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On a universal ultrametric space

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Abstract

For every cardinal τ we construct a universal ultrametric space LW_τ such that any ultrametric space of weight $\leqslant \tau$ can be embedded isometrically in LW_τ. The weight of LW_τ is $\tau^{\aleph_0}$ and we show that for all cardinals $\tau \leqslant c$ and for a wide class of cardinals $> c$ the weight of a universal ultrametric space can not be smaller © 2000 Elsevier Science B.V. All rights reserved.

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In the last 10–15 years a new area of research has evolved in mathematics. That is the theory of ultrametric spaces, i.e., metric spaces in which the strong triangle axiom

$$d(x,z) \leqslant \max\bigl[d(x,y), d(y,z)\bigr] \tag{1}$$

holds. This condition means precisely that any three points x, y, z are vertices of an isosceles triangle with the base being no greater than the sides. In English and German literature these spaces are usually called non-Archimedean [1,2], in French literature they are known as ultrametric [3], the Russian synonym for them is isosceles spaces [5–11]. The development of this new branch of the general theory of metric spaces is due to the following.

First of all, a lot of metric spaces that play important roles in various realms of mathematics turn out to be ultrametric. The most vivid examples are the rings Z_p of p-adic Hensel integers and the fields Q_p of p-adic Henselian numbers in number theory, the Baire space $B_{\aleph_0}$ and generalized Baire spaces B_τ in general topology, non-Archimedean

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normed fields in algebra, a set of words of computer language L equipped with Baire metric in computer science, rings $M(U)$ of of meromorphic functions on an open region U in complex analysis, etc. Finally, such an extensive and rapidly developing area as p-adic analysis is based on ultrametrics: the ground field Ω is ultrametric, and both p-adic integration and function spaces in p-adic analysis are connected with ultrametrics, in view of the theorems on the isosceles property (1) for a Hausdorff exponential of an isosceles space [4,7] and for space of maps into an ultrametric space [7,8].

Second of all, it turns out that there are several deep specific properties of ultrametric spaces that do not hold for general metric spaces. For example, any ultrametric space can be isometrically (!) embedded in Euclidean space (see Theorems 3–5 below); given a series $\sum a_n$ in a complete ultrametric group G, the requirement $|a_n| \to 0$ as $n \to \infty$ is not only necessary but also sufficient (!) for its convergence. These are the specific properties, which form the subject of the theory of ultrametric spaces.

The first step toward a study of non-Archimedean spaces was done by de Groot in [1]. He gave a description of all these spaces up to homeomorphism. However, a study of metric spaces to within a homeomorphism is too rough, homeomorphism preserves neither completeness, nor boundedness, nor Cantor connectedness [6] of a space. That is why a description of ultrametric spaces (and all their uniform one-to-one inverse images) was given in [6] up to uniform homeomorphism. Later on a lot of proximate, uniform [6], metric [5], geometric [9], and categorical [7,8,12,14] properties of these spaces and groups [5,18] were found and applications of the theory to category theory [11], topos theory and intuitionistic logic [10], theory of Boolean algebras [14,15], and computer science [13,17] were described.

The purpose of the paper is to describe all ultrametric spaces up to isometry. Modifying the construction of [5] by the method announced in [16] we construct here, for every cardinal τ, an ultrametric space LW_τ which contains isometrically all ultrametric spaces of weight $\leqslant \tau$. Its weight is $\tau^{\aleph_0}$ and we show that for all cardinals $\tau \leqslant c$ and for a wide class of cardinals $\tau > c$ the weight of a universal ultrametric space can not be smaller.

Main definition. Let M be a set with a base point $a \in M$, $\mathbb{Q}_+$ be the set of positive rationals, and LM be the set of all "eventually a-valued" functions $f : \mathbb{Q}_+ \to M$. (This means that there exists a positive number $X(f)$ such that $f(x) = a$, $\forall x > X(f)$. For simplicity the reader can keep in mind the real line $\mathbb{R}$ instead of M and 0 instead of a.) For any two maps $f, g : \mathbb{Q}_+ \to M$ let us define the distance $\Delta(f, g)$ by the equalities $\Delta(f, f) = 0$ and

$$\Delta(f, g) = \sup\{x \mid f(x) \neq g(x)\}. \tag{2}$$

It is easy to see that Δ is a metric satisfying the strong triangle axiom (1). So (LM, Δ) is an ultrametric space. Without any difficulty the reader can describe spherical neighborhoods, convergent sequences, compact subsets of the space LM and prove that (LM, Δ) is a complete ultrametric space [5,7].

Theorem 1. *Every ultrametric space (M, d) can be isometrically embedded in the space (LM, Δ) of all "eventually a-valued" maps from $\mathbb{Q}_+$ to M for any $a \in M$ chosen as a base point.*

Proof. Let us enumerate the elements of M by ordinal numbers such that $M = \{a_\alpha \mid \alpha < \omega(\tau), a_0 = a\}$, where $\omega(\tau)$ is the initial ordinal of cardinality $\tau = |M| = |\omega(\tau)|$. This makes M a well-ordered set. We shall define an isometry $i : (M, d) \to (LM, \Delta)$ inductively. Denote the image $i(a_\alpha)$ of an element a_α by f_α. Define $f_0(x) = a_0 = a$ for all $x \in \mathbb{Q}_+$, and $f_1(x) = a_1$ for $x \leqslant d(a_0, a_1)$, $f_1(x) = f_0(x) = a_0$ for $x > d(a_0, a_1)$. It is obvious that $d(a_0, a_1) = \Delta(f_0(x), f_1(x))$ even if $d(a_0, a_1)$ is irrational.

Let $\gamma < \omega(\tau)$. Assume by induction that all a_α are already isometrically embedded into the space LM for all $\alpha < \gamma$ in such a way that the range of f_β is a subset of $\{a_\alpha \mid \alpha \leqslant \beta\}$. We shall construct $i(a_\gamma)$ and show that i is isometry on $\{a_0, a_1, \ldots, a_\gamma\}$. To do it compute

$$d_\gamma = \inf\{d(a_\alpha, a_\gamma) \mid \alpha < \gamma\}$$

Consider two cases.

Case 1. There exists $\beta < \gamma$ such that $d(a_\beta, a_\gamma) = d_\gamma$. Then let us define f_γ as follows: $f_\gamma(x) = a_\gamma$ for $x \leqslant d_\gamma$, $f_\gamma(x) = f_\beta(x)$ for $x > d_\gamma$. It follows directly from the definition that $f_\gamma(x) = f_\beta(x)$ over (d_γ, ∞), $f_\gamma(x) = a_\gamma$ on $(0, d_\gamma]$, and $f_\beta(x)$ equals a_γ nowhere. So $\Delta(f_\beta, f_\gamma) = d_\gamma = d(a_\beta, a_\gamma)$.

We need to show that $\Delta(f_\alpha, f_\gamma) = d(a_\alpha, a_\gamma)$ for any $\alpha < \gamma$. Two other cases are to be considered here.

(i) If $d(a_\alpha, a_\beta) < d(a_\beta, a_\gamma)$ then in view of the isosceles property $d(a_\alpha, a_\gamma) = d(a_\beta, a_\gamma)$. By the inductive assumption we have $\Delta(f_\alpha, f_\beta) = d(a_\alpha, a_\beta) < d(a_\beta, a_\gamma) = \Delta(f_\beta, f_\gamma)$. Consequently $f_\alpha(x) = f_\beta(x)$ on $(d(a_\alpha, a_\beta), \infty)$, and therefore on (d_γ, ∞). This implies that $f_\alpha(x) = f_\gamma(x)$ on (d_γ, ∞). On the other hand, $f_\gamma(x) = a_\gamma$ on $(0, d_\gamma]$ while $f_\alpha(x)$ never equals a_γ. Hence $\Delta(f_\alpha, f_\gamma) = d_\gamma = d(a_\beta, a_\gamma) = d(a_\alpha, a_\gamma)$.

(ii) If $d(a_\alpha, a_\beta) \geqslant d(a_\beta, a_\gamma)$ then by the same reasons we get $d(a_\alpha, a_\beta) = d(a_\alpha, a_\gamma)$, $\Delta(f_\alpha, f_\beta) = d(a_\alpha, a_\beta) \geqslant d(a_\beta, a_\gamma) = \Delta(f_\beta, f_\gamma)$. This means that f_α coincides with f_β on $(d(a_\alpha, a_\beta), \infty) \subset (d(a_\beta, a_\gamma), \infty)$ where f_β is equal to f_γ. Therefore $f_\alpha(x) = f_\gamma(x)$ on $(d(a_\alpha, a_\beta), \infty)$ but not over any wider half-infinite interval. Hence $\Delta(f_\alpha, f_\gamma) = d(a_\alpha, a_\beta) = d(a_\alpha, a_\gamma)$.

Case 2. Suppose there is no $\beta < \gamma$ such that $d(a_\beta, a_\gamma) = d_\gamma$. If $d_\gamma > 0$ then set $f_\gamma(x) = a_\gamma$ on $(0, d_\gamma]$. In any case the problem is to define $f_\gamma(x)$ for $x > d_\gamma$. To do this let us choose a sequence $\{b_n\}$ of elements of M such that $b_n = a_{\alpha_n}$, $\alpha_n < \gamma$ and $d(b_n, a_\gamma) < d_\gamma + 1/n$. Denote $i(b_n)$ by $f_n(x)$. For $x > d_\gamma$ we define $f_\gamma(x)$ as follows: $f_\gamma(x) = f_n(x)$, where n satisfies the inequality $d_\gamma + 1/n < x$. Notice that for any m satisfying the similar inequality,

$$\begin{aligned}\Delta(f_n, f_m) = d(b_n, b_m) &\leqslant \max[d(b_n, a_\gamma), d(b_m, a_\gamma)] \\ &< \max[d_\gamma + 1/n, d_\gamma + 1/m] < x.\end{aligned}$$

This implies that $f_n(x) = f_m(x)$. Thus $f_\gamma(x)$ is well-defined.

Let us now show that $f_\gamma(x)$ does not depend on the choice of the sequence $\{b_n\}$. Consider another sequence $\{c_n\}$ such that $c_n = a_{\beta_n}$, $\beta_n < \gamma$ and $d(c_n, a_\gamma) < d_\gamma + 1/n$. Denote $i(c_n)$ by $g_n(x)$. We have that

$$\Delta(f_n, g_n) = d(b_n, c_n) \leqslant \max\bigl[d(b_n, a_\gamma), d(c_n, a_\gamma)\bigr] < d_\gamma + 1/n,$$

which implies that $f_n(x) = g_n(x)$ for all $x > d_\gamma + 1/n$.

Finally we need to prove that $\Delta(f_\alpha, f_\gamma) = d(a_\alpha, a_\gamma)$ for all $\alpha < \gamma$. For every $\alpha < \gamma$ there exists a point b_n such that $d(a_\alpha, a_\gamma) > d_\gamma + 1/n > d(b_n, a_\gamma) > d_\gamma$. Consequently $d(a_\alpha, a_\gamma) = d(a_\alpha, b_n)$. The inductive assumption implies $\Delta(f_\alpha, f_n) = d(a_\alpha, b_n) = d(a_\alpha, a_\gamma) > d_\gamma + 1/n$. By the definition of $f_\gamma(x)$, $f_\gamma(x) = f_n(x)$ on $(d_\gamma + 1/n, \infty)$. On the other hand, $f_\alpha(x) = f_n(x)$ on $(d(a_\alpha, a_\gamma), \infty) \subset (d_\gamma + 1/n, \infty)$. This implies that $f_\alpha(x) = f_\gamma(x)$ on $(d(a_\alpha, a_\gamma), \infty)$ but not on any wider interval. Hence $\Delta(f_\alpha, f_\gamma) = d(a_\alpha, a_\gamma)$. □

Let τ be an arbitrary cardinal. Denote the set of all ordinals of cardinality less than τ by W_τ. Viewed as a pointed set with a base point 0, W_τ gives us the space LW_τ.

Main theorem. *The space LW_τ is a metrically universal space for all ultrametric spaces of weight $\leqslant \tau$, and $|LW_\tau| = w(LW_\tau) = \tau^{\aleph_0}$.*

Proof. Suppose (X, d) is an arbitrary ultrametric space of weight τ and Y is its dense subset of potency τ. There is a one-to-one correspondence between Y and W_τ. It follows from Theorem 1 that there exists an isometric embedding $i : (Y, d) \to (LW_\tau, \Delta)$. Since (LW_τ, Δ) is complete one can extend i to $i* : (X, d) \to LW_\tau$. By the definition of *LM*-space we have $|LW_\tau| = \tau^{\aleph_0}$. To prove that $w(LW_\tau) = \tau^{\aleph_0}$ it is enough to find a discrete subset $Z \subset LW_\tau$ of potency $\tau^{\aleph_0}$. For any $f \in LW_\tau$ let us define the map $F : \mathbb{Q}_+ \to W_\tau$ as follows: $F(x) = 0$ for $x \leqslant 1$, $F(x) = f(x - 1)$ for $x > 1$. The set Z of all these maps is of cardinality $\tau^{\aleph_0}$. By the definition of metric Δ, $\Delta(F, G) > 1$ for any $F, G \in Z$. □

Note 1. If $2 \leqslant \tau \leqslant c$ then $\tau^{\aleph_0} = c$. The weight of a universal ultrametric space cannot be smaller, even for $\tau = 2$, as the following proposition shows.

Proposition. *If an ultrametric space (X, d) contains isometrically all two-point spaces then its weight is not less than continuum, $w(X, d) \geqslant c$.*

It follows directly from the next theorem [16,17].

Theorem 2. *For any ultrametric space (X, d) the set of values of its metric $V = \{d(x, y) \mid x, y \in X\}$ has cardinality no greater than its weight, $|V| \leqslant w(X)$.*

The proof is based on the following lemma of some own interest.

Lemma. *Suppose (X, d) is an ultrametric space, Y is its dense subset, and $V_Y = \{d(x, y) \mid x, y \in Y\}$; then the set $V = \{d(x, y) \mid x, y \in X\}$ coincides with V_Y.*

Proof. Let $(x, y) \notin Y \otimes Y$, $d(x, y) = r > 0$. Enclose x and y in two balls of radius $< r/2$. Take there points x_0 and $y_0 \in Y$. Since the space X is ultrametric and the balls are disjoint $d(x_0, y_0) = d(x, y)$. □

Proof of Theorem 2. For infinite $\tau = w(X)$ we have $|V_Y| \leqslant |Y||Y| = |Y|$, $V = V_Y$. Therefore $|V| = \inf\{|V_Y| \mid Y \text{ is dense in } X\} \leqslant \inf\{|Y| \mid Y \text{ is dense in } X\} = w(X)$. $\square$

Corollary 1. *For any separable ultrametric space the set of values of its metric is at most countable.*

Note 2. These values can be made rational by the mutually uniform, non-expanding, inverse-Lipschitz remetrization which differs from identity by arbitrary small $\varepsilon > 0$ [17].

Note 3. Certainly if one does not require a universal space to be ultrametric its weight can be made much smaller (for example, for $\tau = 2$, $\mathbb{R}$ is universal and it follows from the proposition above that $w(LW_2) = 2^{\aleph_0} = c$). Moreover, Theorems 3–5 below show that Euclidean space and its generalizations are metrically universal for the class of ultrametric spaces. Perhaps it is the most wonderful property of these spaces [9].

Theorem 3 [9]. *Every ultrametric space of weight τ can be isometrically embedded in the generalized Hilbert space H^τ.*

Theorem 4 [9]. *Every ultrametric space of cardinality τ can be isometrically embedded as a closed subset in the algebraically τ-dimensional Euclidean space E^τ, but not in E^σ for $\sigma < \tau$.*

It is natural to compare these theorems with the well-known Nagata–Smirnov theorem (on the homeomorphic embedding in H^τ) and Kuratowski's (and Arens') theorems (on isometric (and closed) embeddings in Banach space (a normed vector space)). Theorems 3 and 4 are extremely nontrivial even for finite ultrametric spaces.

Theorem 5. *Every ultrametric space consisting of $n+1$ points can be isometrically embedded in the n-dimensional Euclidean space E^n. No ultrametric space consisting of $n+1$ points can be isometrically embedded in E^k for $k < n$* [9].

In other words $n+1$ points of an ultrametric space can be considered as vertices of an n-dimensional simplex $\subset E^n$. This enables us to apply the theory of ultrametric spaces to linear and convex programming [13].

Correcting the typographical mistake of the paper [9] we adduce the following corollary giving another application of the theory of ultrametric spaces to number theory [13].

Corollary 2. *Let Z_p be the ring of p-adic Hensel integers, H be the classic Hilbert space. There exists an isometric* (*and closed*) *embedding $i : Z_p \to H$ under which the image $i(Z_p)$ is located on the sphere $S_{r(p)}$ of radius*

$$r(p) = \frac{p}{\sqrt{2(p^2+p+1)}} \to \frac{1}{\sqrt{2}} \quad \text{as } p \to \infty.$$

Note 4. The assignment of the space *LM* to each pointed set M is a covariant functor from the category SET* to the category ULTRAMETR* of all pointed ultrametric spaces and non-expanding maps [7,8,11,12]. It is more interesting to consider it as a functor from the category ORDER of ordered sets to ULTRAMETR* (using only monotone functions in the modified definition of LW_τ). These functors will be described in another paper.

Problem 1. If a weight of a class of ultrametric spaces is a cardinal $\tau \geqslant c$ such that $\tau^{\aleph_0} = \tau$ then a weight of the universal ultrametric space LW_τ is the smallest of possible ones $w(LW_\tau) = \tau^{\aleph_0} = \tau$. However there exist infinite cardinals $\tau > c$ such that $\tau^{\aleph_0} > \tau$ [19,20]. Let the weight of a considered class of ultrametric spaces be such cardinal. *Does there exist a universal ultrametric space (U, d) of the weight smaller than $\tau^{\aleph_0}$*?

Problem 2. The proof of Theorem 1 depends on the Axiom of Choice essentially. *Is the theorem equivalent to the Axiom*?

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References

[1] J. de Groot, Non-Archimedean metrics in topology, Proc. Amer. Math. Soc. 7 (6) (1956) 948–956.
[2] F. Hausdorff, Uber innere Abbildungen, Fund. Math. 23 (1934) 279–291.
[3] M. Krasner, Nombres semi-réels et espaces ultramétriques, C. R. Acad. Sci. Paris 219 (1944) 433–435.
[4] A.J. Lemin, On functor of Hausdorff exponent, in: Colloquium on Topology, Eger, August 1983, Budapest, 1983, p. 59.
[5] A.J. Lemin, On isosceles metric spaces and groups, in: Cardinal Invariants and Mappings of Topological Spaces, Izhevsk, 1984, pp. 26–31 (in Russian).
[6] A.J. Lemin, Proximity on isosceles spaces, Russian Math. Surv. 39 (1) (1984) 143–144.
[7] A.J. Lemin, On stability of the property of a space being isosceles, Russian Math. Surv. 39 (5) (1984) 283–284.
[8] A.J. Lemin, Transition functor to a function space in the uniform topology, Russian Math. Surv. 40 (6) (1985) 133–134.
[9] A.J. Lemin, Isometric embedding of isosceles (non-Archimedean) spaces in Euclidean spaces, Soviet Math. Dokl. 32 (3) (1985) 740–744.
[10] A.J. Lemin, Ultrametric spaces, topos theory and intuitionistic logic, in: Proc. International Conference on Topology, October 1987, Baku, 1987, p. 163.

[11] A.J. Lemin, An application of the theory of isosceles (ultrametric) spaces to the Trnkova–Vinarek theorem, Comment. Math. Univ. Carolin. 29 (3) (1988) 427–434.
[12] A.J. Lemin, Inverse images of metric spaces under non-expanding open mappings, Russian Math. Surv. 43 (3) (1988) 214–215.
[13] A.J. Lemin, Ultrametric spaces and convex programming, in: Proc. Internat. Conf. on Topology, Varna, 1990, pp. 28–29.
[14] A.J. Lemin, Ultrametric spaces and Boolean algebras, in: Proc. 5th Internat. Conf. "Topology and its Applications", Zagreb, 1990, p. 49.
[15] A.J. Lemin, The Smirnov compactification functor is one-to-one over the class of complete first countable spaces, Topology Appl. 38 (1991) 201–204.
[16] A.J. Lemin, V. Lemin, Cayley–MacLane type theorem for ultrametric spaces, in: Proc. VIII Prague Symposium "General Topology and its Relations to Modern Analysis and Algebra", Prague, 1996, pp. 60–61.
[17] A.J. Lemin, V. Lemin, On uniform rationalization of ultrametrics, Topology Proc. 22 (1999) 275–283.
[18] A.J. Lemin, Yu.M. Smirnov, Isometry groups of metric and ultrametric spaces and their subgroups, Russian Math. Surv. 41 (6) (1986) 213–214.
[19] M.E. Rudin, Lectures on Set-Theoretic Topology, Providence, 1975.
[20] T. Jech, Lectures in Set Theory, Lecture Notes in Math., Vol. 217, Springer, Berlin, 1971.
[21] I. Vestfrid, On a universal ultrametric space, Ukrain. Math. Z. 46 (12) (1994) 1700–1706 (in Russian).