Partial and semipartial geometries: an update

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Abstract

The Handbook of Incidence Geometry (Handbook of Incidence Geometry, Buildings and Foundations, North-Holland, Amsterdam, 1995) appeared in 1995. In Chapter 12, *On some rank two geometries*, an almost complete overview was given on the status of the theory on partial and semipartial geometries. Now, 5 years later, it is maybe a good time to give an update of this status. Indeed a lot of things have happened during these years. Moreover we take the opportunity to give complete parameter lists of all known examples of partial and semipartial geometries known so far.
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1. Introduction

An (α,β)-geometry $\mathscr{S} = (\mathscr{P}, \mathscr{L}, I)$ is a connected partial linear space of order (s,t) (i.e. two points are incident with at most one line, each point is incident with $t+1$ ($t \geqslant 1$) lines, and each line is incident with $s+1$ ($s \geqslant 1$) points), with the property that for every anti-flag (x,L) there are either α or β lines through x intersecting L.

The point graph $\Gamma(\mathscr{S})$ of an (α,β)-geometry is the graph with vertex set the set of points of $\mathscr{S}$; two vertices are adjacent if and only if they are different and collinear in $\mathscr{S}$. The *block graph* (also called *line graph* by some authors) of an (α,β)-geometry is the graph whose vertices are the lines, and vertices are adjacent if and only if the corresponding lines are concurrent.

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If $\alpha = \beta$, $\mathscr{S}$ is called a *partial geometry* with parameters s, t, α, which we denote by pg(s, t, α) [1]. In this case the graph $\Gamma(\mathscr{S})$ is a strongly regular graph srg(v, k, λ, μ); more precisely it is a

$$\mathrm{srg}\left((s+1)\,\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

A strongly regular graph Γ with these parameters (which are satisfying $t \geqslant 1$, $s \geqslant 1$, and $1 \leqslant \alpha \leqslant \min\{s+1, t+1\}$) is called a *pseudo-geometric* (s, t, α)-graph. If the graph Γ is indeed the point graph of at least one partial geometry then Γ is called *geometric*.

Note that a graph can be pseudo-geometric for at most one set of values s, t, α and assuming $\alpha \neq s+1$, the cliques of size $s+1$ corresponding to potential lines must be maximal. However, there can exist several non-isomorphic partial geometries with the same graph as point or block graph. A pseudo-geometric graph is called *faithfully geometric* if and only if there is up to isomorphism exactly one partial geometry with this graph as point graph.

Another important family of (α, β)-geometries is given by the so-called $(0, \alpha)$-geometries (i.e. $\beta = 0$). Here the point graph is not necessarily a strongly regular graph. Those $(0, \alpha)$-geometries which have a strongly regular point graph are called *semipartial geometries* and are denoted by spg(s, t, α, μ) and were introduced in [9]. Note that the parameter μ is the parameter of the strongly regular point graph, which counts the number of vertices adjacent to two non-adjacent vertices. If $\alpha = 1$ these semipartial geometries are better known as *partial quadrangles* which are introduced by Cameron [8].

Remarks. (1) Special classes of partial geometries are the generalized quadrangles ($\alpha = 1$) introduced by Tits, see [28]; the 2-$(v, s+1, 1)$ designs ($\alpha = s+1$) and their duals ($\alpha = t+1$); the *Bruck nets* ($\alpha = t$) and dual Bruck nets ($\alpha = s$). In this overview we will restrict ourselves to the so-called *proper partial geometries*, which are the partial geometries with $1 < \alpha < \min\{s, t\}$.

(2) A *proper semipartial geometry* is a semipartial geometry which is not a partial geometry.

(3) For the description of the examples of partial and semipartial geometries known until 1995, we refer to [17]. In the sequel we will give an overview of some new constructions of partial geometries having sometimes new parameters. In Section 4 we will give complete parameter lists of the examples of the proper partial and semipartial geometries known at present.

2. New constructions of partial geometries

2.1. The partial geometry constructed from the Hermitian two-graph

A *two-graph* [30] (Ω, Δ) is a pair of a vertex set Ω and a triple set $\Delta \subset \Omega^{(3)}$, such that each 4-subset of Ω contains an even number of triples of Δ. A two-graph is called

regular whenever each pair of elements of Ω is contained in the same number a of triples of Δ.

Given any graph $\Gamma = (X, \sim)$, one can construct a new graph by using *Seidel-switching*. For this, partition the vertex set X as $X = X_1 \cup X_2$, leave the adjacencies inside X_1 and X_2 as they are and interchange edges and non-edges between vertices of X_1 and X_2. Graphs which can be mapped to each other by Seidel-switching are called *switching equivalent*. It is known [30] that, given v there is a one-to-one correspondence between the two-graphs and the switching classes of graphs on the set of v elements. If the two-graph (Ω, Δ) is regular and if $(\Omega, \sim)$ is any graph in its switching class which has an isolated vertex $\omega \in \Omega$, then $(\Omega \backslash \{\omega\}, \sim)$ is a strongly regular graph.

Let $\mathcal{H}$ be the Hermitian curve in $\mathrm{PG}(2, q)$, q odd, defined by the Hermitian bilinear form $H(x, y)$. The Hermitian two-graph (Ω, Δ) is defined by taking as a vertex set Ω the set of $q^3 + 1$ points of $\mathcal{H}$ and a triple $\{x, y, z\} \in \Omega^{(3)}$ is an element of Δ if and only if $H(x, y)H(y, z)H(z, x)$ is a square (if $q \equiv -1 \pmod 4$) or a non-square (if $q \equiv 1 \pmod 4$) [34]. This two-graph appears to be regular with $a = (q^2 + 1)(q - 1)/2$ and in its switching class there is indeed a graph which has an isolated vertex. This yields a strongly regular graph $\mathcal{H}(q)$ which is an $\mathrm{srg}(q^3, (q^2 + 1)(q - 1)/2, (q - 1)^3/4 - 1, (q^2 + 1)(q - 1)/4)$ and is pseudo-geometric with parameters $s = q - 1$, $t = (q^2 - 1)/2$, $\alpha = (q - 1)/2$. If $q = 3$ this graph is the point graph of the unique generalized quadrangle of order $(2, 4)$. Although it has been proved (computer search) by Spence [33] that $\mathcal{H}(q)$ is not geometric for $q = 5$ and 7 it is remarkable that the graph is indeed geometric if $q = 3^{2m}$ which has been proved by Mathon; we refer to [25] for more details. In [24] Kuijken gives a more geometric construction. Moreover, by making some slight changes in the geometric construction she proves that the graph is also geometric in case q is an odd power of 3.

2.2. Partial geometries from perp-systems

Mathon announced in June 1999 during the 2nd Pythagorean Conference (Samos, Greece) the existence of a set $\mathcal{R}$ of 21 lines of $\mathrm{PG}(5, 3)$ that are pairwise skew (hence form a partial line spread) with the property that every plane of $\mathrm{PG}(5, 3)$ through one of the 21 lines of $\mathcal{R}$ intersects exactly two other lines of $\mathcal{R}$. Actually it is an SPG 1-regulus in the sense of Thas [38] (a brief description can also be found in [17]) with no tangent planes. The construction by R. Mathon is a computer construction. It yields a new partial geometry with parameters $s = 8$, $t = 20$, $\alpha = 2$. Embed $\mathrm{PG}(5, 3)$ as a hyperplane Π in $\mathrm{PG}(6, 3)$. The points of the partial geometry are the 3^6 points of $\mathrm{AG}(6, 3) = \mathrm{PG}(6, 3) \backslash \Pi$, the lines of the partial geometry are the affine planes of $\mathrm{AG}(6, 3)$ having as line at infinity one of the 21 elements of $\mathcal{R}$. Although quite some other nice properties of this SPG 1-regulus $\mathcal{R}$ in $\mathrm{PG}(5, 3)$ are known, there is so far no computer free construction known. However, these properties have led to a new concept, namely *perp-systems* which we shortly describe here. For more details we refer to [16].

Consider a $\mathrm{PG}(N, q)$ equipped with a polarity ρ. Define a *partial perp-system* $\mathcal{R}(r)$ to be any set $\{\pi_1, \ldots, \pi_k\}$ of $k(k > 1)$ mutually disjoint r-dimensional subspaces of

$\mathrm{PG}(N,q)$ such that no π_i^ρ meets an element of $\mathscr{R}(r)$. Hence, each π_i is non-singular with respect to ρ. Note that $N \geqslant 2r + 1$. One easily proves that

$$|\mathscr{R}(r)| \leqslant \frac{q^{(N-2r-1)/2}(q^{(N+1)/2} + 1)}{q^{(N-2r-1)/2} + 1}. \tag{1}$$

We will only deal with systems $\mathscr{R}(r)$ such that equality holds in (1), such a system is called a *perp-system*.

Theorem 1. *Let $\mathscr{R}(r)$ be a perp-system of $\mathrm{PG}(N,q)$ equipped with a polarity ρ and let $\overline{\mathscr{R}(r)}$ denote the union of the point sets of the elements of $\mathscr{R}(r)$. Then $\overline{\mathscr{R}(r)}$ has two intersection sizes with respect to hyperplanes.*

This implies that $\mathscr{R}(r)$ yields a two-weight code and a strongly regular graph $\Gamma^*(\overline{\mathscr{R}(r)})$ [7]. The graph is constructed by embedding $\mathrm{PG}(N,q)$ as a hyperplane Π in $\mathrm{PG}(N+1,q)$. The vertices of the graph are the q^{N+1} points of $\mathrm{AG}(N+1,q) = \mathrm{PG}(N, q) \backslash \Pi$, two vertices are adjacent whenever the line of $\mathrm{PG}(N+1,q)$ joining them is intersecting Π in an element of $\overline{\mathscr{R}(r)}$.

One easily checks that this graph is a pseudo-geometric

$$\left(q^{r+1} - 1, \frac{q^{(N-2r-1)/2}(q^{(N+1)/2} + 1)}{q^{(N-2r-1)/2} + 1} - 1, \frac{q^{r+1} - 1}{q^{(N-2r-1)/2} + 1} \right) \text{-graph.}$$

One can prove some restrictions on the parameters. More precisely one can prove the following theorem:

Theorem 2. *Let $\mathscr{R}(r)$ be a perp-system of $\mathrm{PG}(N,q)$ equipped with a polarity ρ. Then*

- $2r + 1 \leqslant N \leqslant 3r + 2$;
- *if $N = 2r + 1$ then q is odd and $\Gamma^*(\overline{\mathscr{R}(r)})$ is the point graph of a net with q^{r+1} points on a line and $(q^{r+1} + 1)/2$ lines through a point.*
- *assume that $N \neq 2r + 1$ then $(r+1)/(N-2r-1)$ is a positive integer; if N is even then q has to be a square. The graph $\Gamma^*(\overline{\mathscr{R}(r)})$ is the point graph of a partial geometry*

$$\mathrm{pg}\left(q^{r+1} - 1, \frac{q^{(N-2r-1)/2}(q^{(N+1)/2} + 1)}{q^{(N-2r-1)/2} + 1} - 1, \frac{q^{r+1} - 1}{q^{(N-2r-1)/2} + 1} \right).$$

One can construct perp-systems from other perp-systems. More precisely the next theorems are proved in [14].

Theorem 3. *Let $\mathscr{R}(r)$ be a perp-system with respect to some polarity of $\mathrm{PG}(N,q^n)$, then there exists a perp-system $\mathscr{R}'((r+1)n - 1)$ with respect to some polarity of $\mathrm{PG}((N+1)n - 1, q)$.*

Theorem 4. *If the classical polar space P admits a perp-system $\mathscr{R}(r)$, then the polar space Q admits a perp-system $\mathscr{R}(2r+1)$, for*

$$(P, Q) = (\mathrm{H}(2n, q^2), \mathrm{Q}^-(4n+1, q))$$
$$(\mathrm{H}(2n+1, q^2), \mathrm{Q}^+(4n+3, q)),$$
$$(\mathrm{Q}(2n, q^2), \mathrm{Q}^+(4n+1, q)) \text{ for } q \text{ odd},$$
$$(\mathrm{Q}(2n, q^2), \mathrm{Q}(4n, q)) \text{ for } q \text{ even},$$
$$(\mathrm{Q}^-(2n+1, q^2), \mathrm{Q}^-(4n+3, q)),$$
$$(\mathrm{H}(2n, q^2), W_{4n+1}(q)).$$

Remarks. (1) A net with the parameters as in Theorem 2 and coming from a perp-system does exist for every odd q.

(2) If N is maximal i.e. if $N = 3r + 2$ then r is odd and the partial geometry is a

$$\mathrm{pg}(q^{r+1} - 1, (q^{r+1} + 1)(q^{(r+1)/2} - 1), q^{(r+1)/2} - 1).$$

This partial geometry has the parameters of a partial geometry $T_2^*(\mathscr{K})$, with $\mathscr{K}$ a maximal arc of degree $q^{(r+1)/2}$ in a $\mathrm{PG}(2, q^{r+1})$. A selfpolar maximal arc of degree q^n in a $\mathrm{PG}(2, q^{2n})$ is a maximal arc $\mathscr{K}$ such that each point $p \in \mathscr{K}$ is mapped by a polarity ρ of the plane on an exterior line p^ρ of $\mathscr{K}$. If q is even, there exist selfpolar maximal arcs of Denniston type; they yield a perp-system $\mathscr{R}(0)$. Applying Theorem 3 this gives a perp-system with $r = n - 1$ in $\mathrm{PG}(3n - 1, q^2)$ and a perp-system with $r = 2n - 1$ in $\mathrm{PG}(6n - 1, q)$.

(3) The set of 21 lines in $\mathrm{PG}(5, 3)$ found by Mathon is a perp-system $\mathscr{R}(1)$ in $\mathrm{PG}(5, 3)$. The polarity evolved can be either the symplectic polarity or the elliptic orthogonal polarity. In this case $N = 5$ and $r = 1$, hence N is maximal and the partial geometry has the parameters of a $T_2^*(\mathscr{K})$, with $\mathscr{K}$ a maximal arc of degree 3 in $\mathrm{PG}(2, 9)$; however, such a maximal arc does not exist.

(4) So far, there is no example known of a perp-system in $\mathrm{PG}(N, q)$ with $2r + 1 < N < 3r + 2$.

(5) The results in Theorem 4 are results that are of the same type as known results on m-systems, introduced by Thas and Shult [31,32]. There are indeed connections with m-systems. For more details we refer to [14].

2.3. Partial geometries with $t = s + 1$

2.3.1. Derivation of partial geometries

Let Φ be a pg-*spread* of a pg(s, t, α) $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$, that is a (maximal) set of $st/\alpha + 1$ lines partitioning the point set. Assume $t > 1$ and let L be any line of $\mathscr{L} \backslash \Phi$. Let Φ_L be the set of $s + 1$ lines of Φ intersecting L. Then L is called *regular with respect to Φ* if and only if there exists a set of $s + 1$ lines $\mathscr{L}(L) = \{L_0 = L, L_1, \ldots, L_s\}$ that partitions the set $\mathscr{P}(\Phi_L)$ of points covered by Φ_L, and each element of $\mathscr{L} \backslash (\mathscr{L}(L) \cup \Phi)$ is intersecting $\mathscr{P}(\Phi_L)$ in at least one point and at most s points.

It is easy to prove (see [11]) that if a pg(s, t, α) $\mathscr{S}$ has a regular line L with respect to a pg-spread Φ, then $t \geqslant s + 1$. If $t = s + 1$ then every line M not being an element

of the pg-spread Φ neither of $\mathscr{L}(L)$ intersects $\mathscr{P}(\Phi_L)$ in α points. Now assume that Φ is a pg-spread of a $\mathrm{pg}(s, s + 1, \alpha)$ such that every line is regular with respect to Φ. Then $\mathscr{L} \backslash \Phi$ is partitioned in $s(s + 1)/\alpha + 1$ sets $\mathscr{L}_i$ $(i = 1, \ldots, s(s + 1)/\alpha + 1)$ each containing $s + 1$ mutually skew lines. The spread Φ is called a *replaceable spread* and can be used to construct the following incidence structure $\mathscr{S}_\Phi = (\mathscr{P}_\Phi, \mathscr{L}_\Phi, \mathrm{I}_\Phi)$. The elements of $\mathscr{P}_\Phi$ are on the one hand the points of $\mathscr{S}$ and on the other hand the sets $\mathscr{L}_i$ $(i = 1, \ldots, s(s + 1)/\alpha + 1)$, $\mathscr{L}_\Phi = \mathscr{L} \backslash \Phi$. Finally, $p \, \mathrm{I}_\Phi \, L$ is defined by $p \, \mathrm{I} \, L$ if $p \in \mathscr{P}$ and by $L \in p$ if $p \in \{\mathscr{L}_i \mid i = 1, \ldots, s(s + 1)/\alpha + 1\}$. Generalizing a construction of Mathon and Street [26], one can prove (see [11]) that $\mathscr{S}_\Phi$ is a $\mathrm{pg}(s + 1, s, \alpha)$. The partial geometry $\mathscr{S}_\Phi$ (and its dual) is called a *partial geometry derived from $\mathscr{S}$ with respect to Φ*.

Note that the set $\phi = \{\mathscr{L}_i \mid i = 1, \ldots, s(s + 1)/\alpha + 1\}$ is a replaceable spread of $\mathscr{S}_\Phi^D$ and that the derived partial geometry $(\mathscr{S}_\Phi^D)_\phi$ is isomorphic to the partial geometry $\mathscr{S}^D$ [12].

2.3.2. The derived partial geometries of $\mathrm{PQ}^+(4n - 1, q)$ $(q = 2 \text{ or } 3)$

It has been checked by computer (see [26]) that the partial geometry $\mathrm{PQ}^+(7, 2)$ constructed by De Clerck et al. [15] (but other constructions do exist, see [17] for details) has exactly three replaceable spreads yielding (after dualizing) three non-isomorphic partial geometries $\mathrm{pg}(7, 8, 4)$. De Clerck [11] proved this result geometrically for both $q = 2$ and 3. Actually, Mathon and Street [26] have constructed by computer seven new partial geometries $\mathrm{pg}(7, 8, 4)$ by starting from the partial geometry $\mathrm{PQ}^+(7, 2)$ and by using derivation with respect to a suitable replaceable spread. They give in [26] information on the order of the automorphism groups of the geometries as well as information on the point and block graphs of these geometries. They remarked that the point graphs of four of the geometries $\mathrm{pg}(7, 8, 4)$ constructed by them, are isomorphic graphs while their block graphs all are different. Actually that point graph was not a new graph, it is the complement of the graph constructed in [3]. It is an element of the class of graphs called the graphs on a quadric with a hole. Such a graph has vertex set the points of a quadric $\mathrm{Q}^+(2m - 1, q) \backslash M$, M a generator of the quadric and vertices x and y are defined to be adjacent whenever $\langle x, y \rangle \subset \mathrm{Q}^+(2m - 1, q) \backslash M$. This graph is strongly regular for general dimensions and general q.

Klin and Reichard [23,29] found, again by computer, but independently from Mathon and Street, that the complement of the graph on $\mathrm{Q}^+(7, 2)$ with a hole, is indeed the point graph of exactly four partial geometries $\mathrm{pg}(7, 8, 4)$.

In [12] it has been proved that from the eight known partial geometries $\mathrm{pg}(7, 8, 4)$, four of them are the smallest member of a class of $\mathrm{pg}(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$ and all of them are constructed using derivation.

Remarks. (1) For quite a long time it was conjectured that there is only one $\mathrm{pg}(7, 8, 4)$ up to isomorphism. This conjecture has turned out to be false. However, in [16] it has been proved that the point graph of the partial geometry $\mathrm{PQ}^+(7, 2)$ is faithfully geometric. This does not guarantee that the block graph is also faithfully geometric. But, in [27] Panigrahi proves, using combinatorial arguments, that the block graph $\Gamma'(7, 2)$ of the partial geometry $\mathrm{PQ}^+(7, 2)$ is faithfully geometric indeed. Actually the graph $\Gamma'(7, q)$ is the graph $\Gamma^c(\mathrm{Q}^+(7, q))$ with vertices the points on the hyperbolic

quadric $Q^+(7,q)$, two vertices being adjacent if and only if they are on a secant of the quadric (see [22]).

(2) Kantor [22] also proved that if $n \neq 2$, then the block graph of the partial geometry $PQ^+(4n-1,q)$, ($q = 2$ or 3) is not isomorphic to the graph $\Gamma^c(Q^+(4n-1,q))$. Note that the graph $\Gamma^c(Q^+(2m-1,q))$ is pseudo-geometric with parameters $s = q^{m-1}$, $t = q^{m-1} - 1$, $\alpha = q^{m-2}(q-1)$, for any q. The graph $\Gamma^c(Q^+(3,q))$, is the complement of the $(q+1) \times (q+1)$-grid, hence is geometric if and only if there exists a projective plane of order $q+1$. It is not known whether $\Gamma^c(Q^+(5,q))$, $q \geqslant 4$, is geometric. The graph $\Gamma^c(Q^+(5,2))$ is a pseudo-geometric $(4,3,2)$-graph but a $pg(4,3,2)$ does not exist (see for instance [10]). As explained in [27], it can be read off from the computer-aided results of Hall and Roth in [20] that $\Gamma^c(Q^+(5,3))$ is not geometric. As remarked in [27] the graph $\Gamma^c(Q^+(2m-1,q))$ with $m \geqslant 5$ is not geometric for $q = 2$, but the question is still open for $q > 2$. Hence, the fact that the graph $\Gamma^c(Q^+(7,q))$ is geometric for $q = 2$ and 3 is quite remarkable indeed; see also Theorem 7.

(3) Brouwer et al. [2] have proved that the $pg(7,8,4)$ $PQ^+(7,2)$ is embeddable into a Steiner system $S(2,8,120)$. This result has been extended for the three partial geometries directly derived from $PQ^+(7,2)$ in [13].

(4) In some cases derivation of the partial geometry can be rephrased in terms of Seidel switching of graphs. We refer to [13] for the technical details.

3. New constructions of semipartial geometries

3.1. The semipartial geometries $spg(q-1,q^2,2,2q(q-1))$

A very interesting example of semipartial geometry is the semipartial geometry by Metz (private communication). We recall his construction. Let $Q(4,q)$ be a non-singular quadric of the projective space $PG(4,q)$. If we define $\mathscr{P}$ as the set of the elliptic quadrics $Q^-(3,q)$ on $Q(4,q)$, $\mathscr{L}$ as the set of all pencils of such elliptic quadrics which are pairwise tangent in a common point, and I as the natural incidence relation then $\mathscr{S} = (\mathscr{P}, \mathscr{L}, I)$ is an $spg(q-1,q^2,2,2q(q-1))$.

Let $Q^-(5,q)$ be an elliptic quadric of $PG(5,q)$ and p be a point of $PG(5,q)$ not on $Q^-(5,q)$. Let Π be a hyperplane of $PG(5,q)$ not containing p. Let $\bar{\mathscr{P}}_1$ be the projection of the point set of $Q^-(5,q)$ from p on Π and let $\bar{\mathscr{P}}_2$ be the set of points of Π on a tangent of $Q^-(5,q)$ through p. Let $\mathscr{S}$ be the geometry with point set $\mathscr{P} = \bar{\mathscr{P}}_1 \setminus \bar{\mathscr{P}}_2$, whereas the line set $\mathscr{L}$ is the set of all projections on Π of the lines of $\bar{\mathscr{S}}$, excluding the projections completely contained in $\bar{\mathscr{P}}_2$. The incidence is the one of the projective space. Then Hirschfeld and Thas [21] have proved that this is a semipartial geometry $spg(q-1,q^2,2,2q(q-1))$ isomorphic to the one by Metz.

It has been observed by Brown [4] that one does not need necessarily the GQ $Q^-(5,q)$ for this construction. Indeed if a GQ $\mathscr{S}$ of order (s,s^2) contains a subquadrangle $\mathscr{S}'$ of order s, then every point x of $\mathscr{S} \setminus \mathscr{S}'$ is collinear with the $s^2 + 1$ points of an ovoid, denoted by $\mathscr{O}_x$, of $\mathscr{S}'$. The ovoid $\mathscr{O}_x$ is said to be *subtended by* x. If it happens to be that every such subtended ovoid $\mathscr{O}_x$ is also a subtended ovoid $\mathscr{O}_y$ for another point $y \in \mathscr{S} \setminus \mathscr{S}'$, then the ovoid is called *doubly subtended* and is denoted by $\mathscr{O}_{x,y}$. If every subtended ovoid of $\mathscr{S}'$ is doubly subtended, then the sub GQ $\mathscr{S}'$ is called *doubly subtended* in the GQ $\mathscr{S}$.

Theorem 5 (Brown [4]). *Assume $\mathscr{S}$ is a GQ of order (s, s^2) containing a subGQ $\mathscr{S}'$ that is doubly subtended in $\mathscr{S}$; then the incidence structure with points the subtended ovoids of $\mathscr{S}'$, lines the rosettes of subtended ovoids (a rosette is a set of s subtended ovoids containing a common point x and having two by two just x in common), incidence the natural incidence, is a semipartial geometry $\mathrm{spg}(s - 1, s^2, 2, 2s(s - 1))$.*

The generalized quadrangle $\mathrm{Q}(4, q)$ is indeed doubly subtended in $\mathrm{Q}^-(5, q)$ and this yields the construction of Metz. However, Brown, [4] remarks that $\mathrm{Q}(4, q)$ is also doubly subtended in the GQ of order (q, q^2) (q odd) related to the flock $K1$ of Kantor, and hence yields a semipartial geometry.

It is worthwhile to remark that the construction by Hirschfeld and Thas of the semipartial geometry of Metz, implies that for q even, this semipartial geometry $\mathrm{spg}(s - 1, s^2, 2, 2s(s - 1))$ is embedded in $\mathrm{AG}(4, q)$. All semipartial geometries embedded in an affine space $\mathrm{AG}(n, q)$ for $n = 2, 3$ are classified. For $n > 3$ the question is however open. Assuming $q > 2$, then apart from the partial quadrangle $T_3^*(O)$, two models of semipartial geometries embeddable in $\mathrm{AG}(4, q)$ are known. On the one hand there is the semipartial geometry $T_3^*(\mathscr{B})$ with $\mathscr{B}$ a Baer subspace of $\mathrm{PG}(3, q)$, q a square. On the other hand, there is the semipartial geometry $\mathrm{spg}(q - 1, q^2, 2, 2q(q - 1))$ of Metz, q even. Recently, the following results on affine embeddings have been proved. For more details we refer to [5].

Theorem 6. *Let $\mathscr{S}$ be a semipartial geometry $\mathrm{spg}(q - 1, q^2, 2, 2q(q - 1))$ embedded in $\mathrm{AG}(4, q)$. Then $q = 2^h$, and $\mathscr{S}$ is the Hirschfeld–Thas model of the semipartial geometry of Metz.*

We will see in the next section that the semipartial geometry of Metz is part of a bigger family, namely of the family of semipartial geometries constructed from an SPG-system.

3.2. SPG-systems and semipartial geometries

Very recently Thas [39] has generalized the concept of SPG-regulus of a polar space P to SPG-systems of P. Without any doubt this concept will open new perspectives in the near future. We will restrict ourselves here to that part of the theory which yields semipartial geometries with new parameters. It is however important to underline that some of the examples (including the partial geometries $\mathrm{PQ}^+(4n - 1, 2)$ and $\mathrm{PQ}^+(4n - 1, 3)$) can be constructed from SPG-systems.

3.2.1. Definition of an SPG-system and construction of the semipartial geometry

Let $\mathrm{Q}(2n + 2, q)$, $n \geqslant 1$ be a non-singular quadric of $\mathrm{PG}(2n + 2, q)$. An SPG-system of $\mathrm{Q}(2n + 2, q)$ is a set $\mathscr{D}$ of $(n - 1)$-dimensional totally singular subspaces of $\mathrm{Q}(2n + 2, q)$ such that the elements of $\mathscr{D}$ on any non-singular elliptic quadric $\mathrm{Q}^-(2n + 1, q) \subset \mathrm{Q}(2n + 2, q)$ constitute a spread of the quadric $\mathrm{Q}^-(2n + 1, q)$.

Let $\mathrm{Q}^+(2n + 1, q)$ be a non-singular hyperbolic quadric of $\mathrm{PG}(2n + 1, q)$, $n \geqslant 1$. An SPG-system of $\mathrm{Q}^+(2n + 1, q)$ is a set $\mathscr{D}$ of $(n - 1)$-dimensional totally singular

subspaces of $Q^+(2n + 1, q)$ such that the elements of $\mathcal{D}$ on any non-singular quadric $Q(2n, q) \subset Q^+(2n + 1, q)$ constitute a spread of $Q(2n, q)$.

Let $H(2n + 1, q)$ be a non-singular Hermitian variety of $PG(2n + 1, q)$, $n \geqslant 1$, q a square. An SPG-system of $H(2n + 1, q)$ is a set $\mathcal{D}$ of $(n - 1)$-dimensional totally singular subspaces of $H(2n + 1, q)$ such that the elements of $\mathcal{D}$ on any non-singular Hermitian variety $H(2n, q) \subset H(2n + 1, q)$ constitute a spread of $Q(2n, q)$.

One can prove that in each case the number of elements in $\mathcal{D}$ equals the number of points of the polar space.

The construction by Thas of the semipartial geometry is as follows. Let P be one of the above polar spaces, i.e. $Q(2n+2, q)$, $Q^+(2n+1, q)$, $H(2n+1, q)$ $(n \geqslant 1)$. Let $PG(d, q)$ be the ambient space of P. Hence, in the first case $d = 2n + 2$, in the other two cases $d = 2n+1$. Let $\mathcal{D}$ be an SPG-system of P and let P be embedded in a non-singular polar space $\bar{P}$ with ambient space $PG(d + 1, q)$ of the same type as P and with projective index n. Hence for $P = Q(2n+2, q)$, we have $\bar{P} = Q^-(2n+3, q)$; for $P = Q^+(2n+1, q)$, we have $\bar{P} = Q(2n + 2, q)$ and for $P = H(2n + 1, q)$, we have $\bar{P} = H(2n + 2, q)$. If $\bar{P}$ is not symplectic and $y \in \bar{P}$, then let τ_y be the tangent hyperplane of $\bar{P}$ at y; if $\bar{P}$ is symplectic and θ is the corresponding symplectic polarity of $PG(d + 1, q)$, then let $\tau_y = y^\theta$ for any $y \in PG(d + 1, q)$.

For $y \in \bar{P} \backslash P$ let $\bar{y}$ be the set of all points z of $\bar{P} \backslash P$ for which $\tau_z \cap P = \tau_y \cap P$. Note that no two distinct points of $\bar{y}$ are collinear in $\bar{P}$. If P is orthogonal then $|\bar{y}| = 2$ except when $P = Q^+(2n + 1, q)$ and q even, in which case $|\bar{y}| = 1$. If P is Hermitian then $|\bar{y}| = \sqrt{q} + 1$.

Let ξ be any maximal totally singular subspace of $\bar{P}$, not contained in P, such that $\xi \cap P \in \mathcal{D}$ and let $y \in \xi \backslash P$. Further, let $\bar{\xi}$ be the set of all maximal totally singular subspaces η of $\bar{P}$, not contained in P, for which $\xi \cap P = \eta \cap P$ and $\eta \cap \bar{y} \neq \emptyset$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be the incidence structure with $\mathcal{P} = \{\bar{y} \| y \in \bar{P} \backslash P\}$; $\mathcal{L}$ contains all the sets $\bar{\xi}$ as defined above; if $\bar{y} \in \mathcal{P}$ and $\bar{\xi} \in \mathcal{L}$ then $\bar{y} \, I \, \bar{\xi}$ if and only if for some $z \in \bar{y}$ and some $\eta \in \bar{\xi}$, one has that $z \in \eta$.

In [39] it is proved that this incidence structure is a $(0, \alpha)$-geometry of order (s, t) with $s + 1 = q^n$ and $t + 1$ the number of elements in a spread of P. The parameter α equals to q^{n-1} times the number of points of $\bar{P}$ in any set $\bar{y} \in \mathcal{P}$.

Theorem 7. (1) *If P is the polar space $Q(2n+2, q)$ then $\mathcal{S}$ is a semipartial geometry* $spg(q^n - 1, q^{n+1}, 2q^{n-1}, 2q^n(q^n - 1))$.

(2) *If P is the polar space $Q^+(2n + 1, q)$ then the point graph $\Gamma(\mathcal{S})$ is strongly regular if and only if $q = 2$ or $q = 3$. In these cases $\mathcal{S}$ is a partial geometry.*

(3) *If P is the polar space $H(2n + 1, q)$ then $\mathcal{S}$ is a semipartial geometry* $spg(q^n - 1, q^n\sqrt{q}, q^{n-1}(\sqrt{q} + 1), q^{n-1}(q^n - 1)\sqrt{q}(\sqrt{q} + 1))$.

Corollary 1. *Let P be the polar space $Q(2n+2, q)$. The geometry will be denoted by $TQ(2n + 2, q)$.*

If $n = 1$ the SPG-system is the complete set of points of $Q(4, q)$ and the semipartial geometry was known before, it is the semipartial geometry of Metz, see [17].

Assume $n = 2$. The set of lines in all the planes of a spread $\mathscr{R}$ is an SPG-system of $Q(6, q)$. Note that spreads of $Q(6, q)$ are known to exist when $q = p^h$ and $p \in \{2, 3\}$. On the other hand, the line set of the classical generalized hexagon $H(q)$ embedded in $Q(6, q)$, is an SPG-system of $Q(6, q)$. It is proved in [39] that an SPG-system on $Q(6, q)$ is a member of one of these two classes.

For any $n \geqslant 3$, any spread of $Q(2n + 2, q)$ defines an SPG-system. Such a spread is known to exist if q is even.

In [18] Delanote gives a construction of a semipartial geometry with point graph the graph on the internal points of a quadric $Q(4m + 2, 3)$, (vertices are adjacent when non-orthogonal) under the condition of existence of an orthogonal spread. His arguments can easily be generalized for any odd q and in fact, his semipartial geometry is isomorphic to $TQ(2n + 2, q)$ with $n = 2m$.

Corollary 2. *Let P be the polar space $Q^+(2n + 1, q)$; $q = 2$ or 3.*

If $n = 2m - 1$ is odd and $q = 2$ then $Q^+(2n + 1, 2)$ has a spread and the partial geometry is isomorphic to the partial geometry $PQ^+(4m-1, 2)$ of De Clerck et al. [15].

If $n = 2m - 1$ is odd and $q = 3$ then the partial geometry is isomorphic to the partial geometry $PQ^+(4m - 1, 3)$ of Thas, which only exists if $Q^+(4m - 1, 3)$ has a spread; the existence of such a spread is open for $m \geqslant 3$.

Corollary 3. *Let P be the polar space $H(2n + 1, q)$. The geometry will be denoted by $TH(2n + 1, q)$.*

Unfortunately, if $n \geqslant 2$ then no SPG-system of $H(2n + 1, q)$ is known. If $n = 1$, then $\mathscr{D}$ is the set of points of $H(3, q)$ and the semipartial geometry is the one of Thas as described in [17].

4. Parameter lists

4.1. The known partial geometries (up to duality)

Notation	s	t	α	Remarks and references
$\mathscr{S}(\mathscr{K})$	$2^h - 2^m$	$2^h - 2^{h-m}$	$(2^m - 1)(2^{h-m} - 1)$	$0 < m < h$; $h \neq 2$, [35,36]
$T_2^*(\mathscr{K})$	$2^h - 1$	$(2^h + 1)(2^m - 1)$	$2^m - 1$	$0 < m < h$, [35,36]
$\mathscr{M}_3(h)$	$3^h - 1$	$\frac{1}{2}(3^{2h} - 1)$	$\frac{1}{2}(3^h - 1)$	[24,25]
$PQ^+(4n - 1, 2)$ +derivations	$2^{2n-1} - 1$	2^{2n-1}	2^{2n-2}	$1 < n$, [12,15]
$PQ^+(7, 3)$ +derivations	26	27	18	[12,37]
vL-S	5	5	2	[40]
Haemers	4	17	2	[19]
Mathon	8	20	2	[14]

4.2. The known semipartial geometries

Notation	s	t	α	μ	Remarks and references
$M(r)$	1	$r-1$	1	1	Moore graph $r = 2, 3, 7$
$\overline{M(r)}$	$r-1$	$r-1$	$r-1$	$(r-1)^2$	$r = 2, 3, 7$, [9]
$U_{2,3}(n)$	2	$n-3$	2	4	$n \geqslant 4$ [9]
$LP(n,q)$	$q(q+1)$	$\frac{q^{n-1}-1}{q-1} - 1$	$q+1$	$(q+1)^2$	$n \geqslant 4$, [9]
$\overline{W(2n+1,q)}$	q	$q^{2n}-1$	q	$q^{2n}(q-1)$	$n \geqslant 1$, [9]
$NQ^{+}(2n-1,2)$	2	$2^{2n-3}-2^{n-2}-1$	2	$2^{2n-3}-2^{n-1}$	$n \geqslant 3$, [17]
$NQ^{-}(2n-1,2)$	2	$2^{2n-3}+2^{n-2}-1$	2	$2^{2n-3}+2^{n-1}$	$n \geqslant 3$, [17]
$H_q^{(n+1)*} \cong T_{n-1}^*(\mathscr{B})$	q^2-1	$\frac{q^n-1}{q-1} - 1$	q	$q(q+1)$	$n \geqslant 3$, [9]
$T_3^*(\mathcal{O})$	$q-1$	q^2	1	$q(q-1)$	[8]
$T_2^*(\mathscr{U})$	q^2-1	q^3	q	$q^2(q^2-1)$	[9]
$TQ(2n+2,q)$	q^n-1	q^{n+1}	$2q^{n-1}$	$2q^n(q^n-1)$	For $n=1$, see also [4] and Section 3.1. If $n \geqslant 3$ then $q = 2^h$, [39]
$TH(3,q^2)$	q^2-1	q^3	$q+1$	$q(q+1)(q^2-1)$	[17]
$RQ^{-}(2n+3,q)$	$q^{n+1}-1$	q^{n+2}	q^n	$q^{n+1}(q^{n+1}+1)$	q Prime power for $n=1$, $q = 2^h$ for $n \geqslant 2$, [38]
$Gew(56)$	1	9	1	2	$v = 56$, Gewirtz graph
$HS(77)$	1	15	1	4	$v = 77$, Higman–Sims family
$HS(100)$	1	21	1	6	$v = 100$, Higman–Sims family
$T_4^*(\mathscr{K}(11))$	2	10	1	2	$v = 243$; $\mathscr{K}(11)$ the 11-cap in PG$(4,3)$
$T_5^*(\mathscr{K}(56))$	2	55	1	20	$v = 729$; $\mathscr{K}(56)$ the 56-cap in PG$(5,3)$
$T_5^*(\mathscr{K}(78))$	3	77	1	14	$v = 1024$; $\mathscr{K}(78)$ the 78-cap of Hill in PG$(5,4)$

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