GRAF-TYPE THEOREM
FOR LAGUERRE AND LEGENDRE FUNCTIONS

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Abstract—In this paper, we discuss Graf-type series for Laguerre and Legendre functions. Within this context, we indicate the existence of two-variable, one-index generalized functions and discuss their properties. This class of special functions represents a useful tool for dealing with a large number of physical problems, from that relevant to multilevel system dynamics to the study of incommensurate structures.

1. INTRODUCTION

The theory of generalized Bessel functions (GBF) has been developed during recent years mainly for their practical importance in many physical and engineering problems, ranging from non-dipolar scattering to surface diffraction theory [1,2]. The properties of these functions have been studied by non-mathematicians; therefore, their applicative aspects have been emphasized while the relevant, undoubtedly important, mathematical features have not been explored as they should be. The underlying group theoretic structure has not, indeed, been discussed yet, the possibility of studying classes of generalized special functions (Laguerre, hypergeometric, etc.) merely mentioned, and the partial differential equations they satisfy just touched on. The second point is particularly important within the context of “practical” problems like the generation of squeezed states [3] and is also of genuine mathematical interest.

The GBFs introduced in [1,2] are functions of two variables and one parameter and are defined by a series of products of ordinary Bessel functions, namely,

\[ (m)J_n(x, y; t) = \sum_{l=-\infty}^{+\infty} t^l J_l(x) J_{n+m+l}(y), \] (1.1)

where \( m \) and \( n \) are integers and \( t \) is a real or complex parameter such that \( 0 < |t| < \infty \). Together with (1.1), the relevant modified version has been studied and the extension to \( M (> 2) \) variables has been discussed [4].

The series in eq. (1.1) is a generalization of that involved in the Graf theorem which is recovered setting \( m = 1 \). On the other hand, the functions generated by (1.1) setting \( m = \pm 1 \) can be viewed as prototypes of GBF. In fact, as pointed out in [4],

\[ b_{en}(x\sqrt{2}) = (+1)^m J_n(-x, -ix; 1), \] (1.2)

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with $be_n(x)$ being the complex Kelvin function ($be_n(x) = ber_n(x) + ibei_n(x)$) [5]. Just to give a further example, we notice that the function

$$(-1) J_n(iy, x; -i) = \sum_{l=-\infty}^{+\infty} J_{n-l}(x) I_l(y) = (-1)^l I_n(x, y)$$

possesses some interesting features. They satisfy, indeed, the following recurrence relations:

$$\frac{\partial}{\partial x} (-1)^l I_n(x, y) = \frac{1}{2} \left[ (-1)^l I_{n-1}(x, y) - (-1)^l I_{n+1}(x, y) \right],$$

$$\frac{\partial}{\partial y} (-1)^l I_n(x, y) = \frac{1}{2} \left[ (-1)^l I_{n+1}(x, y) + (-1)^l I_{n-1}(x, y) \right],$$

$$2(-1) I_n(x, y) = x \left[ (-1)^l I_{n-1}(x, y) + (-1)^l I_{n+1}(x, y) \right],$$

$$+ y \left[ (-1)^l I_{n-1}(x, y) - (-1)^l I_{n+1}(x, y) \right],$$

and hence the partial differential equation:

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 1 \right) (-1)^l I_n(x, y) = 0.$$

Furthermore, as can be easily checked, $(-1)^l I_n(x, x)$ reduces just to

$$(-1)^l I_n(x, x) = \frac{x^n}{n!}.$$  

The extension to $M$-variables of functions generated by Graf series can be achieved along the lines developed in [1,2,4]. A deeper understanding of the functions generated by Graf series may certainly provide further insight into the structure of generalized special functions; therefore, this paper is aimed at stating the Graf-like theorems for the Laguerre and hypergeometric functions. Before entering into the details of the central topic of the present paper, we devote the remaining part of the introduction to some propaedeutic remarks necessary to the understanding of what follows.

**a) Laguerre function (LF)**

The Laguerre function is defined by [6]

$$\phi_n(x^2) = \sqrt{\frac{n!}{(n+l)!}} x^l e^{-x^2/2} L_n^l(x^2),$$

where $L_n^l(\cdot)$ denotes generalized Laguerre polynomials [7]. The recurrence properties of LF can be directly derived from the definition (1.7), thus getting [6]

$$\frac{x^2 + l}{x} \phi_n(x^2) = \sqrt{n + l} \phi_n^{l-1}(x^2) + \sqrt{n + l + 1} \phi_n^{l+1}(x^2),$$

$$\frac{d}{dx} \phi_n(x^2) = \sqrt{n + l} \phi_n^{l-1}(x^2) - \sqrt{n + l + 1} \phi_n^{l+1}(x^2).$$

Furthermore, it is well known that [5]

$$\lim_{n\to\infty} \phi_n^l(x^2) = J_l(2\sqrt{n}x).$$

Accordingly, it is easy to check that in the large $n$ limit the relations (1.8) reproduce the recurrence relations of cylindrical Bessel functions.
Modified Laguerre functions (MLF) can be associated to (1.7), according to the following relation:

\[ X_{n}^{l}(x^2) = (-i)^{l} \phi_{n}^{l}(-x^2). \]  

The large \( n \) limit of the MLF is therefore

\[ \lim_{n \to \infty} X_{n}^{l}(x^2) = (-i)^{l} J_{l}(2i\sqrt{n}x) = I_{l}(2\sqrt{n}x). \]  

The recurrence relations of \( X_{n}^{l}(\cdot) \) are easily inferred and read

\[ \frac{-x^2 + l}{x} X_{n}^{l}(x^2) = \sqrt{n + l} X_{n-1}^{l}(x^2) - \sqrt{n + l + 1} X_{n+1}^{l}(x^2) \]  

\[ \frac{d}{dx} X_{n}^{l}(x^2) = \sqrt{n + l} X_{n-1}^{l}(x^2) + \sqrt{n + l + 1} X_{n+1}^{l}(x^2). \]  

b) Legendre function (LeF)

Let us now turn to a more general function, often encountered in two-level dynamics problems [6], namely,

\[ \phi_{n+,n-}^{l}(x^2) = \left[ \frac{(n_+ + l + 1)}{(n_- + l)} \right]^{1/2} x^l (1 - x^2)^{n_+ - n_- - l/2} F_{1}(1, -2l; n_+ - n_- + 1; 1 - x^2) \]

\[ = \sqrt{\frac{n_+! n_-!}{(n_+ + l)! (n_- - l)!}} x^l (1 - x^2)^{n_+ - n_- - l/2} P_{n+}^{l}(1 - 2x^2), \]  

where \( F_{1}(\cdot) \) denotes the hypergeometric function [5,6], and \( P_{n+}^{l}(\cdot) \) the Jacobi polynomial.

The structure of the recurrence relations satisfied by \( \phi_{n+,n-}^{l}(x^2) \) is more complicated than that of the corresponding relations satisfied by LeF; they explicitly write as

\[ \frac{l - (2l + n_+ - n_-)x^2}{x(1 - x^2)^{1/2}} \phi_{n+,n-}^{l}(x^2) = \sqrt{(n_+ + l)(n_- - l + 1)} \phi_{n+,n-}^{l-1}(x^2) \]

\[ + \sqrt{(n_+ + l + 1)(n_- - l)} \phi_{n+,n-}^{l+1}(x^2) \]  

\[ (1 - x^2)^{1/2} \frac{d}{dx} \phi_{n+,n-}^{l}(x^2) = \sqrt{(n_+ + l)(n_- - l + 1)} \phi_{n+,n-}^{l-1}(x^2) \]

\[ - \sqrt{(n_+ + l + 1)(n_- - l)} \phi_{n+,n-}^{l+1}(x^2). \]  

The functions \( \phi_{n+,n-}^{l}(x^2) \) in the large \( n \)-limit reduce to

\[ \lim_{n \to \infty} \phi_{n+,n-}^{l}(x^2) = \phi_{n+}^{l}(n_- x^2). \]  

It is therefore natural to introduce their modified version \( X_{n+,n-}^{l}(x^2) \) according to

\[ X_{n+,n-}^{l}(x^2) = (-i)^{l} \phi_{n+,n-}^{l}(-x^2), \]  

whose recurrence relations are easily proved to be

\[ \frac{l + (2l + n_+ - n_-)x^2}{x(l + x^2)^{1/2}} X_{n+,n-}^{l}(x^2) = \sqrt{(n_+ + l)(n_- - l + 1)} X_{n+,n-}^{l-1}(x^2) \]

\[ - \sqrt{(n_+ + l + 1)(n_- - l)} X_{n+,n-}^{l+1}(x^2) \]  

\[ (1 + x^2)^{1/2} \frac{d}{dx} X_{n+,n-}^{l}(x^2) = \sqrt{(n_+ + l)(n_- - l + 1)} X_{n+,n-}^{l-1}(x^2) \]

\[ + \sqrt{(n_+ + l + 1)(n_- - l)} X_{n+,n-}^{l+1}(x^2). \]  

In the next sections, we will discuss the generalized functions associated to Graf series for \( \phi_{n+}^{l}(x^2) \) and \( \phi_{n+,n-}^{l}(x^2) \).

\*Needless to say, in the large \( n \) limit, eqs. (1.12) reproduce the recurrence relations of the \( I_{n}(\cdot) \) functions.
2. GRAF SERIES AND HAGUERRE FUNCTIONS

As already mentioned, the Graf theorem for Bessel functions states that [8]

\[
\sum_{l=-\infty}^{+\infty} t^l J_l(x) J_{l+1}(y) = \left[ \frac{y - x/t}{y - xt} \right]^{n/2} J_n(\xi_-(x, y; t)),
\]

(2.1)

where

\[
\xi_-(x, y; t) = \sqrt{x^2 + y^2 - xy \left( t + \frac{1}{t} \right)}.
\]

(2.2)

When \( t = 1 \), eq. (2.1) gives the Neumann addition theorem, namely

\[
\sum_{l=-\infty}^{+\infty} J_l(x) J_{n\pm l}(y) = J_n(y + x),
\]

(2.3)

and when \( n = 0 \), eq. (2.1) takes the simpler form

\[
\sum_{l=-\infty}^{+\infty} t^l J_l(x) J_l(y) = J_0(\xi_-(x, y; t)).
\]

(2.4)

In [7] it has been proved that the generalization of the above equation to LF reads

\[
\sum_{l=-n}^{+\infty} t^l \phi_n^l(x^2) \phi_n^l(y^2) = \exp \left[ \frac{xy}{2} \left( t - \frac{1}{t} \right) \right] \phi_n(\xi_2^2),
\]

(2.5)

which clearly reduces to (2.4) in the very large \( n \) limit.

The most direct generalization of (2.1) to LF seems to be of the type

\[
A_{m,n}(x, y; t) = \sum_{l=-n}^{+\infty} t^l \phi_n^l(x^2) \phi_{n-m}^l(y^2).
\]

(2.6)

However, we are looking for a generalized form which

1) reduces to (2.1) when \( n \) is very large;
2) reduces to (2.5) when \( m = 0 \);
3) possesses recurrence relations connecting the nearest-neighbor \( m \) and \( n \) indices.

The function \( A_{m,n} \) satisfies the first two requirements but not the third, as is easily verified. On the other hand, the summation

\[
(1) \phi_n^{(m)}(x, y; t) = \sum_{l=-\infty}^{\infty} t^l \phi_n^l(x^2) \phi_{n-m}^l(y^2)
\]

(2.7)

satisfies the first two points and obeys the following recursive relations:

\[
\begin{align*}
\frac{\partial}{\partial t} (1) \phi_n^{(m)} &= \frac{x^2}{t} (1) \phi_n^{(m)} + \sqrt{n} \frac{x}{x^2} (1) \phi_n^{(m-1)} + \sqrt{n + 1} \frac{x}{x^2} (1) \phi_{n+1}^{(m+1)} \\
\frac{\partial}{\partial x} (1) \phi_n^{(m)} &= t\sqrt{n + 1} (1) \phi_{n+1}^{(m+1)} - \frac{1}{t} \sqrt{n} (1) \phi_n^{(m-1)} \\
\frac{\partial}{\partial y} (1) \phi_n^{(m)} &= \sqrt{n - m + 1} (1) \phi_{n-1}^{(m-1)} - \sqrt{n - m} (1) \phi_n^{(m+1)} \\
(m - y^2)(1) \phi_n^{(m)} &= y \left[ \sqrt{n - m + 1} (1) \phi_n^{(m-1)} + \sqrt{n - m} (1) \phi_n^{(m+1)} \right] - t \frac{\partial}{\partial t} (1) \phi_n^{(m)}.
\end{align*}
\]

(2.8)
relating contiguous functions, as required. The next step will be concerned with the derivation of an explicit form for \((1)\phi^{(m)}_n(x, y; t)\). Recalling eqs. (2.1), (2.5), and the limits for large \(n\), it is natural to assume the following form:

\[
(1)\phi^{(m)}_n(x, y; t) = \exp \left\{ \frac{xy}{2} \left( t - \frac{1}{t} \right) \right\} \cdot \left[ f(x, y; t) \right]^m \phi^{(m)}_{n-m}(\xi^2). \quad (2.9)
\]

The functional form of \(f(x, y; t)\) is easily specified. Deriving, e.g., (2.9) with respect to \(z\) and then using (2.8) we get

\[
\frac{\partial}{\partial z} (1)\phi^{(m)}_n = \left[ \frac{y}{2} \left( t - \frac{1}{t} \right) - \frac{\xi^2}{f} \frac{\partial}{\partial x} f \right] (1)\phi^{(m)}_n \\
+ \sqrt{n+1} \left\{ \frac{\xi}{f^2} \frac{\partial}{\partial x} f \frac{1}{2} \frac{1}{f\xi} \left[ 2x - y \left( t + \frac{1}{t} \right) \right] \right\} (1)\phi^{(m+1)}_{n+1} \\
+ \frac{\sqrt{n}}{\xi} \left\{ - \frac{1}{t} + \frac{f}{2\xi} \left[ 2x - y \left( t + \frac{1}{t} \right) \right] \right\} (1)\phi^{(m-1)}_{n-1},
\]

which once compared with the second of (2.8) yields

\[
\xi - \frac{\partial}{\partial x} f = \frac{yf}{2\xi} \left( t - \frac{1}{t} \right) \\
\xi - \frac{\partial}{\partial x} f = - \left\{ \frac{1}{t} + \frac{f}{2\xi} \left[ 2x - y \left( t + \frac{1}{t} \right) \right] \right\} \\
\xi - \frac{\partial}{\partial x} f = \frac{f}{2\xi} \left[ 2x - y \left( t + \frac{1}{t} \right) \right] + tf^2. \quad (2.11)
\]

Finally, equating the r.h.s. of the first two eqs. (2.11) we end up with

\[
f(x, y; t) = \left( \frac{y - x/t}{y - xt} \right), \quad (2.12)
\]

thus specifying the Graf-type series rule for LF. Generalized forms including MLF will be discussed in the concluding remarks.

### 3. GRAF SERIES FOR LEGENDRE FUNCTIONS

In this section, we will discuss generalized Graf series for LF. In [7], the following series rule has been derived

\[
\sum_{l=-n_+}^{n_-} t^l \phi^{(m)}_{n_+,n_-}(x^2) \phi^{(m)}_{n_+,n_-}(y^2) = \frac{\sqrt{1 - x^2}(1 - y^2) + xyt}{\sqrt{(1 - x^2)(1 - y^2) + 2x^2 t^2}} \phi^{(n_- - n_+)}_{n_+,n_-}(\eta^2), \quad (3.1)
\]

where

\[
\eta^2(x, y; t) = x^2(1 - y^2) + y^2(1 - y^2) - \frac{t^2 + 1}{t} - x y \sqrt{(1 - x^2)(1 - y^2)}. \quad (3.2)
\]

For the same reasons leading to the form (2.7) for the generalized Laguerre function (GLF), we will consider as generalized LF the following one:

\[
(1)\phi^{(m)}_{n_+,n_-}(x, y; t) = \sum_{l=-n_+}^{n_-} t^l \phi^{l+m}_{n_+,n_-}(x^2) \phi^{l+m}_{n_+,n_-}(y^2), \quad (3.3)
\]
whose recurrence relations are obtained after a tedious amount of algebra. Combining (1.14) and the Gauss relations for contiguous indices hypergeometric functions, we end up with the following rather intrigued relations:

\[
\frac{\partial}{\partial t} (1)^{\phi_{n_+,n_-}} = (n_- - n_+ - \frac{x^2}{t}) (1)^{\phi_{n_+,n_-}} \\
+ \sqrt{n_-(n_+ + 1)} x (1 - x^2)^{\frac{1}{2}} (1)^{\phi_{n_+,n_-+1,n_-}} \\
+ \sqrt{n_+(n_- + 1)} \frac{x}{t} (1 - x^2)^{\frac{1}{2}} (1)^{\phi_{n_+,n_-+1,n_-}} \\
(1 - x^2)^{\frac{1}{2}} \frac{\partial}{\partial x} (1)^{\phi_{n_+1,n_-}} = \sqrt{n_-(n_+ + 1)} t (1)^{\phi_{n_+,n_-+1,n_-}} \\
- \sqrt{n_+(n_- + 1)} \frac{1}{t} (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
\frac{\partial}{\partial y} (1)^{\phi_{n_+1,n_-}} = \sqrt{(n_- + m)(n_- - m + 1)} (1)^{\phi_{n_+,n_-+1,n_-}} \\
- \sqrt{(n_+ - m)(n_- + m + 1)} (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
[(n_- - n_+)(x^2 - y^2) + m(1 - 2y^2)] (1)^{\phi_{n_+1,n_-}} = x(1 - x^2)^{\frac{1}{2}} \left[ \sqrt{n_-(n_+ + 1)} (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
+ \frac{1}{t} \sqrt{n_+(n_- + 1)} (1)^{\phi_{n_+,n_-+1,n_-+1}} \right] + y(1 - y^2)^{\frac{1}{2}} \\
\cdot \left[ \sqrt{(n_- + m)(n_- - m + 1)} (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
+ \sqrt{(n_+ - m)(n_- + m + 1)} (1)^{\phi_{n_+,n_-+1,n_-+1}} \right].
\]

The explicit form of the series (3.3) will be obtained using an ansatz analogous to (2.9). Let us write indeed

\[
(1)^{\phi_{n_+,n_-}}(x,y;t) = [R(x,y;t)]^{n_- - n_+} [g(x,y;t)]^m \cdot (1)^{\phi_{n_+,n_-+1,n_-+1}},
\]

where

\[
R(x,y;t) = \frac{x}{(1 - x^2)(1 - y^2)} \frac{1}{x^2 + y^2 + x^2 y^2}.
\]

According to the above-discussed procedure, we derive eq. (3.5) with respect to x, thus obtaining

\[
\frac{\partial}{\partial x} (1)^{\phi_{n_+,n_-}}(x,y;t) = (n_- - n_+) \left[ \frac{1}{R} \frac{\partial}{\partial x} R - \frac{y^2}{g} \frac{\partial}{\partial x} f \right] (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
+ \sqrt{n_+(n_- + 1)} \left[ \eta(1 - \eta^2) \frac{\partial}{\partial x} g + \frac{g}{(1 - \eta^2)^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \eta \right] (1)^{\phi_{n_+,n_-+1,n_-+1}} \\
+ \sqrt{n_-(n_+ + 1)} \left[ \frac{1}{g^2} \eta (1 - \eta^2) \frac{\partial}{\partial x} g - \frac{1}{g} (1 - \eta^2)^{\frac{1}{2}} \frac{\partial}{\partial \eta} \eta \right] (1)^{\phi_{n_+,n_-+1,n_-+1}},
\]

which, compared with the second of (3.4), yields

\[
\eta^2 \frac{\partial}{\partial x} g = \frac{g}{R} \frac{\partial}{\partial x} R \\
\frac{\eta^2}{x^2 + y^2 + x^2 y^2} \frac{\partial}{\partial x} g = \frac{\eta}{(1 - \eta^2)^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \eta \\
\frac{\eta^2}{x^2 + y^2 + x^2 y^2} \frac{\partial}{\partial x} g = \frac{\eta^2}{(1 - \eta^2)^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \eta.
\]

Equating the r.h.s. of the first two relations in (3.8), it follows that

\[
g(x,y;t) = R(x,y;t) \left[ \frac{y \sqrt{1 - x^2 - \frac{x^2}{y^2}} \sqrt{1 - y^2}}{y \sqrt{1 - x^2 - x t \sqrt{1 - y^2}}} \right]^{\frac{1}{2}},
\]

thus finally yielding a Graf-type addition formula for LoE.
4. CONCLUDING REMARKS

In this note, we have generalized Graf type series to Laguerre and Legendre functions. The results obtained seem to be interesting in themselves and also in view of a wider approach to the theory of generalized special functions, in the sense discussed in the introduction. Some further simple consequences can be obtained from the relations derived in the previous sections. A Neumann-type "theorem" can be stated as a particular case of eqs. (2.9) and (3.5), indeed,

\[
\sum_{l=m}^{\infty} \phi_n^l(x^2) \phi_{n-m}^{l+m}(y^2) = \phi_n^m(x-y)^2.
\]

Using polar coordinates in (2.9) we also find the interesting relation

\[
(4.1)
\]

\[
\frac{1}{2} \rho \sin \theta \exp \left\{ \frac{i}{2} \rho^2 \sin 2\theta \right\}.
\]

The sum rules for MLF can be easily derived, in fact,

\[
(1) X_n^m(x, y; t) = (-i)^m (1) \phi_n^m(i x, i y; -t)
\]

\[
= \exp \left\{ \frac{1}{2} xy \left( t - \frac{1}{t} \right) \right\} \left[ \frac{y + x/t}{y + x/t} \right]^{m/2} X_n^{(m)}(\xi_+^2), \quad \left( \xi_+^2 = x^2 + y^2 + xy \left( t + \frac{1}{t} \right) \right).
\]

Along the same lines, mixed sum rules can be calculated; combining indeed X and \( \phi \) functions we get

\[
(1) \phi_n^m(x, y; t) = \sum_{l=-n}^{\infty} t^l \phi_n^l(x^2) X_n^{l+m}(y^2) = (-i)^m (1) \phi_n^m(x, iy; -it)
\]

\[
= (-i)^m \exp \left\{ \frac{1}{2} xy \left( t + \frac{1}{t} \right) \right\} \left[ \frac{y - x/t}{y + x/t} \right]^{m/2} \phi_n^{m} \left[ x^2 - y^2 - xy \left( t - \frac{1}{t} \right) \right]
\]

and

\[
(1) X_n^m(x, y; t) = \sum_{l=-n}^{\infty} t^l X_n^l(x^2) \phi_n^{l-m}(y^2) = (1) \phi_n^m(i x, y; -it)
\]

\[
= \exp \left\{ \frac{1}{2} xy \left( t + \frac{1}{t} \right) \right\} \left[ \frac{y + x/t}{y - x/t} \right]^{m/2} \phi_n^{m} \left[ -x^2 + y^2 - xy \left( t - \frac{1}{t} \right) \right].
\]

Finally, it can be realized that similar relations, although in a more intriguing form, can be obtained for LeFs.

In conclusion, in this paper we just introduced and studied some of the relevant properties of Graf-type series for Laguerre and Legendre functions. A more detailed analysis and some applications to physical and mathematical problems will be discussed elsewhere.

REFERENCES