.e. $[f_1, z_1] = [f_2, z_2]$).

The proofs of surjectivity and naturality are left to the reader.

### 5.4. Action γ

We consider the removal of a coincidence $(x, y)$ where $x \neq y$, the other case involving removal of a coincidence of the form $(x, x)$ is left to the reader. For each $B \in \mathbf{B}$ define $\psi_B : \overline{P'}B \to \overline{P}B$ by $[[f, z]] \mapsto [f, z]$. Showing this map is well defined is equivalent to proving $[[f_1, z_1]] = [[f_2, z_2]] \Rightarrow [f_1, z_1] = [f_2, z_2]$. So suppose $[[f_1, z_1]] = [[f_2, z_2]]$ then by Proposition 2.1 either $f_1 = f_2$ and $z_1 = z_2$ or there exist $u'_1, \dots, u'_s \in Q'B$ with $u'_1 = (f_1, z_1)$ and $u'_s = (f_2, z_2)$ such that $\forall i = 1, \dots, s-1$ there exists a morphism $h_i : B_i \to B$ with $d'_i R'_{B_i} e'_i$ or $e'_i R'_{B_i} d'_i$ where $Q'h_i(d'_i) = u'_i$ and $Q'h_i(e'_i) = u'_{i+1}$. We will now show that $d'_i R'_{B_i} e'_i$ implies that either $d'_i = e'_i$ or that we can find $c_1, \dots, c_t$ with $c_1 = d'_i$ and $c_t = e'_i$ such that $[c_j] = [c_{j+1}]$ for all $j = 1, \dots, t-1$. Since $R'_{B_i} = K'_{B_i} \cup L'_{B_i} \cup N'_{B_i}$ we have three cases to consider

1. If $d'_i K'_{B_i} e'_i$ where $d'_i = (g, z)$, $e'_i = (1_{B_i}, w)$ and $w \in P'g(z)$ then there are four subcases to consider
   (a) If $z \neq x$ and $w \neq x$ then $w \in Pg(z)$ and so $d'_i K_{B_i} e'_i$.
   (b) If $z = x$ and $w \neq x$ then either $w \in Pg(z)$ in which case $d'_i K_{B_i} e'_i$ or $w \in Pg(y)$. But then we have $(1_B, x)L_B(1_B, y)$ and $(g, y)K_{B_i}(1_{B_i}, w)$.
   (c) If $z \neq x$ and $w = x$ then either $w \in Pg(z)$ in which case we have $d'_i K_{B_i} e'_i$ or $z \in (Pg)^{-1}(y)$. But then we have $(g, z)K_{B_i}(1_{B_i}, y)$ and $(1_{B_i}, y)L_{B_i}(1_{B_i}, x)$.
   (d) If $z = x$ and $w = x$ then either

   $w \in Pg(z)$ but then $d'_i K_{B_i} e'_i$.

   $y \in Pg(y)$ but then $(1_{B_i}, x)L_{B_i}(1_{B_i}, y)$ and $(g, y)K_{B_i}(1_{B_i}, y)$.

   $x \in Pg(y)$ but then $(1_{B_i}, x)L_{B_i}(1_{B_i}, y)$ and $(g, y)K_{B_i}(1_{B_i}, x)$.

   $y \in Pg(x)$ but then $(1_{B_i}, x)L_{B_i}(1_{B_i}, y)$ and $(1_{B_i}, y)K_{B_i}(g, x)$.
2. If $d'_i L'_{B_i} e'_i$ where $d'_i = (1_{B_i}, z)$ and $e'_i = (1_{B_i}, w)$ then there are four subcases to consider
   (a) If $z \neq x$ and $w \neq x$ then $d'_i L_{B_i} e'_i$.
   (b) If $z \neq x$ and $w = x$ then either $d'_i L_{B_i} e'_i$ or we have $z \in S_{B_i}(y)$ and so $(1_{B_i}, z)L_{B_i}(1_{B_i}, y)$ and $(1_{B_i}, y)L_{B_i}(1_{B_i}, x)$.
   (c) If $z = x$ and $w \neq x$ then $d'_i L_{B_i} e'_i$ or we have $w \in S_B(y)$ and so $(1_{B_i}, w)L_{B_i}(1_{B_i}, y)$ and $(1_{B_i}, y)L_{B_i}(1_{B_i}, x)$.
   (d) If $z = x$ and $w = x$ then $d'_i = e'_i$.
3. If $d'_i N'_{FA_2} e'_i$ then either $d'_i N_{FA_2} e'_i$ or there are three subcases to consider
   (a) $(1_{FA_2}, x)N'_{FA_2}(Ff, z)$ where $x = \mu'_{A_2}(Xf(m))$, $y = \mu_{A_2}(Xf(m))$ and $z = \mu_{A_1}(m) \neq y$. We then have $(1_{FA_2}, x)L_{FA_2}(1_{FA_2}, y)$ and $(1_{FA_2}, y)N_{FA_2}(Ff, z)$.
   (b) $(1_{FA_2}, z)N'_{FA_2}(Ff, x)$ where $z = \mu_{A_2}(Xf(m)) \neq y$, $x = \mu'_{A_1}(m)$ and $y = \mu_{A_1}(m)$. We then have $(1_{FA_2}, z)N_{FA_2}(Ff, y)$ and $(1_{FA_1}, y)L_{FA_1}(1_{FA_1}, x)$.
   (c) $(1_{FA_2}, x)N'_{FA_2}(Ff, x)$ where $x = \mu'_{A_2}(Xf(m))$, $y = \mu_{A_2}(Xf(m))$, $x = \mu'_{A_2}(m)$ and $y = \mu_{A_2}(m)$. We then have $(1_{FA_2}, y)N_{FA_2}(Ff, y)$ and $(1_{FA_2}, y)L_{FA_2}\ (1_{FA_2}, x)$.

This then shows that $[d'_i] = [e'_i]$ for all $i$, and hence that $[f_1, z_1] = [f_2, z_2]$.

Next we must show that the maps $\psi_B$ for each $B \in \mathbf{B}$ are injective. This is equivalent to showing that $[f_1, z_1] = [f_2, z_2] \Rightarrow [[f_1, z_1]] = [[f_2, z_2]]$ where $z_1, z_2 \neq y$. So suppose $[f_1, z_1] = [f_2, z_2]$ then either $f_1 = f_2$ and $z_1 = z_2$ or there exists $u_1, \ldots, u_{s-1}$ with $u_1 = (f_1, z_1)$ and $u_s = (f_2, z_2)$ such that $\forall i = 1, \ldots, s-1$ there exists a morphism $h_i : B_i \to B$ with $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i$ where $Qh_i(d_i) = u_i$ and $Qh_i(e_i) = u_{i+1}$. We define the map $\nu_B : QB \to Q'B$ by $(f, z) \mapsto (f, z)$ when $z \neq y$ and $(f, y) \mapsto (f, x)$. Given $c \in QB$ we denote $\nu_B(c)$ by $c'$. We now show that $d_i R_{B_i} e_i$ implies that either $d_i = e_i$ or that we can find $c'_1, \ldots, c'_t$ with $c'_1 = d'_1$ and $c'_t = e'_i$ such that $[[c'_j]] = [[c'_{j+1}]]$ for all $j = 1, \ldots, t-1$. Since $R_{B_i} = K_{B_i} \cup L_{B_i} \cup N_{B_i}$ we have three cases to consider

1. If $d_i K_{B_i} e_i$ with $d_i = (g, w_1)$, $e_i = (1B_i, w_2)$ and $w_2 \in Pg(w_1)$ then we have four subcases

(a) If $w_1 \neq y$ and $w_2 \neq y$ then $d_i' = d_i$, $e_i' = e_i$ and $d_i' K_{B_i}' e_i'$.
(b) If $w_1 = y$ and $w_2 \neq y$ then $d_i' = (g, x)$, $e_i' = e_i$ and $w_2 \in P' g(x)$ thus $d_i' K_{B_i}' e_i'$.
(c) If $w_1 \neq y$ and $w_2 = y$ then $d_i' = d_i$, $e_i' = (1_{B_i}, x)$ and $x \in P' g(w_1)$ thus $d_i' K_{B_i}' e_i'$.
(d) If $w_1 = y$ and $w_2 = y$ then $d_i' = (g, x)$, $e_i' = (1, x)$ and $x \in P' g(x)$ (since $y \in P g(y)$) thus $d_i' K_{B_i}' e_i'$.

2. If $d_i L_{B_i} e_i$ with $d_i = (1_{B_i}, w_1)$ and $e_i = (1_{B_i}, w_2)$ where $(w_1, w_2) \in S_{B_i}$ then we have four subcases to consider

(a) If $w_1 \neq y$ and $w_2 \neq y$ then $d_i' = d_i$, $e_i' = e_i$ and $d_i' L_{B_i}' e_i'$.
(b) If $w_1 = y$ and $w_2 \neq y$ then $d_i' = (1_{B_i}, x)$, $e_i' = e_i$ and $w_2 \in S_{B_i}(y) \Rightarrow w_2 \in S_{B_i}'(x)$ and thus $d_i' L_{B_i}' e_i'$.
(c) If $w_1 \neq y$ and $w_2 = y$ then $d_i' = d_i$, $e_i' = (1_{B_i}, x)$ and $w_1 \in S_{B_i}(y) \Rightarrow w_2 \in S_{B_i}'(x)$ and thus $d_i' L_{B_i}' e_i'$.
(d) If $w_1 = y$ and $w_2 = y$ then $d_i' = e_i'$.

3. If $d_i N_{FA_2} e_i$ then there are four possibilities

(a) $d_i' = d_i$, $e_i' = e_i$ and then $d_i' N_{FA_2}' e_i'$.
(b) $d_i = (1_{FA_2}, y)$, $e_i = (Ff, z)$ where $y = \mu_{A_2}(Xf(m))$ and $z = \mu_{A_1}(m) \neq y$. We then have $x = \mu_{A_2}'(Xf(m))$ and so $[[1_{FA_2}, x]] = [[Ff, z]]$.
(c) $d_i = (1_{FA_2}, z)$, $e_i = (Ff, y)$ where $z = \mu_{A_2}(Xf(m)) \neq y$ and $y = \mu_{A_1}(m)$. We then have $x = \mu_{A_1}'(m)$ and so $[[1_{FA_2}, z]] = [[Ff, x]]$.
(d) $d_i = (1_{FA_2}, y)$, $e_i = (Ff, y)$ where $y = \mu_{A_2}(Xf(m))$ and $y = \mu_{A_2}(m)$. We then have $x = \mu_{A_2}'(Xf(m))$ and $x = \mu_{A_2}'(m)$ and so $[[1_{FA_2}, x]] = [[Ff, x]]$.

The maps $\psi_B$ will now be shown to be surjective. It should be clear that anything of the form $[f, z]$ where $z \neq y$ lies in the image of these maps (i.e. $[[f, z]] \Rightarrow [f, z]$) so it is sufficient to show that we can find elements of $Q'B$ for some $B \in \mathbf{B}$ which map to elements of the form $[f, y]$. This is easy though since $(1_B, y) L_B (1_B, x)$ and so $[[f, x]] \Rightarrow [f, x] = [f, y]$. Naturality is straightforward and can be proved in the same manner as before.

### 5.5. *Action* $\delta$

For each $B \in \mathbf{B}$ define $\psi_B : \overline{P}B \to \overline{P'}B$ by $[f, z] \mapsto [[f, z]]$. Showing this map is well defined is equivalent to showing $[f_1, z_1] = [f_2, z_2] \Rightarrow [[f_1, z_1]] = [[f_2, z_2]]$. So suppose $[f_1, z_1] = [f_2, z_2]$ then by Proposition 2.1 either $f_1 = f_2$ and $z_1 = z_2$ or there exist $u_1, \ldots, u_s \in QB$ with $u_1 = (f_1, z_1)$ and $u_s = (f_2, z_2)$ such that $\forall i = 1, \ldots, s-1$ there exists a morphism $h_i : B_i \to B$ with $d_i R_{B_i} e_i$ or $e_i R_{B_i} d_i$ where $Qh_i(d_i) = u_i$ and $Qh_i(e_i) = u_{i+1}$. We now show that $d_i R_{B_i} e_i$ implies that either $d_i = e_i$ or that we can find $c_1', \ldots, c_t'$ with $c_1' = d_i$ and $c_t' = e_i$ such that $[[c_j']] = [[c_{j+1}']]$ for all $j = 1, \ldots, t-1$. Since $R_{B_i} = K_{B_i} \cup L_{B_i} \cup N_{B_i}$ we have three cases to consider

1. If $d_i K_{B_i} e_i$ then either $d_i K_{B_i}' e_i$, or $d_i K_{B_i} e_i$ has the form $(g, x) K_{B_i} (1_{B_i}, z)$ but then we have $(g, x) K_{B_i}' (1_{B_i}, y)$ and $(1_{B_i}, y) L_{B_i}' (1_{B_i}, z)$.

2. If $d_i L_{B_i} e_i$ then $d_i L'_{B_i} e_i$.
3. If $d_i N_{B_i} e_i$ then $d_i N'_{B_i} e_i$.

Showing the maps are injective is equivalent to showing that $[[f_1, z_1]] = [[f_2, z_2]] \Rightarrow [f_1, z_1] = [f_2, z_2]$. So suppose that $[[f_1, z_1]] = [[f_2, z_2]]$ then by Proposition 2.1 either $f_1 = f_2$ and $z_1 = z_2$ or there exist $u'_1, \ldots, u'_s \in Q'B$ with $u'_1 = (f_1, z_1)$ and $u'_s = (f_2, z_2)$ such that $\forall i = 1, \ldots, s-1$ there exists $h_i : B_i \to B$ with $d'_i R'_{B_i} e'_i$ or $e'_i R'_{B_i} d'_i$ where $Q'h_i(d'_i) = u'_i$ and $Q'h_i(e'_i) = u'_{i+1}$. We will now show that $d'_i R'_{B_i} e'_i$ implies that either $d'_i = e'_i$ or that we can find $c_1, \ldots, c_t$ with $c_1 = d'_i$ and $c_t = e'_i$ such that $[c_j] = [c_{j+1}]$ for all $j = 1, \ldots, t-1$. Since $R'_{B_i} = K'_{B_i} \cup L'_{B_i} \cup N'_{B_i}$ we have three cases to consider

1. If $d'_i K'_{B_i} e'_i$ then $d'_i K_{B_i} e'_i$ since $P'g \subseteq Pg \quad (\forall g)$.
2. If $d'_i L'_{B_i} e'_i$ then either $d'_i L_{B_i} e'_i$ or the relation has the form $(1_{B_i}, y) L'_{B_i} (1_{B_i}, z)$ in which case we note that $(g, x) R_{B_i} (1_{B_i}, z)$ and $(g, x) R_{B_i} (1_{B_i}, y)$ which implies that $[1_{B_i}, z] = [1_{B_i}, y]$.
3. If $d'_i N'_{B_i} e'_i$ then $d'_i N_{B_i} e'_i$.

Surjectivity and naturality are straightforward and can be proved in the same manner as before. □

## 6. An algorithm for computing $\overline{P}$

The algorithm described in this section is non-deterministic in that at each step there may be several courses of action.

**Definition 6.1.** A *run of the algorithm* consists of a sequence of the four actions $\alpha$, $\beta$, $\gamma$ and $\delta$ applied to an initial (finite) presentation $P$ thus generating a sequence of presentations

$$P = P_0 \mapsto P_1 \mapsto P_2 \mapsto \cdots.$$

It is said to *terminate* if there exists $t \geq 0$ such that the presentation $P_t$ is invariant under all four actions.

By Proposition 4.1 the presentation $P_t$ reached upon termination must be the restriction to $G_{\mathbf{B}}$ of some functor $\mathbf{B} \longrightarrow \mathbf{Sets}$. By Proposition 3.2 this functor is naturally isomorphic to $\overline{P_t}$, then by Theorem 5.1 and induction we have $\overline{P_t}$ naturally isomorphic to $\overline{P}$. In each case the isomorphism is compatible with the associated $\mu$ natural transformations. So it should be easy to see that by applying the algorithm and reaching the terminating state $P_t$ we have effectively calculated $\overline{P}$ from $P$.

It is not clear that every run of this algorithm should terminate. Clearly if $\overline{P}$ is not finite then termination is impossible. What about when $\overline{P}$ is finite? In order to ensure termination in this case some conditions will be imposed on the sequence of actions. From now on all presentations considered will be finite.

First we number of all the elements in the starting presentation with natural numbers starting at 1. Then each time a new element is created by action $\alpha$ during a run of the algorithm it is labelled with the next largest number available, we call this number the *rank* of the element. Elements in the starting presentation will be called *initial elements*.

**Definition 6.2.** A sequence $\eta_1, \eta_2, \ldots$ of the four actions $\alpha$, $\beta$, $\gamma$ and $\delta$ is said to be a *fair interleaving* if it satisfies the following conditions

1. For each action $\eta = \alpha, \beta, \gamma$ or $\delta$ and each $n \geq 1$, there exists $m$ such that $m > n$ and $\eta_m = \eta$ (i.e. no action is left out of the sequence indefinitely).
2. When applying action $\alpha$ the element involved is always chosen to have minimal rank.
3. When applying action $\gamma$ the element of highest rank in the coincidence is deleted.
4. For all $n \geq 1$ there exists $m$ such that $m > n$ and $P_m$ is invariant under the actions $\beta$, $\gamma$ and $\delta$.

The first three conditions are easy to implement. To see that the fourth is also straightforward we prove the following proposition.

**Proposition 6.1.** *Suppose $\eta_1, \eta_2, \ldots$ is a sequence of the three actions $\beta$, $\gamma$ and $\delta$ such that for each action $\eta = \beta, \gamma$ or $\delta$ and each $n \geq 1$ there exists $m$ such that $m > n$ and $\eta_m = \eta$. Let $P = P_0 \mapsto P_1 \mapsto P_2 \mapsto \cdots$ be the associated sequence of presentations then there exists $t$ such that $P_t$ is invariant under the actions $\beta$, $\gamma$ and $\delta$.*

**Proof.** Given a finite presentation $P$ the total number of elements is finite. Thus the total number of possible coincidences (pairings of elements) is also finite. To each coincidence which is created during the course of the algorithm we assign a number. This number will be the place in the sequence where that coincidence is first created. (Note: the same coincidence may be added many times.) Choose the maximal such number (this is a position in the sequence after which no new coincidences are created). Now because none of the actions are indefinitely left out we continue to delete coincidences and we also know that each coincidence deleted can never be added back, thus there must be a point in the sequence (after a finite number of steps) where all the coincidences have been deleted and after which no coincidences can be created. This then means that we have reached an invariant presentation, since both action $\beta$ and $\delta$ involve the addition of coincidences. $\square$

It follows from Proposition 6.1 that if we ensure that during any run of the algorithm we regularly stop applying action $\alpha$ and just allow actions $\beta$, $\gamma$ and $\delta$ to operate then we will always reach a presentation invariant under these three actions, thus implementing the fourth condition in Definition 6.2.

**Theorem 6.1.** *Given a presentation $P$ where $\overline{P}$ is finite then any fair interleaving of the four actions $\alpha$, $\beta$, $\gamma$ and $\delta$ applied to $P$ must terminate.*

**Proof.** Let $\eta_1, \eta_2, \ldots$ be any fair interleaving of the four actions. Let $P = P_0, P_1, P_2, \ldots$ be the corresponding sequence of presentations. Since $\overline{P}$ is finite, the collection of elements in the set $\sqcup_{B \in \mathbf{B}} \overline{P} B$ is finite. We can thus write down a list of representatives $(f_1, z_1), \ldots, (f_m, z_m)$ where $f_i \in \mathbf{B}$ for each $i$ and $z_i$ is an initial element for each $i$. Since $\overline{P}$ is a functor it follows that for each morphism $g : B_1 \to B_2$ in $G_{\mathbf{B}}$ and each $[f_i, z_i]$ where $\mathrm{cod}(f_i) = B_1$, then there exists $j$ with $\overline{P} g[f_i, z_i] = [f_j, z_j]$ (i.e. $[g f_i, z_i] = [f_j, z_j]$).

By Proposition 2.1 either $gf_i = f_j$ and $z_i = z_j$ or there exists $u_1, \ldots, u_s \in QB$ with $u_1 = (gf_i, z_i)$ and $u_s = (f_j, z_j)$ such that for each $t = 1, \ldots, s-1$ there exists $h_t : B_t \to B$ with $d_t R_{B_i} e_t$ or $e_t R_{B_i} d_t$ where $Qh_t(d_t) = u_i$ and $Qh_t(e_t) = u_{i+1}$. Similarly if $z_i \in PB$ then there exists $j$ such that $[1_B, z_i] = [f_j, z_j]$ and we can apply Proposition 2.1 again to conclude that either $f_j = 1_B$ and $z_i = z_j$ or that there exists a sequence $u_1, \ldots, u_s$ with the usual properties. Collect together all of the sequences of elements $u_i$ that can be found in these two ways and observe that there are only finitely many of them. Now define the length of any morphism in **B** to be the minimal length of all the morphisms in $\mathcal{F}G_{\mathbf{B}}$ corresponding to it (take the length of identity arrows to be zero). Let $l$ be the maximum length of any morphism from **B** occurring in the first component of a member of any of these sequences. We now study the properties of these sequences and how they interact with the four actions $\alpha$, $\beta$, $\gamma$ and $\delta$.

First we define some terminology. Given a presentation $P$ (with associated functor $Q$) and $u_1, \ldots, u_s \in QB$ such that for each $t = 1, \ldots, s-1$ there exists $h_t : B_t \to B$ with $d_t R_{B_i} e_t$ or $e_t R_{B_i} d_t$ where $Qh_t(d_t) = u_i$ and $Qh_t(e_t) = u_{i+1}$. Then we call such a collection $\{u_1, \ldots, u_s\} \in QB$ a *chain on* $Q$.

Suppose we apply action $\alpha$ or $\beta$ to $P$ giving us a new presentation $P'$, then $\{u_1, \ldots, u_s\} \in Q'B$ is a chain on $Q'$. This follows because $R_{B_i} \subseteq R'_{B_i}$ for each $B \in \mathbf{B}$ (see Section 5 for a more detailed discussion of this point). If we apply action $\gamma$ to $P$ to remove a coincidence $(x, y)$ then $\{u'_1, \ldots, u'_s\} \in Q'B$ is a chain on $Q'$ where we define $u'_i = (g, z)$ if $u_i = (g, z)$ and $z \neq y$ or $u'_i = (g, x)$ if $u_i = (g, y)$. This follows from the fact that $x$ inherits all of the properties that the element $y$ originally had, e.g. if $z \in Pg(y)$ then $z \in P'g(x)$, if $z \in S_B(y)$ then $z \in S'_B(x)$ etc. Finally if we apply action $\delta$ to $P$ we may have to modify the chain slightly. Suppose that $d_i K_{B_i} e_i$ (where $Qh_i(d_i) = u_i$ and $Qh_i(e_i) = u_{i+1}$ for some morphism $h_i$) then either $d_i K'_{B_i} e_i$ or $d_i K_{B_i} e_i$ has the form $(g, x) K_{B_i} (1_{B_i}, z)$ but then we have $(g, x) K'_{B_i} (1_{B_i}, y)$ and $(1_{B_i}, y) L'_{B_i} (1_{B_i}, z)$. So we can replace $u_i$ and $u_{i+1}$ in the chain with $(h_i g, x)$, $(h_i, y)$ and $(h_i, z)$. The other relations are not a problem since $L_{B_i} \cup M_{B_i} \subseteq L'_{B_i} \cup M'_{B_i}$.

The important thing to note in all four cases is that the maximal length of morphisms occurring in the first components of any elements in a chain does not increase when the action is applied, thus it is always bounded above by the quantity $l$ that we defined earlier.

We now turn our attention back to the sequence of presentations. We call two elements $x$ and $y$ *path connected* if there exist elements $u_1, \ldots, u_s$ in the presentation with $u_1 = x$ and $u_s = y$ such that for each $i = 1, \ldots, s-1$ either there exists $h_i \in G_{\mathbf{B}}$ with $Ph_i(u_i) = u_{i+1}$ or $(u_1, u_{i+1})$ is a coincidence. The collection of morphisms involved in any connection between $x$ and $y$ forms a morphism in $\mathcal{F}G_{\mathbf{B}}$ which we call a *path from* $x$ to $y$.

It is straightforward to prove by induction that given a presentation in the sequence $P_0, P_1, P_2, \ldots$ then any element in this presentation is either initial or path connected to an initial element. It follows that if the presentation is invariant under action $\gamma$ then any element $y$ is either initial or there exists a morphism $g \in \mathcal{F}G_{\mathbf{B}}$ and an initial element $x$ such that $y \in Pg(x)$.

Observe that conditions 2 and 3 in Definition 6.2 ensure that given any morphism $g$ in $\mathcal{F}G_{\mathbf{B}}$ with domain $A$ where $P_0(A) \neq \emptyset$ then there exists a presentation $P_m$ in the sequence and element $x$ such that $x$ is path connected to an initial element and the associated path is $g$. In particular if we let $n$ be the number of elements in $P_0$ and let $m$ be the maximum number of morphisms with a common domain that occur in $G_{\mathbf{B}}$ then after applying action $\alpha$ a total of $nm^l$ times we can conclude that every path of length less than or equal to $l$ occurs as the path connection of some element in the presentation to an initial element. In summary, if $m$ is chosen large enough then $P_m$ will satisfy

1. Invariance under actions $\beta, \gamma$ and $\delta$ (apply condition 4 of Definition 6.2).
2. Every element $y$ in the presentation $P_m$ is either initial or there exists a morphism $g \in \mathcal{F}G_{\mathbf{B}}$ and an initial element $x$ such that $y \in Pg(x)$.
3. Given a morphism $g \in \mathcal{F}G_{\mathbf{B}}$ with length less than or equal to $l$ and $x$ an initial element then $Pg(x) \neq \emptyset$.

It will now be shown that this presentation $P_m$ is invariant under action $\alpha$.

Recall from the start of the proof that $(f_1, z_1), \ldots, (f_m, z_m)$ are a list of representatives of elements in the set $\amalg_{B \in \mathbf{B}} \overline{P}B$. It is easy to see that we can write down a list of representatives for $\overline{P}_m$ of the form $(f_1, z'_1), \ldots, (f_m, z'_m)$ where each $z'_i$ is still an initial element. This is because an initial element can only be replaced by another initial element when a coincidence is removed, during the course of the algorithm. (See condition 3 of Definition 6.2.) By definition the length of each $f_i$ is less that or equal to $l$, and so it follows that $P_m f_i(z'_i) \neq \emptyset$. In fact since $P_m$ is invariant under action $\delta$ there can be no non-determinacy thus $P_m f_i(z'_i)$ defines exactly one element.

Earlier we noted that for each $i$ and each applicable morphism $g$ in $G_{\mathbf{B}}$ there was a chain connecting the elements $(gf_i, z_i)$ and $(f_j, z_j)$ (for some $j$). Using the properties of chains in relation to the actions it follows that there is a chain connecting $(gf_i, z'_i)$ and $(f_j, z'_j)$. It was also proved that the length of the morphisms in the first components remained bounded above by $l$. Hence if $u_i = (k_i, w_i)$ is an element of the chain then $P_m k_i(w_i)$ is a uniquely defined element of the presentation $P_m$. Using the invariance of $P_m$ under actions $\beta, \gamma$ and $\delta$ it can be shown that $P_m k_i(w_i) = P_m k_{i+1}(w_{i+1})$ and thus $P_m g f_i(z'_i) = P_m f_j(z'_j)$. It follows that the set of elements

$$G = \{P_m f_i(z'_i) \mid 1 \leq i \leq m\}$$

is closed under the action of the morphisms. Other chains were also considered between elements $(1_B, z_i)$ and the representatives $(f_i, z_i)$. Carrying everything through as before we deduce that all of the initial elements $z'_i$ in $P_m$ are included in the set $G$. But then from the construction of $P_m$ we know that all its elements are either initial or lie in the image of an initial element. Thus the closure of $G$ ensures that it contains all elements of $P_m$. Therefore $P_m$ is invariant under action $\alpha$ since $P_m g(x)$ is defined for all elements $x$ and applicable morphisms $g$. □

## 7. Left Kan extensions

So far we have described an algorithm which starts with an arbitrary presentation $P$ and computes the associated functor $\overline{P} : \mathbf{B} \longrightarrow \mathbf{Sets}$ and natural transformation $\overline{\mu} : X \stackrel{\cdot}{\longrightarrow} \overline{P}F$, terminating exactly when the answer is finite. As will be shown, by choosing $P$ carefully we can ensure that $(\overline{P}, \overline{\mu})$ is in fact the left Kan extension of $X$ along $F$. First we state a result concerning the structure of left Kan extensions.

**Proposition 7.1.** *Given functors $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $X : \mathbf{A} \longrightarrow$ **Sets** where $\mathbf{A}$ and $\mathbf{B}$ are finitely generated categories, we define a functor $L : \mathbf{B} \longrightarrow$ **Sets** as follows. For each object $B \in \mathbf{B}$*

$$LB = \left[ \sum_{A \in \mathbf{A}} \mathbf{B}(FA, B) \times XA \right] \Big/ \sim$$

*where $\sim$ is the smallest equivalence relation such that for all $f : A \to A'$ in $\mathbf{A}$, $g : FA' \to B$ in $\mathbf{B}$ and $x \in XA$*

$$(gFf, x) \sim (g, Xf(x)).$$

*For each morphism $h : B \to B'$ in $\mathbf{B}$ define*

$$Lh : LB \to LB' : [g, x] \mapsto [hg, x]$$

*where the equivalence class of $(g, x)$ with respect to $\sim$ has been denoted $[g, x]$. Now define the natural transformation $\mu : X \stackrel{\cdot}{\longrightarrow} LF$ by*

$$\mu_A : XA \to LFA : x \mapsto [1_{FA}, x].$$

*Then $L$ and $\mu$ form the left Kan extension of $X$ along $F$.*

**Proof.** The proof is a relatively straightforward exercise and can be found in Walters (1991). □

**Proposition 7.2.** *Define a presentation $P$ as follows:*

1. $PB = \sum_{A \in F^{-1}B} XA$ *for each $B \in \mathbf{B}$.*
2. $Pg = \emptyset$ *for all morphisms $g \in G_{\mathbf{B}}$.*
3. $SB = \emptyset$ *for each $B \in \mathbf{B}$.*
4. $\mu_A : XA \to PFA$ *is taken to be the inclusion mapping $x \mapsto x$ for each $A \in \mathbf{A}$.*

*Then $(\overline{P}, \overline{\mu})$ satisfy the universal property that given any functor $U : \mathbf{B} \longrightarrow$ **Sets** and natural transformation $\eta : X \stackrel{\cdot}{\longrightarrow} UF$, there exists a unique natural transformation $\psi : \overline{P} \stackrel{\cdot}{\longrightarrow} U$ such that $\psi F \circ \mu = \eta$. (This is the defining property of a left Kan extension.)*

**Proof.** From Proposition 7.1 above and the definition of $\overline{P}$ this follows immediately. □

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