Proof of two conjectures of Móricz on double trigonometric series

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Abstract


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1. Introduction

Let \( \{b_n\}_{n=1}^{\infty} \) be a sequence of real numbers such that \( \lim_{n \to \infty} b_n = 0 \) and \( \sum_{k=1}^{\infty} |b_k - b_{k+1}| \) converges. It is well known that the sine series \( \sum_{k=1}^{\infty} b_k \sin kx \) converges for all \( x \in [0, \pi] \). Boas [1] proved that \( \sum_{k=1}^{\infty} \frac{b_k}{k} \) converges if and only if \( \lim_{\delta \to 0^+} \int_{\delta}^{\pi} \sum_{k=1}^{\infty} b_k \sin kx \, dx \) exists. Consequently, he proved that if \( \sum_{k=0}^{\infty} a_k \) is an absolutely convergent series of real numbers and \( \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k = 0 \), then \( \lim_{\delta \to 0^+} \int_{\delta}^{\pi} \frac{1}{x} \left( \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right) \, dx \) exists if and only if \( \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{a_0}{2} + \sum_{j=1}^{k} a_j \right) \) converges.

The object of this paper is to use a unified method to extend the aforementioned results to higher dimensions; in particular, we give affirmative answers to two conjectures of Móricz [3] concerning double trigonometric series.

This paper is organized as follows. In Section 2 we deal with the one-dimensional case separately since it is straightforward and we get a good prospect of the general situation. In Section 3 we give some preliminaries on which all our subsequent reasoning depends. In Section 4 we obtain a crucial result concerning rectangular multiple series; see Theorem 4.3 for details. In Sections 5 and 6, we apply Theorem 4.3 to extend the above-mentioned Boas’ results from one-dimensional to \( m \)-dimensional trigonometric series.

2. One-dimensional case

The main result of this section is Theorem 2.2 whose proof depends on the following well-known Dirichlet’s test.
Theorem 2.1. Let \( \{u_k\}_{k=1}^{\infty} \) and \( \{v_k\}_{k=1}^{\infty} \) be two sequences of real numbers. If \( \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} u_k \right| \) is finite, \( \lim_{n \to \infty} v_n = 0 \) and \( \sum_{k=1}^{\infty} |v_k - v_{k+1}| \) converges, then \( \sum_{k=1}^{\infty} u_k v_k \) converges and
\[
\sum_{k=1}^{\infty} u_k v_k = \sum_{k=1}^{\infty} (v_k - v_{k+1}) \sum_{j=1}^{k} u_j.
\]

Theorem 2.2. Let \( \{c_k\}_{k=1}^{\infty} \) be a sequence of real numbers such that \( \lim_{n \to \infty} c_n = 0 \) and \( \sum_{k=1}^{\infty} |c_k - c_{k+1}| \) converges. If \( \{\Phi_n\}_{n=1}^{\infty} \subset C[0, \pi] \) and
\[
\sup_{n \in \mathbb{N}} \|\Phi_n\|_{C[0, \pi]} + \sup_{0 < x < \pi} \sup_{n \in \mathbb{N}} \left| \frac{\Phi_n(x) - \Phi_n(0)}{nx} \right| + \sup_{x \in [0, \pi]} \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} x \Phi_k(x) \right| < \infty,
\]
then \( \sum_{k=1}^{\infty} \frac{c_k \Phi_k(x)}{k} \) converges for all \( x \in (0, \pi] \). Moreover,
\[
\lim_{\delta \to 0^+} \left\{ \sum_{k=1}^{\lfloor \frac{1}{\delta} \rfloor} \frac{c_k \Phi_k(0)}{k} - \sum_{k=\lfloor \frac{1}{\delta} \rfloor + 1}^{\infty} \frac{c_k \Phi_k(\delta)}{k} \right\} = 0.
\]

Proof. The first assertion is a consequence of Dirichlet’s test. To prove (1) we may assume that the sequence \( \{c_k\}_{k=1}^{\infty} \) is decreasing and
\[
\sup_{n \in \mathbb{N}} \|\Phi_n\|_{C[0, \pi]} + \sup_{0 < x < \pi} \sup_{n \in \mathbb{N}} \left| \frac{\Phi_n(x) - \Phi_n(0)}{nx} \right| + \sup_{x \in [0, \pi]} \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} x \Phi_k(x) \right| \leq \frac{1}{2}.
\]
Let \( \delta \in (0, 1) \) and let \( \lfloor \frac{1}{\delta} \rfloor \) be the greatest integer less or equal to \( \frac{1}{\delta} \). Then
\[
\left| \sum_{k=1}^{\lfloor \frac{1}{\delta} \rfloor} \frac{c_k \Phi_k(0)}{k} - \sum_{k=1}^{\infty} \frac{c_k \Phi_k(\delta)}{k} \right| \leq \sum_{k=1}^{\lfloor \frac{1}{\delta} \rfloor} \frac{c_k \Phi_k(\delta) - \Phi_k(0)}{k} + \sum_{k=\lfloor \frac{1}{\delta} \rfloor + 1}^{\infty} \frac{c_k \Phi_k(\delta)}{k}
\]
\[
\leq \frac{1}{\lfloor \frac{1}{\delta} \rfloor} \sum_{k=1}^{\lfloor \frac{1}{\delta} \rfloor} c_k + \frac{1}{\lfloor \frac{1}{\delta} \rfloor} \sum_{n=\lfloor \frac{1}{\delta} \rfloor + 1}^{\infty} \frac{n \Phi_k(\delta)}{k}
\]
\[
\leq \frac{1}{\lfloor \frac{1}{\delta} \rfloor} \sum_{k=1}^{\lfloor \frac{1}{\delta} \rfloor} c_k + 2c_{\lfloor \frac{1}{\delta} \rfloor}.
\]
It is now clear that the assumption \( \lim_{n \to \infty} c_n = 0 \) implies (1). The proof is complete. \( \square \)

The following results of Boas are immediate consequences of Theorem 2.2. Incidentally, we correct a small error (cf. [1, p. 792, lines 1–2]) in the proof of [1, Theorem 3(b)].

Theorem 2.3. (See [1, Theorem 3(b)].) Let \( \{b_k\}_{k=1}^{\infty} \) be a sequence of real numbers such that \( \lim_{k \to \infty} b_k = 0 \) and \( \sum_{k=1}^{\infty} |b_k - b_{k+1}| \) converges. Then \( \sum_{k=1}^{\infty} b_k k \sin kx \) converges if and only if \( \lim_{\delta \to 0^+} \int_{\delta}^{\pi} b_k \sin kx \, dx \) exists.

Proof. We apply Theorem 2.2 with \( c_k = b_k \) and \( \Phi_k(x) = \int_{\delta}^{\pi} k \sin kt \, dt \) to conclude that \( \sum_{k=1}^{\infty} b_k \frac{1 - (-1)^k}{k} \) converges if and only if \( \lim_{\delta \to 0^+} \sum_{k=1}^{\infty} b_k \int_{\delta}^{\pi} k \sin kt \, dt \) exists. Since it is well known that \( \int_{\delta}^{\pi} b_k \sin kx \, dx = \sum_{k=1}^{\infty} b_k \sin kx \) for all \( 0 < \delta < \pi \) and the series \( \sum_{k=1}^{\infty} (-1)^k b_k \) converges by Dirichlet’s test, the theorem is proved. \( \square \)

Theorem 2.4. (See [1, Theorem 3(a)].) Let \( \sum_{k=0}^{\infty} a_k \) be an absolutely convergent series of real numbers. If \( \sum_{k=1}^{\infty} a_k = 0 \), then \( \lim_{\delta \to 0^+} \int_{\delta}^{\pi} \frac{1}{2} (a_0 + \sum_{k=\lfloor \pi \delta \rfloor}^{\infty} a_k) \cos kx \, dx \) exists if and only if \( \sum_{k=1}^{\infty} \frac{1}{k} (a_0 + \sum_{r=1}^{k} a_r) \) converges.
Proof. Let $\lambda_0 := \frac{1}{2}$ and $\lambda_k := 1$ ($k \in \mathbb{N}$). If $x \in (0, \pi]$, then our assumptions and Theorem 2.1 yield

$$
\frac{1}{x} \sum_{k=0}^{\infty} \lambda_k a_k \cos kx = -\frac{1}{x} \sum_{k=1}^{\infty} a_k (1 - \cos kx)
$$

$$
= -\frac{1}{x} \sum_{k=1}^{\infty} \left( \sum_{j=k}^{\infty} a_j \right) (\cos(k - 1)x - \cos kx)
$$

$$
= \frac{1}{x} \sum_{k=1}^{k-1} \left( \sum_{j=0}^{k-1} \lambda_j a_j \right) (\cos(k - 1)x - \cos kx)
$$

$$
= a_0 (1 - \cos x) + G(x) \sum_{k=1}^{\infty} \left( \sum_{j=0}^{k} \lambda_j a_j \right) \sin \left( k + \frac{1}{2} \right) x,
$$

where $G(x) := \frac{2 \sin \frac{x}{2}}{x}$. Therefore

$$
\lim_{\delta \to 0^+} \left\{ \int_{\delta}^{\pi} \frac{1}{x} \sum_{k=0}^{\infty} \lambda_k a_k \cos kx \, dx - \int_{\delta}^{\pi} G(x) \sum_{k=1}^{\infty} \left( \sum_{j=0}^{k} \lambda_j a_j \right) \sin \left( k + \frac{1}{2} \right) x \, dx \right\} \text{ exists.}
$$

(2)

In view of (2) it remains to show that $\sum_{k=1}^{\infty} \frac{a_0}{k} + \sum_{r=1}^{n} a_r$ converges if and only if

$$
\lim_{\delta \to 0^+} \int_{\delta}^{\pi} G(x) \sum_{k=1}^{\infty} \left( \sum_{j=0}^{k} \lambda_j a_j \right) \sin \left( k + \frac{1}{2} \right) x \, dx \text{ exists.}
$$

Direct computations yield

$$
\sup_{n \in \mathbb{N}} \sup_{x \in [0, \pi]} \left| \int_{0}^{x} n G(t) \sin \left( n + \frac{1}{2} \right) t \, dt \right| \leq (1 + \|G'\|_{L^1[0, \pi]}) \sup_{n \in \mathbb{N}} \sup_{x \in [0, \pi]} \left| \frac{n \cos (n + \frac{1}{2}) x}{n + \frac{1}{2}} \right| < \infty,
$$

$$
\sup_{x \in (0, \pi) \cap \mathbb{N}} \frac{1}{nx} \left| \int_{0}^{x} G(t) \sin \left( n + \frac{1}{2} \right) t \, dt \right| < \infty,
$$

and

$$
\sup_{x \in [0, \pi] \cap \mathbb{N}} \sup_{n \in \mathbb{N}} \left| \int_{0}^{x} k G(t) \sin \left( k + \frac{1}{2} \right) t \, dt \right| \leq 2\pi \|G'\|_{L^1[0, \pi]} + 2\pi^2 < \infty,
$$

so we can apply Theorem 2.2 with $c_k = \sum_{j=1}^{k} \lambda_j a_j$ and $\Phi_k(x) = \int_{x}^{\pi} k G(t) \sin (k + \frac{1}{2}) t \, dt$ to conclude that $\sum_{k=1}^{\infty} \frac{1}{k} \left( \sum_{j=0}^{k} \lambda_j a_j \right) \int_{x}^{\pi} k G(t) \sin (k + \frac{1}{2}) t \, dt$ converges if and only if $\lim_{\delta \to 0^+} \int_{\delta}^{\pi} \frac{1}{x} \left( \frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_k a_k \cos kx \right) \, dx$ exists. Since $\left| \frac{1}{1 - \int_{\delta}^{\pi} G(x) \sin \left( k + \frac{1}{2} \right) x \, dx \right| = O\left( \frac{1}{k^2} \right)$ for all $k \in \mathbb{N}$ and the assumption $\sum_{k=0}^{\infty} |\lambda_k a_k| < \infty$ holds, the theorem follows. $\Box$

Theorem 2.5. (See [1, Theorem 1(a)].) If $\sum_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then $\lim_{\delta \to 0^+} \int_{\delta}^{\pi} \frac{1}{x} \sum_{k=1}^{\infty} b_k \sin kx \, dx$ exists.

Proof. The proof is similar to that of Theorem 2.4. $\Box$

Theorem 2.6. (See [1, Theorem 1(b)].) Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\lim_{n \to \infty} a_n = 0$ and $\sum_{k=0}^{\infty} |a_k - a_{k+1}|$ converges. Then $\lim_{\delta \to 0^+} \int_{\delta}^{\pi} \left( \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right) \, dx$ exists.

Proof. The proof is similar to that of Theorem 2.3. $\Box$
3. Some preliminaries concerning multiple rectangular series

Unless specified otherwise, \( m \geq 2 \) is always a fixed positive integer. Each point \((x_1, \ldots, x_m)\) in \( \mathbb{R}^m \) is usually denoted by the corresponding bold letter \( x \). For \( p, q \in \mathbb{N}_0^m \), write \( p \preceq q \) or \( q \succeq p \) if and only if \( p_i \leq q_i \) for each \( i \in \{1, \ldots, m\} \).

**Definition 3.1.** (See [2].) Let \( \{u_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers. We consider the following (formal) multiple series:

\[
\sum_{k \in \mathbb{N}^m} u_k := \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} u_{k_1,\ldots,k_m}.
\]

(i) For each \( n \in \mathbb{N}^m \), the rectangular \( n \) partial sum of (3) is given by

\[
\sum_{1 \leq k \leq n} u_k := \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} u_{k_1,\ldots,k_m}.
\]

(ii) The multiple series (3) converges in Pringsheim’s sense to a real number \( s \) if for each \( \varepsilon > 0 \) there exists an integer \( N(\varepsilon) \in \mathbb{N} \) such that

\[
\left| \sum_{1 \leq k \leq n} u_k - s \right| < \varepsilon
\]

whenever \( \min\{n_1, \ldots, n_m\} \geq N(\varepsilon) \).

(iii) The multiple series (3) converges regularly if for each \( \varepsilon > 0 \) there exists \( N(\varepsilon) \in \mathbb{N} \) such that

\[
\left| \sum_{p \preceq k \preceq q} u_k \right| < \varepsilon
\]

whenever \( p, q \in \mathbb{N}^m \) with \( q \succeq p \) and \( \max\{p_1, \ldots, p_m\} \geq N(\varepsilon) \).

**Theorem 3.2.** (See [2, Theorem 1].) Let \( \{u_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers. The multiple series \( \sum_{k \in \mathbb{N}^m} u_k \) is regularly convergent if and only if

(i) \( \sum_{k \in \mathbb{N}^m} u_k \) converges in Pringsheim’s sense, and

(ii) for each choice of the index \( j \in \{1, \ldots, m\} \) and for all fixed integral values of \( c_j \), the \( (m-1) \)-multiple series

\[
\sum_{k \in \mathbb{N}^m \atop k_j = c_j} u_k
\]

are regularly convergent.

In order to formulate a multidimensional analogue of Theorem 2.1, we need some notations. Let \( \{v_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers. For \( j \in \{1, \ldots, m\} \), set \( \Delta_j(v_k) := v_k \) and

\[
\Delta_j(v_{k_1,\ldots,k_n}) := v_{k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_n} - v_{k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_n}.
\]

For \( \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, m\} \), set

\[
\Delta_{(j_1,\ldots,j_s)}(v_k) := \Delta_{j_1} \left( \cdots \left( \Delta_{j_s}(v_k) \right) \cdots \right).
\]

Set \( \|x\| = \max_{i=1,\ldots,m} |x_i| \) (\( x \in \mathbb{R}^m \)). The following theorem is essential [2, Theorem 3].
Theorem 3.3. Let \( \{v_k: k \in \mathbb{N}^m\} \) be a multiple sequences of real numbers such that \( \lim_{|n| \to \infty} v_n = 0 \) and the multiple series \( \sum_{k \in \mathbb{N}^m} |\Delta_{[1,\ldots,m]}(v_k)| \) converges. If the rectangular partial sums of the series (3) are bounded, then the multiple series \( \sum_{k \in \mathbb{N}^m} u^k v^k \) converges regularly and
\[
\sum_{k \in \mathbb{N}^m} u^k v^k = \sum_{k \in \mathbb{N}^m} \left\{ \Delta_{[1,\ldots,m]}(v_k) \sum_{1 \leq j \leq k} u^j \right\};
\]
the multiple series on the right being absolutely convergent.

4. A multidimensional analogue of Theorem 2.2

The main result of this section is Theorem 4.3, which is an \( m \)-dimensional analogue of Theorem 2.2. We need the following lemmas.

Lemma 4.1. Let \( \{c_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers such that \( \lim_{|n| \to \infty} c_n = 0 \) and the multiple series \( \sum_{k \in \mathbb{N}^m} |\Delta_{[1,\ldots,m]}(c_k)| \) converges. Then there exist two multiple sequences \( \{c_{1,k}: k \in \mathbb{N}^m\}, \{c_{2,k}: k \in \mathbb{N}^m\} \) of non-negative numbers such that

(a) \( c_k = c_{1,k} - c_{2,k} \) for all \( k \in \mathbb{N}^m \),
(b) \( \lim_{|n| \to \infty} c_{1,n} = \lim_{|n| \to \infty} c_{2,n} = 0 \), and
(c) for \( i = 1, 2, 0 \leq \Delta_{[1,\ldots,m]}(c_{i,k}) \leq |\Delta_{[1,\ldots,m]}(c_k)| \) for all \( k \in \mathbb{N}^m \).

Proof. For each \( i = 1, 2 \) and \( k \in \mathbb{N}^m \), we set
\[
c_{i,k} := \frac{1}{2} \left\{ \sum_{r \geq k} \left( |\Delta_{[1,\ldots,m]}(c_r)| + (-1)^{i-1} |\Delta_{[1,\ldots,m]}(c_r)| \right) \right\}.
\]
Then it is clear that assertions (a)-(c) hold. \( \square \)

Lemma 4.2. If \( u_1, \ldots, u_m \) and \( v_1, \ldots, v_m \) are real numbers, then
\[
\left| \prod_{i=1}^m u_i - \prod_{i=1}^m v_i \right| \leq \sum_{j=1}^m \left( \prod_{i=1}^{j-1} |u_i| \right) \left( \prod_{i=j+1}^m |v_i| \right) |v_j - u_j|.
\]

Proof. Use induction on \( m \). \( \square \)

Theorem 4.3. Let \( \{c_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers such that \( \lim_{|n| \to \infty} c_n = 0 \) and \( \sum_{k \in \mathbb{N}^m} |\Delta_{[1,\ldots,m]}(c_k)| \left( \ln(\|k\| + 1) \right)^{m-1} \) converges. If \( \{\Phi_{i,n}\}_{n=1}^\infty \subset C[0, \pi] \) \( (i = 1, \ldots, m) \) and
\[
\max_{i=1,\ldots,m} \left\{ \sup_{n \in \mathbb{N}} \|\Phi_{i,n}\| + \sup_{x_i \in (0, \pi)} \sup_{n \in \mathbb{N}} \left| \Phi_{i,n}(x_i) - \Phi_{i,n}(0) \right| + \sup_{x_i \in [0, \pi]} \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^n x_i \Phi_{i,k}(x_i) \right| \right\} \leq \infty,
\]
then the multiple series \( \sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^m \frac{\Phi_{i,k}(x_i)}{k_i} \) converges regularly for all \( x \in (0, \pi)^m \) and
\[
\lim_{\delta \to 0} \left\{ \sum_{1 \leq k \leq \left\lceil \frac{1}{\delta} \right\rceil} c_k \prod_{i=1}^m \frac{\Phi_{i,k}(0)}{k_i} - \sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^m \frac{\Phi_{i,k}(\delta_i)}{k_i} \right\} = 0.
\]

Moreover,
\[
\sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^m \frac{\Phi_{i,k}(0)}{k_i} \text{ converges regularly } \iff \lim_{\delta \to 0} \sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^m \frac{\Phi_{i,k}(\delta_i)}{k_i} \text{ exists.}
\]
Proof. In view of Lemma 4.1 we may assume that

\[ \lim_{\|n\| \to \infty} c_n = 0 \quad \text{and} \quad \Delta_{\{1, \ldots, m\}}(c_k) \geq 0 \quad (k \in \mathbb{N}^m). \]  

(7)

Since \( \min_{k \in \mathbb{N}^m} \ln(\|k\| + 1) \geq \ln 2 > \frac{1}{2} \), the first assertion is an easy consequence of Theorem 3.3. To prove (5) we may assume that

\[ \max_{i=1, \ldots, m} \left\{ \sup_{n \in \mathbb{N}} \|\Phi_{i,n}\| \infty + \sup_{x_i \in (0, \pi)} \sup_{n \in \mathbb{N}} \frac{\Phi_{i,n}(x_i) - \Phi_{i,n}(0)}{nx_i} \right\} + \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^n x_i \Phi_{i,k}(x_i) \right| \leq \frac{1}{2}. \]

Set \( \Psi(x) := \prod_{i=1}^m \Phi_{i,k_i}(x_i) \) and let \( \delta \in (0, 1)^m \) be given. We claim that

\[ \left| \sum_{1 \leq k \leq \left(\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_m}\right)} c_k \Psi_k(\delta) - \sum_{k \in \mathbb{N}^m} c_k \Psi_k(\delta) \right| \]

\[ \leq \sum_{i=1}^m \frac{1}{\delta_i} \sum_{1 \leq k_i \leq \left[ \frac{1}{\delta_i} \right]} k_i c_k \left( \prod_{j=1}^m k_j \right) + \sum_{\emptyset \neq \Gamma \subseteq \{1, \ldots, m\}} 12^m \sum_{k \in \mathbb{N}^m} \left( \Delta_{\{1, \ldots, m\}}(c_k) \right) \left( \ln(\|k\| + 1) \right)^{m-1}. \]  

(8)

Set \( \Gamma' := \{1, \ldots, m\} \setminus \Gamma \) (\( \Gamma \subseteq \{1, \ldots, m\} \)). Then, by the triangle inequality,

\[ \left| \sum_{1 \leq k \leq \left(\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_m}\right)} c_k \Psi_k(\delta) - \sum_{k \in \mathbb{N}^m} c_k \Psi_k(\delta) \right| \]

\[ \leq \sum_{1 \leq k \leq \left(\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_m}\right)} c_k \left| \Psi_k(\delta) - \Psi_k(\delta) \right| + \sum_{\emptyset \neq \Gamma' \subseteq \{1, \ldots, m\}} \sum_{\emptyset \neq \Gamma \subseteq \{1, \ldots, m\}} \left| \sum_{k \in \mathbb{N}^m} c_k \Psi_k(\delta) \right| \]

\[ := S + \sum_{\emptyset \neq \Gamma \subseteq \{1, \ldots, m\}} |T_{\Gamma}|. \]  

(9)

A direct application of Lemma 4.2 yields

\[ S \leq \sum_{i=1}^m \frac{1}{\delta_i} \sum_{1 \leq k_i \leq \left[ \frac{1}{\delta_i} \right]} k_i c_k \left( \prod_{j=1}^m k_j \right). \]  

(10)

On the other hand, for each fixed non-empty \( \Gamma \subseteq \{1, \ldots, m\} \),

\[ T_{\Gamma} = \sum_{k \in \mathbb{N}^m} \sum_{\begin{array}{c} j \in \mathbb{N}^m \\ j_i > k_i \forall i \in \Gamma' \\ j_i = k_i \forall i \in \Gamma \\ k_i \leq \left[ \frac{1}{\delta_i} \right] \forall \delta_i \end{array}} c_j \Psi_j(\delta). \]

so that

\[ |T_{\Gamma}| \leq \sum_{k \in \mathbb{N}^m} \left| \sum_{\begin{array}{c} j \in \mathbb{N}^m \\ j_i > k_i \forall i \in \Gamma' \\ j_i = k_i \forall i \in \Gamma \\ k_i \geq 1 \forall \delta_i \end{array}} c_j \prod_{i \in \Gamma} \frac{\Phi_{i,j_i}(\delta_i)}{j_i} \right| \prod_{i \in \Gamma'} \frac{1}{k_i}. \]
Using (7) and Theorem 3.3 we get

\[
|T_\Gamma| \leq \sum_{k \in \mathbb{N}^m} ^m \sum_{j \in \mathbb{N}^m} ^m c_j \prod_{i \in \Gamma} ^m \frac{2}{\delta_i} \prod_{i \in \Gamma'} \frac{1}{k_i}.
\]  

(11)

Let \( R_\Gamma \) denote the right-hand side of (11). Then

\[
R_\Gamma \leq 4^m \sum_{k \in \mathbb{N}^m} ^m \sum_{r \in \mathbb{N}^m} ^m \Delta_{\{1, \ldots, m\}}(c_r) \prod_{i \in \Gamma} ^m \frac{1}{k_i} \quad \text{(by (7))}
\]

\[
= 4^m \Delta_{\{1, \ldots, m\}}(c_r) \prod_{i \in \Gamma} ^m \sum_{r_i \geq \frac{1}{k_i}} ^m \frac{1}{k_i} \quad \text{(using (7) and interchanging the order of summation)}
\]

\[
\leq 12^m \sum_{k \in \mathbb{N}^m} ^m \left( \Delta_{\{1, \ldots, m\}}(c_k) \left( \ln \left( \|k\| + 1 \right) \right) \right)^{m-1}
\]

(12)

because \( \sum_{k=1}^{n} \frac{1}{k} \leq 3 \ln(n + 1) \quad (n \in \mathbb{N}) \) and the cardinality of \( \Gamma \) is at most \( m \). Combining (9)–(12) yields (8) to be proved.

Following the proof of (12) we get

\[
\sum_{k \in \mathbb{N}^m} ^m \frac{k_i c_k \prod_{j=1}^{m} k_j}{\prod_{j=1}^{m} k_j} \leq 3^{m-1} \sum_{k \in \mathbb{N}^m} ^m \left( \Delta_{\{1, \ldots, m\}}(c_k) \left( \ln \left( \|k\| + 1 \right) \right) \right)^{m-1}
\]

(13)

for \( i = 1, \ldots, m \). Hence the assumption

\[
\sum_{k \in \mathbb{N}^m} ^m |\Delta_{\{1, \ldots, m\}}(c_k)| \left( \ln \left( \|k\| + 1 \right) \right)^{m-1} < \infty
\]

and (8) yield (5) to be proved.

It is now easy to obtain (6) as a consequence of Theorem 3.2. We infer from (5) that \( \sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^{m} \frac{\Phi_{\delta_i}(0)}{k_i} \) converges in the sense of Pringsheim if and only if

\[
\lim_{\delta \to 0} \sum_{\delta \in (0, \pi]^m} \sum_{k \in \mathbb{N}^m} c_k \prod_{i=1}^{m} \frac{\Phi_{\delta_i}(\delta_i)}{k_i}
\]

exists. From the proof of (13) it is clear that for \( j \in \{1, \ldots, m\} \) and for all fixed integral values of \( d_j \), the \((m - 1)\)-multiple series

\[
\sum_{k_j = d_j} ^m \frac{c_k}{\prod_{i=1}^{m} k_i}
\]

are regularly convergent. An appeal to Theorem 3.2 completes the proof. \( \square \)
5. An integrability theorem for multiple sine series

The main result of this section is Theorem 5.2, which gives an affirmative answer to a conjecture of Móricz [3, Remark 1(i)]. We need a lemma.

Lemma 5.1. Let \( \{b_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers such that \( \lim_{\|n\| \to \infty} b_n = 0 \) and \( \sum_{k \in \mathbb{N}^m} |\Delta_{[1,\ldots,m]}(b_k)(\ln(\|k\| + 1))^{m-1} \) converges. If \( \Gamma \subseteq \{1, \ldots, m\} \) is non-empty, then

\[
\sum_{k \in \mathbb{N}^m} b_k \prod_{i \in \Gamma} \frac{(-1)^{k_i}}{k_i} \prod_{i \in \Gamma^c} \frac{1}{k_i}
\]

is regularly convergent.

Proof. In view of Lemma 4.1, we may assume that \( \Delta_{[1,\ldots,m]}(b_k) \geq 0 \) for all \( k \in \mathbb{N}^m \). If \( p, q \in \mathbb{N}^m \) with \( q \geq p \), then a single summation by parts gives

\[
\left| \sum_{p \leq k \leq q} b_k \prod_{i \in \Gamma} \frac{(-1)^{k_i}}{k_i} \prod_{i \in \Gamma^c} \frac{1}{k_i} \right| \leq 2 \sum_{p \leq k \leq q} b_k \prod_{i=1}^{m} \frac{1}{k_i} \text{ for some } \ell \in \Gamma
\]

\[
\leq 3^m \sum_{k \in \mathbb{N}^m} (\Delta_{[1,\ldots,m]}(b_k))(\ln(\|k\| + 1))^{m-1}.
\]

It is now easy to check that the lemma holds. \( \square \)

Theorem 5.2. Let \( \{b_k: k \in \mathbb{N}^m\} \) be a multiple sequence of real numbers such that \( \lim_{\|n\| \to \infty} b_n = 0 \) and \( \sum_{k \in \mathbb{N}^m} |\Delta_{[1,\ldots,m]}(b_k)(\ln(\|k\| + 1))^{m-1} \) converges. Then the multiple series \( \sum_{k \in \mathbb{N}^m} \frac{b_k}{\prod_{i=1}^{m} k_i} \) converges regularly if and only if

\[
\lim_{\delta \to 0} \int_{\delta}^{\pi} \frac{1}{\prod_{i=1}^{m} \sin i \pi} \sum_{k \in \mathbb{N}^m} b_k \prod_{i=1}^{m} k_i dx
\]

exists. \( \quad (14) \)

Proof. We apply Theorem 4.3 with \( c_k = b_k \) and \( \Phi_{i,k}(x) = \int_{x}^{\pi} k \sin kt \, dt \) \( (i = 1, \ldots, m) \) to conclude that \( (14) \) holds if and only if the multiple series

\[
\sum_{k \in \mathbb{N}^m} \frac{b_k}{\prod_{i=1}^{m} k_i} \prod_{i=1}^{m} \int_{0}^{\pi} k_i \sin k_i x_i \, dx_i
\]

converges regularly. \( \quad (15) \)

To this end, it remains to show that \( (15) \) holds if and only if \( \sum_{k \in \mathbb{N}^m} \frac{b_k}{\prod_{i=1}^{m} k_i} \) is regularly convergent. But this assertion is a consequence of Lemma 5.1, since

\[
\sum_{k \in \mathbb{N}^m} \left\{ \frac{b_k}{\prod_{i=1}^{m} k_i} \prod_{i=1}^{m} \int_{0}^{\pi} k_i \sin k_i x_i \, dx_i - \frac{b_k}{\prod_{i=1}^{m} k_i} \right\}
\]

can be written as a finite sum of regularly convergent multiple series

\[
\sum_{k \in \mathbb{N}^m} \left\{ \frac{b_k}{\prod_{i=1}^{m} k_i} \prod_{i=1}^{m} \int_{0}^{\pi} k_i \sin k_i x_i \, dx_i - \frac{b_k}{\prod_{i=1}^{m} k_i} \right\} = \sum_{\emptyset \neq I \subseteq \{1, \ldots, m\}} \sum_{k \in \mathbb{N}^m} b_k \prod_{i \in I} \frac{(-1)^{k_i-1}}{k_i} \prod_{i \in I^c} \frac{1}{k_i}.
\]

The proof is complete. \( \square \)
6. An integrability theorem for multiple cosine series

Let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The main result of this section is Theorem 6.3, which gives an affirmative answer to another conjecture of Móricz [3, Remark 3(iii)]. We need two lemmas.

**Lemma 6.1.** Let \( \sum_{k \in \mathbb{N}_0^m} a_k \) be an absolutely convergent multiple series of real numbers such that

\[
\sum_{k \in \mathbb{N}_0^m} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = 0 \quad \text{for all } x \in [0, \pi)^m \setminus (0, \pi)^m.
\]

If \( x \in [0, \pi]^m \), then

\[
\sum_{k \in \mathbb{N}_0^m} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = (-1)^m \sum_{k \in \mathbb{N}_0^m} a_{k+1} \prod_{i=1}^m (1 - \cos (k + 1) x_i).
\]

**Proof.** For the case when \( m = 1 \), it is easy to check that (16) holds. Since

\[
\sum_{k \in \mathbb{N}_0^m} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = \sum_{k_1=0}^{\infty} \left\{ \sum_{k_2=0}^{\infty} \cdots \left\{ \sum_{k_m=0}^{\infty} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i \right\} \cdots \right\},
\]

the lemma follows. \( \square \)

**Lemma 6.2.** Let \( \{a_k : k \in \mathbb{N}_0^m\} \) be given as in Lemma 6.1, let \( \Gamma \subset \{1, \ldots, m\} \), and let \( \bigcup_{\ell \in \Gamma} \{c_\ell\} \subset \mathbb{N}_0^m \). Then

\[
\sum_{k \in \mathbb{N}_0^m} \left( \prod_{i \in \Gamma'} \lambda_i \right) (a_k) = 0.
\]

**Proof.** We may assume that \( \Gamma \) is non-empty. In this case, we have

\[
\sum_{r \in \mathbb{N}_0^m} \left\{ \prod_{i \in \Gamma} \lambda_i \cos r_i t_i \right\} \left\{ \sum_{k \in \mathbb{N}_0^m} \left( \prod_{i \in \Gamma'} \lambda_i \cos k_i (0) \right) (a_k) \right\} = 0
\]

whenever \( 0 < t_i \leq \pi \) for each \( i \in \Gamma \). Since

\[
\sum_{r \in \mathbb{N}_0^m} \left| \sum_{k \in \mathbb{N}_0^m} \left( \prod_{i \in \Gamma'} \lambda_i \cos k_i (0) \right) (a_k) \right| \leq \sum_{k \in \mathbb{N}_0^m} |a_k| < \infty,
\]

it is clear that the lemma holds. \( \square \)

**Theorem 6.3.** Let \( \{a_k : k \in \mathbb{N}_0^m\} \) be a multiple sequence of real numbers such that the multiple series \( \sum_{k \in \mathbb{N}_0^m} |a_k| |\ln(\|k\| + 2)|^{m-1} \) converges and \( \sum_{k \in \mathbb{N}_0^m} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = 0 \) for all \( x \in [0, \pi)^m \setminus (0, \pi)^m \). Let

\[
f_1(x) := \begin{cases} \prod_{i=1}^m (x_i - 1)^{-1} \sum_{k \in \mathbb{N}_0} a_k \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i & \text{if } x \in (0, \pi)^m, \\ 0 & \text{otherwise}. \end{cases}
\]

Then the multiple series \( \sum_{k \in \mathbb{N}_0^m} \prod_{i=1}^m \frac{1}{k_i} \sum_{0 \leq j \leq k} a_j \prod_{i=1}^m \lambda_{ij} \) converges regularly if and only if

\[
\lim_{\delta \to 0} \int_{\delta \in [0, \pi]^m} f_1(x) \, dx \text{ exists.}
\]
Proof. Let $x \in (0, \pi]^m$ and set $A_k := \sum_{0 \leq j \leq k} a_j \prod_{i=1}^m \lambda_{ij}$. Then

$$f_1(x) = \frac{(-1)^m}{\prod_{i=1}^m x_i} \sum_{k \in \mathbb{N}_0^m} a_{k+1} \prod_{i=1}^m (1 - \cos(k_i + 1)x_i) \quad \text{(by Lemma 6.1)}$$

$$= \frac{1}{\prod_{i=1}^m x_i} \sum_{k \in \mathbb{N}_0^m} \Delta_{\{1, \ldots, m\}}(A_k) \prod_{i=1, j_i = 0}^m (\cos j_i x_i - \cos(j_i + 1)x_i)$$

$$= \frac{1}{\prod_{i=1}^m x_i} \sum_{k \in \mathbb{N}_0^m} A_k \prod_{i=1}^m (\cos k_i x_i - \cos(k_i + 1)x_i)$$

$$= \left\{ \prod_{i=1}^m \frac{2 \sin \frac{x_i}{2}}{x_i} \right\} \sum_{\Gamma \subseteq \{1, \ldots, m\}} \sum_{k \in \mathbb{N}_0^m} A_k \left\{ \prod_{i \in \Gamma} \sin k_i x_i \cos \frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos k_i x_i \sin \frac{x_i}{2} \right\}$$

$$+ \left\{ \prod_{i=1}^m \frac{\sin x_i}{x_i} \right\} \sum_{i=1}^m A_k \prod_{i=1}^m \sin k_i x_i$$

$$= g_1(x) + g_2(x) \quad \text{(say).}$$

We write $B_k := \Delta_{\{1, \ldots, m\}}(A_k)$. Then, by multiple summation by parts,

$$\sup_{\Gamma \subseteq \{1, \ldots, m\}} \int_{[0, \pi]^m} \left| \sum_{k \in \mathbb{N}_0^m} A_k \left\{ \prod_{i \in \Gamma} \sin k_i x_i \cos \frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos k_i x_i \sin \frac{x_i}{2} \right\} \right| dx$$

$$\leq \sup_{\Gamma \subseteq \{1, \ldots, m\}} \int_{[0, \pi]^m} \left| \sum_{k \in \mathbb{N}_0^m} |B_k| \left\{ \prod_{i \in \Gamma} \sin j_i x_i \cos \frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos j_i x_i \sin \frac{x_i}{2} \right\} \right| dx$$

$$= \mathcal{O} \left( \sum_{k \in \mathbb{N}_0^m} |a_k| \left( \ln(\|k\| + 2) \right)^{m-1} \right)$$

$$< \infty,$$

so $g_1 \in L^1([0, \pi]^m)$. To complete the proof, it remains to prove that the following statements are equivalent:

(a) $\lim_{\delta \to 0} \int_{x \in (0, \pi]^m} g_2(x) \, dx$ exists.

(b) $\sum_{k \in \mathbb{N}_0^m} A_k \prod_{i=1}^m \int_0^\pi \sin x_i \sin k_i x_i \frac{x_i}{x_i} \, dx_i$ converges regularly.

(c) $\sum_{k \in \mathbb{N}_0^m} A_k \prod_{i=1}^m \frac{1}{k_i}$ converges regularly.
(a) ⇔ (b). Direct computations yield

\[
\sup_{n \in \mathbb{N}} \sup_{x \in [0, \pi]} \left| \int_{0}^{x} \frac{n \sin t \sin nt}{t} \, dt \right| \leqslant 2 < \infty,
\]

\[
\sup_{x \in (0, \pi)} \sup_{n \in \mathbb{N}} \left| \frac{1}{nx} \int_{0}^{x} \frac{n \sin t \sin nt}{t} \, dt \right| < \infty,
\]

and

\[
\sup_{x \in [0, \pi]} \sup_{n \in \mathbb{N}} \left| x \sum_{k=1}^{n} \int_{0}^{\pi} \frac{k \sin t \sin kt}{t} \, dt \right| \leqslant 2 + 2\pi < \infty,
\]

so Theorem 4.3 implies that (a) is equivalent to (b).

Finally (b) is equivalent to (c) since

\[
\sum_{k \in \mathbb{N}^{m}} |A_k| \left| \prod_{i=1}^{m} \frac{1}{k_i} - \prod_{i=1}^{m} \int_{0}^{\pi} \frac{x_i \sin k_i x_i}{x_i} \, dx_i \right|
\]

\[
\leqslant \sum_{k \in \mathbb{N}^{m}} |A_k| \left\{ \sum_{j=1}^{m} \left( \prod_{i=1}^{j-1} \frac{1}{k_i} \right) \left( \prod_{i=j+1}^{m} \frac{1}{k_i} \right) \int_{0}^{\pi} \frac{\cos k_j x_j}{k_j} \, dx_j \right\} \frac{1}{k_j} \left( \prod_{i=1}^{j-1} \frac{1}{k_i} \right) \left( \prod_{i=j+1}^{m} \frac{1}{k_i} \right)
\]

(by Lemma 4.2)

\[
= \sum_{j=1}^{m} \sum_{k \in \mathbb{N}^{m}} O \left\{ |A_k| \left( \prod_{i=1}^{j-1} \frac{1}{k_i} \right) \left( \prod_{i=j+1}^{m} \frac{1}{k_i} \right) \int_{0}^{\pi} \frac{\cos k_j x_j}{k_j} \, dx_j \right\}
\]

(by integration by parts)

\[
= \sum_{j=1}^{m} \sum_{k \in \mathbb{N}^{m}} O \left\{ |A_k| \left( \prod_{i=1}^{j-1} \frac{1}{k_i} \right) \left( \prod_{i=j+1}^{m} \frac{1}{k_i} \right) \frac{1}{k_j^2} \right\}
\]

(by integration by parts again)

\[
= \sum_{j=1}^{m} O \left( \sum_{k \in \mathbb{N}^{m}} \sum_{r \geq k} \Delta_{1, \ldots, m}(A_r) \left\{ \left( \prod_{i=1}^{j-1} \frac{1}{k_i} \right) \left( \prod_{i=j+1}^{m} \frac{1}{k_i} \right) \frac{1}{k_j^2} \right\} \right)
\]

\[
= O \left( \sum_{k \in \mathbb{N}^{m}} |\Delta_{1, \ldots, m}(A_k)| (\ln(\|k\| + 1))^{m-1} \right)
\]

< \infty.

The proof is complete. \( \Box \)

**References**

