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Heat transfer in a liquid film over an unsteady stretching surface with viscous dissipation in presence of external magnetic field

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ABSTRACT

This paper presents a mathematical analysis of MHD flow and heat transfer to a laminar liquid film from a horizontal stretching surface. The flow of a thin fluid film and subsequent heat transfer from the stretching surface is investigated with the aid of similarity transformation. The transformation enables to reduce the unsteady boundary layer equations to a system of non-linear ordinary differential equations. Numerical solution of resulting non-linear differential equations is found by using efficient shooting technique. Boundary layer thickness is explored numerically for some typical values of the unsteadiness parameter *S* and Prandtl number *Pr*, Eckert number *Ec* and Magnetic parameter *Mn*. Present analysis shows that the combined effect of magnetic field and viscous dissipation is to enhance the thermal boundary layer thickness.

1. Introduction

In recent years the analysis of flow of a thin liquid film have attracted the attention of number of researchers because of its possible applications in many branches of science and technology. The knowledge of flow and heat transfer within a thin liquid film is crucial in understanding the coating process and design of various heat exchangers and chemical processing equipments. Other applications include wire and fiber coating, food stuff processing reactor fluidization, transpiration cooling and so on. The prime aim in almost every extrusion application is to maintain the surface quality of the extrudate. All coating processes demand a smooth glossy surface to meet the requirements for best appearance and optimum service properties such as low friction, transparency and strength. The problem of extrusion of thin surface layers needs special attention to gain some knowledge for controlling the coating product efficiently.

A class of flow problems with obvious relevance to polymer extrusion is the flow induced by the stretching motion of a flat elastic sheet. For example, in a melt spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet, which is thereafter solidified through rapid quenching or gradual cooling by direct contact with water or the coolant liquid. The stretching provides a unidirectional orientation to the extrudate, thereby improving the fluid mechanical properties. The quality of the final product greatly depends on the rate of cooling and the stretching rate. The choice of an appropriate cooling liquid is crucial as it has a direct impact on rate of cooling and care must be taken to exercise optimum stretching rate otherwise sudden stretching may spoil the properties desired for the final outcome. Some industrially important liquids like synthetic oils, dilute polymeric solutions such as 5.4% of polyisobutylene in cetane can be used as effective coolant liquids (see [1]). Since the quality of the final product in such extrusion processes depend

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Nomenclature

b	stretching rate [s^{-1}]
U	sheet velocity [$m\ s^{-1}$]
x	horizontal co-ordinate [m]
y	vertical co-ordinate [m]
u	horizontal velocity component [$m\ s^{-1}$]
v	vertical velocity component [$m\ s^{-1}$]
T	temperature [K]
t	time [s]
h	film thickness [m]
S	Unsteadiness parameter, $\frac{\alpha}{b}$
C_p	specific heat [$J\ kg^{-1}K^{-1}$]
f	dimensionless stream function, Eq. (10)
Pr	Prandtl number, $\frac{\upsilon}{k}$
Ec	Eckert number, $\frac{U^2}{C_p(T_s-T_0)}$
Mn	Magnetic parameter, $\frac{\sigma B_0^2}{\rho b}$
q	heat flux, $-k\frac{\partial T}{\partial y}$ [$Js^{-1}m^{-2}$]
Re_x	local Reynolds number
Nu_x	local Nusselt number, Eq. (23)

Greek symbols

α	constant [s^{-1}]
β	dimensionless film thickness
η	similarity variable, Eq. (12)
θ	dimensionless temperature, Eq. (11)
k	thermal diffusivity [$m^2\ s^{-1}$]
μ	dynamic viscosity [$kg\ m^{-1}\ s^{-1}$]
υ	kinematic viscosity [$m^2\ s^{-1}$]
ρ	density [$kg\ m^{-3}$]
τ	shear stress, $\mu\partial u/\partial y$ [$kg\ m^{-1}\ s^{-2}$]
ψ	stream function [$m^2\ s^{-1}$]

Subscripts

o	origin
ref	reference value
s	sheet
x	local value

Superscripts

$'$	first derivative
$''$	second derivative
$'''$	third derivative

considerably on the flow and heat transfer characteristics of a thin liquid film over a stretching sheet, the analysis and fundamental understanding of the momentum and thermal transports for such processes are very important.

Crane [2] was the first among others to consider the steady two-dimensional flow of a Newtonian fluid driven by a stretching elastic flat sheet which moves in its own plane with a velocity varying linearly with the distance from a fixed point. The pioneering works of Crane [2] are subsequently extended by many authors to explore various aspects of the flow and heat transfer occurring in an infinite domain of the fluid surrounding the stretching sheet (see [3–16]). Wang [17,18] has studied the three dimensional flow due to a stretching surface and that due to a stretching surface in a rotating fluid. The hydrodynamics of the finite fluid domain (thin liquid film), over a stretching sheet was first considered by Wang [19] who reduced the unsteady Nervier–Stokes equations to a non-linear ordinary differential equations by means of similarity transformation and solved the same using a kind of multiple shooting method (see Robert and Shipman [20]). However, Liao [21] has used homotopy analysis method to deal with the non-linear problems of similar kind. Wang [22] himself has used homotopy analysis method to reinvestigate the thin film flow over a stretching sheet. Of late the works of Wang [19] to the case of finite fluid domain are extended by several authors [23–28] for fluids of both Newtonian and non-Newtonian kinds using various velocity and thermal boundary conditions.

There are extensive works in literature concerning the production of thin fluid film either on a vertical wall achieved through the action of gravity or that over a rotating disc achieved through the action of centrifugal forces. The problem of laminar-film condensation on a vertical plate was first considered by Sparrow and Gregg [29] based on the boundary layer theory and similarity transformation. They [30] have also considered the heat and mass transfer in a liquid film on a rotating disk. The melting from a horizontal rotating disk was investigated by Wang [31] where the non-linear governing equations were solved by perturbation method and numerical integration. Dandapat and Ray [32,33] have studied the thin liquid film flow over a rotating horizontal disk. Kumari and Nath [34] studied the unsteady MHD film over a rotating infinite disk. Dandapat et al. [35] have investigated liquid film flow over an unsteady stretching sheet. Abbas et al. [36] have studied the flow of a second grade fluid film over an unsteady stretching sheet.

Most of the aforementioned studies have neglected the combined effect of viscous dissipation and magnetic field on the heat transfer which is important in view point of desired properties of the outcome. In the present study we include the same for heat transfer analysis in a thin liquid film from an unsteady stretching sheet.

2. Mathematical formulation

2.1. Governing equations and Boundary conditions

Let us consider a thin elastic sheet which emerges from a narrow slit at the origin of a Cartesian co-ordinate system for investigations as shown schematically in Fig. 1. The continuous sheet at $y = 0$ is parallel with the x-axis and moves in its own plane with the velocity

$$U(x,t) = \frac{bx}{(1-\alpha t)}, \tag{1}$$

where b and α are both positive constants with dimension per time. The surface temperature T_s of the stretching sheet is assumed to vary with the distance x from the slit as

$$T_s(x,t) = T_0 - T_{\mathrm{ref}}\left[\frac{bx^2}{2\upsilon}\right](1-\alpha t)^{-\frac{3}{2}}, \tag{2}$$

where T_0 is the temperature at the slit and T_{ref} can be taken as a constant reference temperature such that $0 \leqslant T_{\mathrm{ref}} \leqslant T_0$. The term $\frac{bx^2}{\upsilon(1-\alpha t)}$ can be recognized as the Local Reynolds number based on the surface velocity U. The expression (1) for the velocity of the sheet $U(x,t)$ reflects that the elastic sheet which is fixed at the origin is stretched by applying a force in the positive x-direction and the effective stretching rate $\frac{b}{(1-\alpha t)}$ increase with time as $0 \leqslant \alpha < 1$. With the same analogy the expression for the surface temperature $T_s(x,t)$ given by Eq. (2) represents a situation in which the sheet temperature decreases from T_0 at the slit in proportion to x^2 and such that the amount of temperature reduction along the sheet increases with time. The applied transverse magnetic field is assumed to be of variable kind and is chosen in its special form as

$$B(x,t) = B_0(1-\alpha t)^{-\frac{1}{2}}. \tag{3}$$

The particular form of the expressions for $U(x, t)$, $T_s(x,t)$ and $B(x, t)$ are chosen so as to facilitate the construction of a new similarity transformation which enables in transforming the governing partial differential equations of momentum and heat transport into a set of non-linear ordinary differential equations.

Consider a thin elastic liquid film of uniform thickness $h(t)$ lying on the horizontal stretching sheet (Fig. 1). The x-axis is chosen in the direction along which the sheet is set to motion and the y-axis is taken perpendicular to it. The fluid motion within the film is primarily caused solely by stretching of the sheet. The sheet is stretched by the action of two equal and

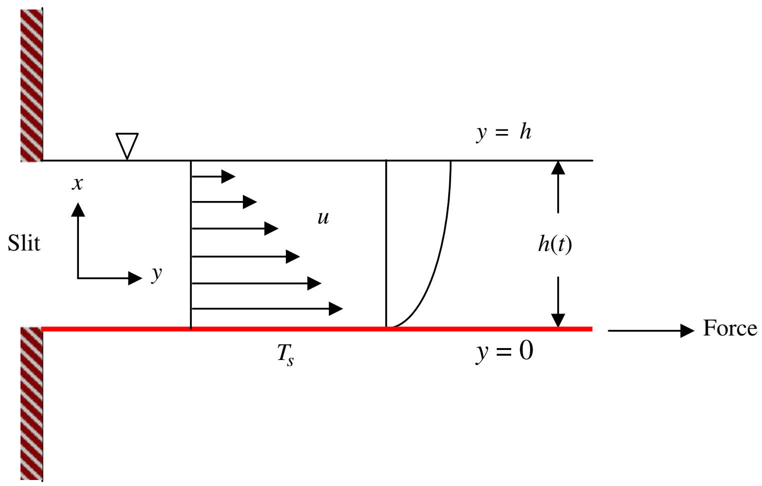

Fig. 1. Schematic representation of a liquid film on an elastic sheet.

opposite forces along the x-axis. The sheet is assumed to have velocity U as defined in Eq. (1) and the flow field is exposed to the influence of an external transverse magnetic field of strength B as defined in Eq. (3). We have neglected the effect of latent heat due to evaporation by assuming the liquid to be nonvolatile. Further the buoyancy is neglected due to the relatively thin liquid film, but it is not so thin that intermolecular forces come into play. The velocity and temperature fields of the liquid film obey the following boundary layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}u, \tag{5}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{6}$$

The pressure in the surrounding gas phase is assumed to be uniform and the gravity force gives rise to a hydrostatic pressure variation in the liquid film. In order to justify the boundary layer approximation, the length scale in the primary flow direction must be significantly larger than the length scale in the cross stream direction. We choose the representative measure of the film thickness to be $\left(\frac{\upsilon}{b}\right)^{\frac{1}{2}}$ so that the scale ratio is large enough i.e., $\frac{x}{\left(\frac{\upsilon}{b}\right)^{\frac{1}{2}}}\gg 1$. This choice of length scale enables us to employ the boundary layer approximations. Further it is assumed that the induced magnetic field is negligibly small.

The associated boundary conditions are given by

$$u=U,\quad v=0,\quad T=T_s\quad \text{at}\quad y=0, \tag{7}$$

$$\frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=0\quad \text{at}\quad y=h, \tag{8}$$

$$v=\frac{dh}{dt}\quad \text{at}\quad y=h. \tag{9}$$

At this juncture we make a note that the mathematical problem is implicitly formulated only for $x\geqslant 0$. Further it is assumed that the surface of the planar liquid film is smooth so as to avoid the complications due to surface waves. The influence of interfacial shear due to the quiescent atmosphere, in other words the effect of surface tension is assumed to be negligible. The viscous shear stress $\tau=\mu\left(\frac{\partial u}{\partial y}\right)$ and the heat flux $q=-k\left(\frac{\partial T}{\partial y}\right)$ vanish at the adiabatic free surface (at $y = h$).

2.2. Similarity transformations

We now introduce dimensionless variables f and θ and the similarity variable η as

$$f(\eta)=\frac{\psi(x,y,t)}{\left(\frac{\upsilon b}{1-\alpha t}\right)^{\frac{1}{2}}x}, \tag{10}$$

$$\theta(\eta)=\frac{T_0-T(x,y,t)}{T_{\text{ref}}\left(\frac{bx^2}{2\upsilon(1-\alpha t)^{-\frac{3}{2}}}\right)}, \tag{11}$$

$$\eta=\left(\frac{b}{\upsilon(1-\alpha t)}\right)^{\frac{1}{2}}y. \tag{12}$$

The physical stream function $\psi(x, y, t)$ automatically assures mass conservation given in Eq. (4). The velocity components are readily obtained as

$$u=\frac{\partial\psi}{\partial y}=\left(\frac{bx}{1-\alpha t}\right)f'(\eta), \tag{13}$$

$$v=-\frac{\partial\psi}{\partial x}=-\left(\frac{\upsilon b}{1-\alpha t}\right)^{\frac{1}{2}}f(\eta). \tag{14}$$

The mathematical problem defined through Eqs. (4)–(9) transforms exactly into a set of ordinary differential equations and associated boundary conditions as follows

$$S\left(f'+\frac{\eta}{2}f''\right)+(f')^2-ff''=f'''-Mnf', \tag{15}$$

$$Pr\left[\frac{S}{2}(3\theta+\eta\theta')+2\theta f'-\theta' f\right]=\theta''+EcPrf''^2, \tag{16}$$

$$f'(0)=1,\quad f(0)=0,\quad \theta(0)=1, \tag{17}$$

$$f''(\beta)=0,\quad \theta'(\beta)=0, \tag{18}$$

$$f(\beta)=\frac{S\beta}{2}. \tag{19}$$

Here $S \equiv \frac{\alpha}{b}$ is the dimensionless measure of the unsteadiness and the prime indicates differentiation with respect to η. Further, β denotes the value of the similarity variable η at the free surface so that Eq. (12) gives

$$\beta = \left(\frac{b}{\upsilon(1-\alpha t)}\right)^{\frac{1}{2}} h. \tag{20}$$

Yet β is an unknown constant, which should be determined as an integral part of the boundary value problem. The rate at which film thickness varies can be obtained by differentiating Eq. (20) with respect to t, in the form

$$\frac{dh}{dt} = -\frac{\alpha\beta}{2}\left(\frac{\upsilon}{b(1-\alpha t)}\right)^{\frac{1}{2}}. \tag{21}$$

Thus the kinematic constraint at $y = h(t)$ given by Eq. (9) transforms into the free surface condition (21). It is noteworthy that the momentum boundary layer equation defined by Eq. (15) subject to the relevant boundary conditions (17)–(19) is decoupled from the thermal field; on the other hand the temperature field $\theta(\eta)$ is coupled with the velocity field $f(\eta)$. Since the sheet is stretched horizontally the convection least affects the flow (i.e., buoyancy effect is negligibly small) and hence there is a one-way coupling of velocity and thermal fields.

The local skin friction coefficient, which of practical importance, is given by

$$C_f \equiv \frac{-2\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho U^2} = -2Re_x^{-\frac{1}{2}} f''(0), \tag{22}$$

and the heat transfer between the surface and the fluid conventionally expressed in dimensionless form as a local Nusselt number is given by

$$Nu_x \equiv -\frac{x}{T_{\mathrm{ref}}}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{2}(1-\alpha t)^{\frac{-1}{2}}\theta'(0)Re_x^{\frac{3}{2}}, \tag{23}$$

where $Re_x = \frac{Ux}{\upsilon}$ the local Reynolds number and T_{ref} denotes the same reference temperature (temperature difference) as in Eq. (2).

We now march on to find the solution of the boundary value problem (15)–(19).

3. Numerical solution

The non-linear differential Eqs. (15) and (16) with appropriate boundary conditions given in (17)–(19) are solved numerically, by the most efficient numerical shooting technique with fourth order Runge–Kutta algorithm (see Refs. [37,38]). The non-linear differential Eqs. (15) and (16) are first decomposed to a system of first order differential equations in the form

$$\begin{aligned}
&\frac{df_0}{d\eta} = f_1, \\
&\frac{df_1}{d\eta} = f_2, \\
&\frac{df_2}{d\eta} = S\left(f_1 + \frac{\eta}{2}f_2\right) + (f_1)^2 - f_0 f_2 + Mnf_1, \\
&\frac{d\theta_0}{d\eta} = \theta_1, \\
&\frac{d\theta_1}{d\eta} = Pr\left[\frac{S}{2}(3\theta_0 + \eta\theta_1) + 2\theta_0 f_1 - \theta_1 f_0\right] - EcPrf_2^2.
\end{aligned} \tag{24}$$

Corresponding boundary conditions take the form,

$$f_1(0) = 1, \quad f_0(0) = 0, \quad \theta_0(0) = 1, \tag{25}$$

$$f_2(\beta) = 0, \quad \theta_1(\beta) = 0, \tag{26}$$

$$f_0(\beta) = \frac{S\beta}{2}. \tag{27}$$

Here $f_0(\eta) = f(\eta)$ and $\theta_0(\eta) = \theta(\eta)$. The above boundary value problem is first converted into an initial value problem by appropriately guessing the missing slopes $f_2(0)$ and $\theta_1(0)$. The resulting IVP is solved by shooting method for a set of parameters appearing in the governing equations and a known value of S. The value of β is so adjusted that condition (27) holds. This is done on the trial and error basis. The value for which condition (27) holds is taken as the appropriate film thickness and the IVP is finally solved using this value of β. The step length of $h = 0.01$ is employed for the computation purpose. The convergence criterion largely depends on fairly good guesses of the initial conditions in the shooting technique. The iterative process is terminated until the relative difference between the current and the previous iterative values of $f(\beta)$ matches with the value of $\frac{S\beta}{2}$ up to a tolerance of 10^{-6}. Once the convergence in achieved we integrate the resultant ordinary differential equations using standard fourth order Runge–Kutta method with the given set of parameters to obtain the required solution.

Table 1
Comparison of values of skin friction coefficient $f''(0)$ with $Mn = 0.0$.

S	Wang [22]			Present results	
	β	$f''(0)$	$\frac{f''(0)}{\beta}$	β	$f''(0)$
0.4	5.122490	−6.699120	−1.307785	4.981455	−1.134098
0.6	3.131250	−3.742330	−1.195155	3.131710	−1.195128
0.8	2.151990	−2.680940	−1.245795	2.151990	−1.245805
1.0	1.543620	−1.972380	−1.277762	1.543617	−1.277769
1.2	1.127780	−1.442631	−1.279177	1.127780	−1.279171
1.4	0.821032	−1.012784	−1.233549	0.821033	−1.233545
1.6	0.576173	−0.642397	−1.114937	0.576176	−1.114941
1.8	0.356389	−0.309137	−0.867414	0.356390	−0.867416

Note: Wang [22] has used different similarity transformation due to which the value of $\frac{f''(0)}{\beta}$ in his paper is the same as $f''(0)$ of our results.

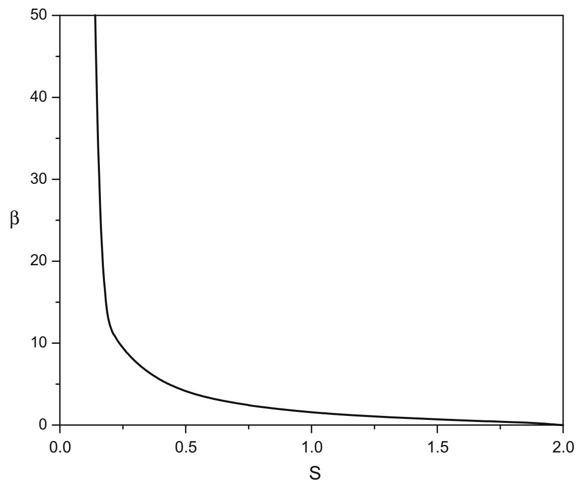

Fig. 2. Variation of film thickness β with unsteadiness parameter S with $Mn = 0.0$.

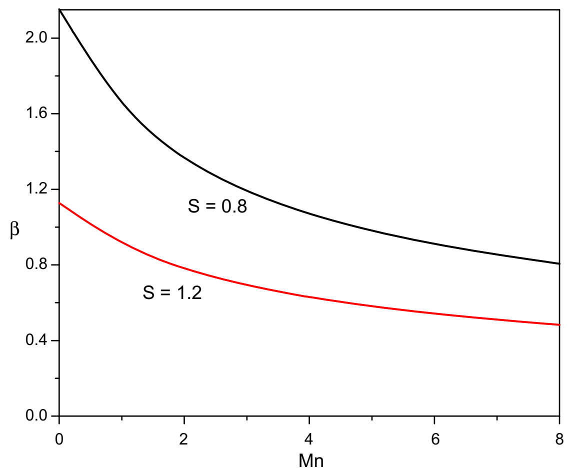

Fig. 3. Variation of film thickeness β with magnetic parameter Mn.

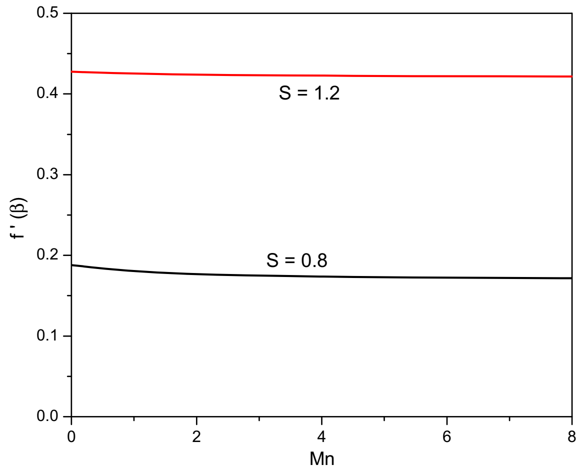

Fig. 4. Variation of free surface velocity $f'(\beta)$ with magnetic parameter *Mn*.

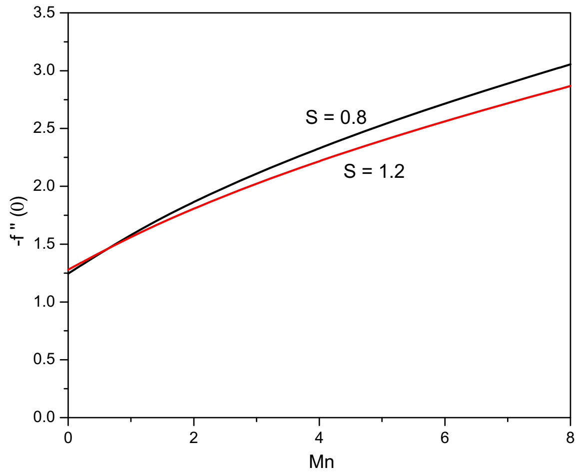

Fig. 5. Variation of wall shear stress parameter $-f''(0)$ with magnetic parameter *Mn*.

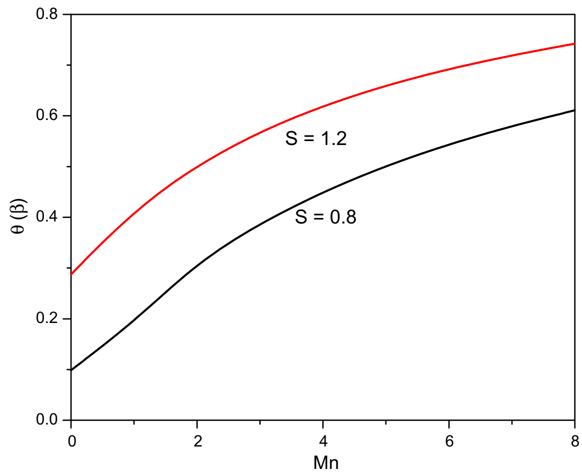

Fig. 6. Variation of free surface temperature $\theta(\beta)$ with the magnetic parameter *Mn*.

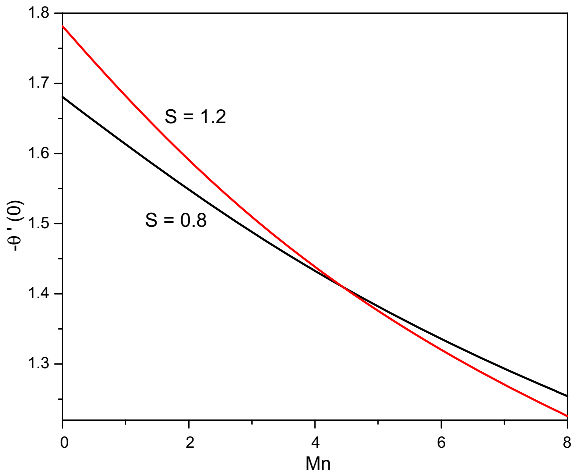

Fig. 7. Variation of wall dimensionless heat flux $-\theta'(0)$ with magnetic parameter *Mn*.

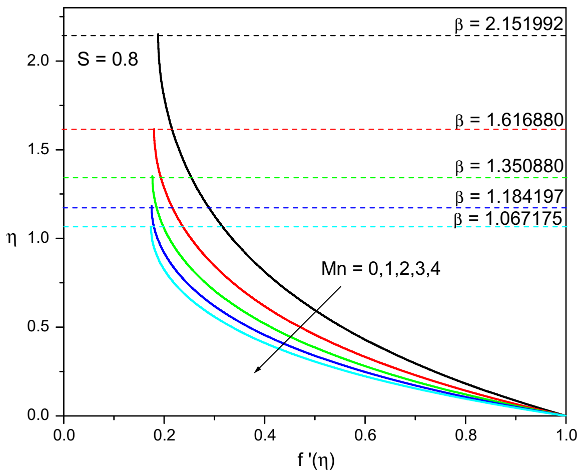

Fig. 8a. Variation in the velocity profiles $f'(\eta)$ for different values of magnetic parameter *Mn* with $S = 0.8$.

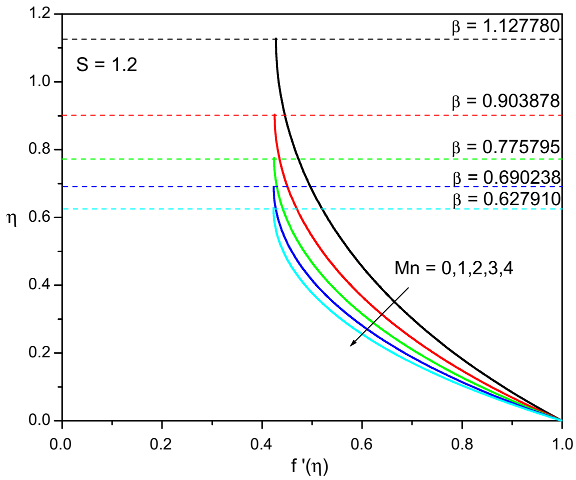

Fig. 8b. Variation in the velocity profiles $f'(\eta)$ for different values of magnetic parameter *Mn* with $S = 1.2$.

4. Results and discussion

The exact solution do not seem to be feasible for a complete set of Eqs. (15) and (16) because of the non-linear form of the momentum and thermal boundary layer equations. This fact forces one to obtain the solution of the problem numerically. Appropriate similarity transformation is adopted to transform the governing partial differential equations of flow and heat transfer into a system of non-linear ordinary differential equations. The resultant boundary value problem is solved by the efficient shooting method. It is note worthy to mention that the solution exists only for a small range of values of unsteadiness parameter $0 \leqslant S \leqslant 2$. Moreover, when $S \to 0$ the solution approaches to the analytical solution obtained by Crane [2] with infinitely thick layer of fluid ($\beta \to \infty$). The other limiting solution corresponding to $S \to 2$ represents a liquid film of infinitesimal thickness ($\beta \to 0$). The numerical results are obtained for $0 \leqslant S \leqslant 2$. Present results are compared with some of the earlier published results in some limiting cases which are tabulated in Tables 1 and 2. The effects of magnetic parameter on various fluid dynamic quantities are shown in Figs. 2–9 for different unsteadiness parameter.

Fig. 2 shows the variation of film thickness β with the unsteadiness parameter. It is evident from this plot that the film thickness β decreases monotonically when S is increased from 0 to 2. This result concurs with that observed by Wang [22]. The variation of film thickness β with respect to the magnetic parameter Mn is projected in Fig. 3 for different values of unsteadiness parameter. It is clear from this plot that the increasing values of magnetic parameter decreases the film thickness. The result holds for different values of unsteadiness parameter S.

The variation of free surface velocity $f(\beta)$ with respect to Mn is shown in Fig. 4. The free surface velocity behaves almost as a constant function of Mn as can be seen from Fig. 4. The effect of Mn on the wall shear stress parameter $-f''(0)$ is illustrated

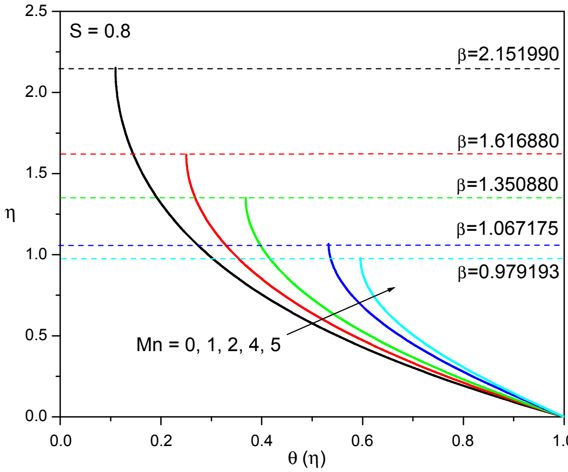

Fig. 9a. Variation of dimensionless temperature $\theta(\eta)$ with the magnetic parameter with $S = 0.8$.

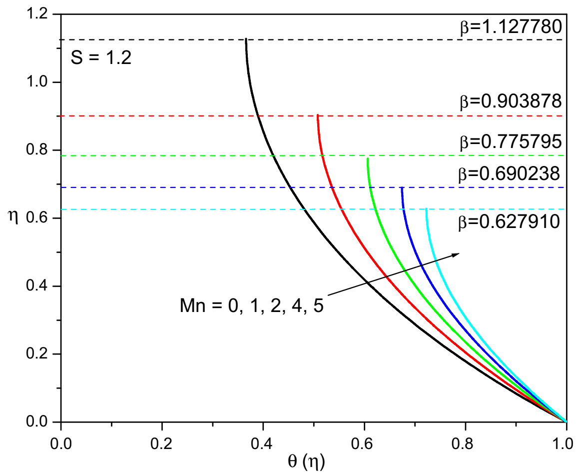

Fig. 9b. Variation of dimensionless temperature $\theta(\eta)$ with the magnetic parameter with $S = 1.2$.

Table 2
Comparison of values of surface temperature $\theta(\beta)$ and wall temperature gradient $-\theta'(0)$ with $Mn = Ec = 0.0$.

Pr	Wang [22]			Present results	
	$\theta(\beta)$	$-\theta'(0)$	$\frac{-\theta'(0)}{\beta}$	$\theta(\beta)$	$-\theta'(0)$
S = 0.8 and β = 2.15199					
0.01	0.960480	0.090474	0.042042	0.960438	0.042120
0.1	0.692533	0.756162	0.351378	0.692296	0.351920
1	0.097884	3.595790	1.670913	0.097825	1.671919
2	0.024941	5.244150	2.436884	0.024869	2.443914
3	0.008785	6.514440	3.027170	0.008324	3.034915
S = 1.2 and β = 1.127780					
0.01	0.982331	0.037734	0.033458	0.982312	0.033515
0.1	0.843622	0.343931	0.304962	0.843485	0.305409
1	0.286717	1.999590	1.773032	0.286634	1.773772
2	0.128124	2.975450	2.638324	0.128174	2.638431
3	0.067658	3.698830	3.279744	0.067737	3.280329

Note: Wang [22] has used different similarity transformation due to which the value of $\frac{-\theta'(0)}{\beta}$ in his paper is the same as $-\theta'(0)$ of our results.

Table 3
Values of β, $f(\beta)$, $f'(\beta)$, $f''(\beta)$, $f'''(\beta)$ for various values of S, Mn.

S	Mn	β	$f(\beta)$	$f'(\beta)$	$f''(\beta)$	$f'''(\beta)$
0.0	0.0	∞	1.000000	0.000000	0.000000	0.000000
0.002	0.0	1001.066	1.001065	0.000024	0.000000	0.000000
0.1	0.0	20.123800	1.006196	0.000007	0.000000	0.000001
0.2	0.0	10.121550	1.012155	0.001566	0.000000	0.000316
0.4	0.0	4.981455	0.996291	0.029495	0.000000	0.012668
0.6	0.0	3.131710	0.939513	0.094580	0.000000	0.065693
0.8	0.0	2.151990	0.860797	0.187840	0.000000	0.185556
1.0	0.0	1.543617	0.771808	0.300925	0.000000	0.391481
1.2	0.0	1.127780	0.676668	0.427548	0.000000	0.695854
1.4	0.0	0.821033	0.574723	0.563274	0.000000	1.105860
1.6	0.0	0.576176	0.460940	0.705119	0.000000	1.625384
1.8	0.0	0.356390	0.320751	0.851124	0.000000	2.256437
1.9	0.0	0.237020	0.225167	0.925253	0.000000	2.614076
1.99	0.0	0.000010	0.000010	1.000000	0.000000	2.990000
2.00	0.0	0.000001	0.000000	1.000000	0.000000	3.000000
0.8	0.0	2.151990	0.860797	0.187840	0.000000	0.185556
0.8	1.0	1.616880	0.646752	0.179716	0.000000	0.355787
0.8	2.0	1.350880	0.540352	0.176488	0.000000	0.525313
0.8	3.0	1.184197	0.473679	0.174751	0.000000	0.694590
0.8	4.0	1.067175	0.426870	0.173665	0.000000	0.863753
0.8	5.0	0.979192	0.391677	0.172922	0.000000	1.032852
0.8	6.0	0.909925	0.363970	0.172381	0.000000	1.201908
0.8	7.0	0.853552	0.341421	0.171971	0.000000	1.370947
0.8	8.0	0.806512	0.322605	0.171648	0.000000	1.539967
1.2	0.0	1.127780	0.676668	0.427548	0.000000	0.695854
1.2	1.0	0.903878	0.542327	0.425050	0.000000	1.115778
1.2	2.0	0.775795	0.465478	0.423849	0.000000	1.535964
1.2	3.0	0.690238	0.414143	0.423142	0.000000	1.956246
1.2	4.0	0.627910	0.376746	0.422676	0.000000	2.376569
1.2	5.0	0.579900	0.347940	0.422346	0.000000	2.796919
1.2	6.0	0.541450	0.324870	0.422100	0.000000	3.217286
1.2	7.0	0.509757	0.305854	0.421909	0.000000	3.637665
1.2	8.0	0.483048	0.289829	0.421759	0.000000	4.058062

in Fig. 5. Clearly, increasing values of Mn results in increasing the wall shear stress. Fig. 6 demonstrates the effect of Mn on the free surface temperature $\theta(\beta)$. From this plot it is evident that the free surface temperature increases monotonically with Mn. Fig. 7 highlights the effect of Mn on the dimensionless wall heat flux $-\theta'(0)$. It is found from this plot that the dimensionless wall heat flux $-\theta'(0)$ decreases with the increasing values of Mn. The effect of Mn on $f(\beta)$, $-f''(0)$, $\theta(\beta)$ and $-\theta'(0)$ is observed to be same for different values of unsteadiness parameter S.

The effect of Mn on the axial velocity is depicted in Figs. 8a and 8b for two different values of S. From these plots it is clear that the increasing values of magnetic parameter decreases the axial velocity. This is due to the fact that applied transverse magnetic field produces a drag in the form of Lorentz force thereby decreasing the magnitude of velocity. The drop in horizontal velocity as a consequence of increase in the strength of magnetic field is observed for both the values of $S = 0.8$ and $S = 1.2$.

Figs. 9a and 9b depicts the effect of Mn on temperature profiles for two different values of S. The results show that the thermal boundary layer thickness increases with the increasing values of Mn. The increasing frictional drag due to the Lorentz force is responsible for increasing the thermal boundary layer thickness.

Further, the effects of Prandtl number Pr and the Eckert number Ec are not affected by the magnetic parameter Mn. Hence we omit the discussion on the results of Pr and Ec for reasons of space as they are extensively studied by Chen [39,40].

Tables 1 and 2 give the comparison of present results with that of Wang [22]. With out any doubt, from these tables, we can claim that our results are in excellent agreement with that of Wang [22] under some limiting cases. The values of $f(\beta)$, $f'(\beta)$, $f''(\beta)$, $f'''(\beta)$ at the free surface, obtained through the actual computation, are tabulated in Table 3. The values tabulated in this table are very important as they serve the purpose of validating the momentum Eq. (15) in dimensionless form (thereby Eq. (5)), at the free surface $\eta = \beta$.

5. Conclusions

A theoretical study of the boundary layer behavior in a liquid film over an unsteady stretching sheet is carried out including the effects of variable transverse magnetic field. The results on viscous dissipation coincide with that of Chen [40]. Present investigation reveals that the magnetic field effect has a considerable impact in controlling the flow and heat transfer. The effect of transverse magnetic field on a viscous incompressible conducting fluid is to suppress the velocity field which in turn causes the enhancement of the temperature field.

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