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On some fixed point results in b-metric, rectangular and b-rectangular metric spaces

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Abstract. In this paper we consider, discuss, improve and generalize recent fixed point results for mappings in b-metric, rectangular metric and b-rectangular metric spaces established by Đukić et al. (2011), George and Rajagopalan (2013) and Roshan et al. (2015). Also, we prove a Geraghty type theorem in the setting of b-metric spaces as well as a Boyd–Wong type theorem in the framework of b-rectangular metric spaces, in both cases, without using Hausdorff assumption. One example is given to support the results.

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1. INTRODUCTION AND PRELIMINARIES

It is well known that the Banach contraction principle [5] is a fundamental result in the fixed point theory, which has been used and extended in many different directions. Also, there are several generalizations of usual metric spaces. Three well known generalizations of (usual) metric spaces are b-metric spaces [4,7] or metric type spaces—MTS by some authors

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[22,15,19,28], generalized metric spaces (g.m.s.) [6] or rectangular metric spaces [9,12,16, 17,21] and rectangular b-metric space [11] or a b-generalized metric space (b-g.m.s.) [26].

The following definitions are consistent with [4,7,6] and [11,26], respectively:

Definition 1.1 *([4,7])***.** Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq s\,[d(x, y) + d(y, z)]$ (b-triangular inequality).

In this case, the pair (X, d) is called a b-metric space (metric type space).

Further, for all definitions of notions as b-convergence, b-completeness, b-Cauchy in the setting of b-metric spaces see [1,4,7,8,3,22,14,15,19,20,23,24,27].

Definition 1.2 *([6])***.** Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(r1) $d(x, y) = 0$ if and only if $x = y$,
(r2) $d(x, y) = d(y, x)$,
(r3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then (X, d) is called a rectangular or generalized metric spaces (g.m.s.).

For all definitions of notions regarding this new class of generalized metric spaces see [6,9,10,12,16,17,21,26] and references in [17].

Definition 1.3 *([11,26,18])***.** Let X be a nonempty set, $s \geq 1$ be a given real number and let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x and y:

(rb1) $d(x, y) = 0$ if and only if $x = y$,
(rb2) $d(x, y) = d(y, x)$,
(rb3) $d(x, y) \leq s\,[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality).

Then (X, d) is called a b-rectangular metric space or a b-generalized metric space (b-g.m.s.).

Note that every metric space is a rectangular metric space (g.m.s.) and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse is not necessarily true ([11], Examples 2.4. and 2.5.). Also, every metric space is a b-metric space (metric type space) and every b-metric space is a b-rectangular metric space.

Note also that every b-metric space with coefficient s is a b-rectangular metric space with coefficient s^2 but the converse is not necessarily true ([11], Examples 2.7).

Hence we have the following diagram

$$\begin{tikzcd}
\boxed{\text{metric spaces}} \arrow[r] \arrow[d] & \boxed{b\text{-metric spaces}} \arrow[d] \\
\boxed{\text{rectangular metric spaces}} \arrow[r] & \boxed{b\text{-rectangular metric spaces}}
\end{tikzcd}$$

where arrows stand for inclusions. The inverse inclusions do not hold.

Note that the limit of a sequence in a b-rectangular metric space (the same as in a rectangular metric space (g.m.s.)) is not necessarily unique and also every rectangular metric convergent sequence in a b-rectangular metric space is not necessarily b-rectangular metric space-Cauchy ([11], Examples 2.7).

The following four crucial lemmas are useful in proving all main results in [27] and [26] (see also [25]):

Lemma 1.1 *([27], Lemma 1.6).* *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y), \tag{1.1}$$

in particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x,z). \tag{1.2}$$

Lemma 1.2 *([27], Lemma 1.7).* *Let (X,d) be a b-metric space with $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{1.3}$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences

$$\begin{aligned} &d\left(x_{m(k)},x_{n(k)}\right), d\left(x_{m(k)},x_{n(k)+1}\right), d\left(x_{m(k)+1},x_{n(k)}\right) \quad \text{and} \\ &d\left(x_{m(k)+1},x_{n(k)+1}\right), \end{aligned} \tag{1.4}$$

it holds:

$$\varepsilon \leq \liminf_{n\to\infty} d\left(x_{m(k)},x_{n(k)}\right) \leq \limsup_{n\to\infty} d\left(x_{m(k)},x_{n(k)}\right) \leq s\varepsilon, \tag{1.5}$$

$$\frac{\varepsilon}{s} \leq \liminf_{n\to\infty} d\left(x_{m(k)},x_{n(k)+1}\right) \leq \limsup_{n\to\infty} d\left(x_{m(k)},x_{n(k)+1}\right) \leq s^2\varepsilon, \tag{1.6}$$

$$\frac{\varepsilon}{s} \leq \liminf_{n\to\infty} d\left(x_{m(k)+1},x_{n(k)}\right) \leq \limsup_{n\to\infty} d\left(x_{m(k)+1},x_{n(k)}\right) \leq s^2\varepsilon, \tag{1.7}$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{n\to\infty} d\left(x_{m(k)+1},x_{n(k)+1}\right) \leq \limsup_{n\to\infty} d\left(x_{m(k)+1},x_{n(k)+1}\right) \leq s^3\varepsilon. \tag{1.8}$$

Lemma 1.3 *([26], Lemma 1).* *Let (X,d) be a b-g.m.s. with $s \geq 1$ and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 1.4 *([26], Lemma 2).* *Let (X,d) be a b-g.m.s. with $s \geq 1$.*

(a) *Suppose that sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *are such that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$, *with* $x \neq y$, *and* $x_n \neq x, y_n \neq y$ *for* $n \in \mathbb{N}$. *Then we have*

$$\frac{1}{s}d\left(x, y\right) \leq \liminf_{n\to\infty} d\left(x_n, y_n\right) \leq \limsup_{n\to\infty} d\left(x_n, y_n\right) \leq sd\left(x, y\right). \tag{1.9}$$

(b) *If* $y \in X$ *and* $\{x_n\}$ *is a Cauchy sequence in* X *with* $x_n \neq x_m$ *for infinitely many* $m, n \in \mathbb{N}, n \neq m$, *converging to* $x \neq y$, *then*

$$\frac{1}{s}d\left(x, y\right) \leq \liminf_{n\to\infty} d\left(x_n, y\right) \leq \limsup_{n\to\infty} d\left(x_n, y\right) \leq sd\left(x, y\right), \tag{1.10}$$

for all $x \in X$.

2. Main results

In our first result we generalize, complement and improve recent Geraghty type result [13] from ([29], Theorem 3.8) for b-metric spaces. For the use in b-metric spaces (with the given $s > 1$) we will consider the class of functions $\mathcal{B}_s$, where $\beta \in \mathcal{B}_s$ if $\beta : [0, +\infty) \to [0, \frac{1}{s})$ and has the property

$$\beta\left(t_n\right) \to \frac{1}{s} \text{ implies } t_n \to 0. \tag{2.1}$$

An example of a function in $\mathcal{B}_s$ is given by $\beta\left(t\right) = \frac{1}{s}e^{-t}$ for $t > 0$ and $\beta\left(0\right) \in [0, \frac{1}{s})$.

In the following result we generalize and improve Theorem 3.8. from [29]. Note that we do not assume that the b-metric d is continuous.

Theorem 2.1. *Let* (X, d) *be a b-metric space with* $s > 1$, *and let* $f, g : X \to X$ *be two self maps such that* $f\left(X\right) \subseteq g\left(X\right)$, *one of these two subsets of* X *being complete. If for some function* $\beta \in \mathcal{B}_s$,

$$d\left(fx, fy\right) \leq \beta\left(d\left(gx, gy\right)\right) d\left(gx, gy\right) \tag{2.2}$$

holds for all $x, y \in X$, *then* f *and* g *have a unique point of coincidence* ω. *Moreover, for each* $x_0 \in X$, *a corresponding Jungck sequence* $\{y_n\}$ *can be chosen such that* $\lim_{n\to\infty} y_n = \omega$.

If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Proof. Let us prove that the point of coincidence of f and g is unique (if it exists). Suppose that ω_1 and ω_2 are distinct points of coincidence of f and g. From this follows that there exist two points u_1 and u_2 such that $fu_1 = gu_1 = \omega_1$ and $fu_2 = gu_2 = \omega_2$. Then (2.2) implies that

$$\begin{aligned} d\left(\omega_1, \omega_2\right) &= d\left(fu_1, fu_2\right) \leq \beta\left(d\left(gu_1, gu_2\right)\right) d\left(gu_1, gu_2\right) \\ &= \beta\left(d\left(\omega_1, \omega_2\right)\right) d\left(\omega_1, \omega_2\right) < \frac{1}{s}d\left(\omega_1, \omega_2\right) < d\left(\omega_1, \omega_2\right), \end{aligned} \tag{2.3}$$

which is a contradiction.

In order to prove that f and g have a point of coincidence, take an arbitrary point $x_0 \in X$ and, using that $f(X) \subseteq g(X)$, choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.4}$$

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $gx_{n_0+1} = y_{n_0} = y_{n_0+1} = fx_{n_0+1}$ and f and g have a point of coincidence. Suppose, further, that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_{n+1}$, $y = x_n$ in (2.2) we obtain that

$$\begin{aligned} d(y_{n+1}, y_n) &= d(fx_{n+1}, fx_n) \\ &\leq \beta(d(gx_{n+1}, gx_n))\, d(gx_{n+1}, gx_n) < \frac{1}{s} d(y_n, y_{n-1}) \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$. From (2.5) further follows that $d(y_{n+1}, y_n) \to 0$, as $n \to \infty$.

Let us prove that $\{y_n\}$ is a b-Cauchy sequence in b-metric space (X, d). According to Lemma 1.2 if $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$n(k) > m(k) > k, \quad d\left(y_{m(k)}, y_{n(k)}\right) \geq \varepsilon \quad \text{and} \quad d\left(y_{m(k)}, y_{n(k)-1}\right) < \varepsilon \tag{2.6}$$

for all positive k. Now, putting $x = x_{m(k)+1}, y = x_{n(k)}$ in (2.2) we obtain

$$\begin{aligned} d\left(y_{m(k)+1}, y_{n(k)}\right) &\leq \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) d\left(y_{m(k)}, y_{n(k)-1}\right) \\ &\leq \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) \varepsilon, \end{aligned} \tag{2.7}$$

that is,

$$\frac{d\left(y_{m(k)+1}, y_{n(k)}\right)}{\varepsilon} \leq \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) < \frac{1}{s}, \tag{2.8}$$

whenever $n(k) > m(k) > k$ for all positive k.

Hence, by (1.7) of Lemma 1.2 and (2.8) we have

$$\begin{aligned} \frac{1}{s} &= \frac{1}{\varepsilon} \cdot \frac{\varepsilon}{s} \leq \liminf_{n\to\infty} \frac{d\left(y_{m(k)+1}, y_{n(k)}\right)}{\varepsilon} \leq \liminf_{n\to\infty} \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) \\ &\leq \limsup_{n\to\infty} \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) \leq \frac{1}{s}, \end{aligned} \tag{2.9}$$

that is,

$$\lim_{k\to\infty} \beta\left(d\left(y_{m(k)}, y_{n(k)-1}\right)\right) = \frac{1}{s}. \tag{2.10}$$

From this further implies that $d\left(y_{m(k)}, y_{n(k)-1}\right) \to 0$, as $k \to \infty$. However, this is impossible, because by (1.5) of Lemma 1.2 we obtain that

$$\frac{\varepsilon}{s} \leq \frac{1}{s} d\left(y_{m(k)}, y_{n(k)}\right) \leq d\left(y_{m(k)}, y_{n(k)-1}\right) + d\left(y_{n(k)-1}, y_{n(k)}\right) \to 0, \tag{2.11}$$

as $k \to \infty$. Therefore, $\{y_n\}$ is a b-Cauchy sequence.

Suppose, e.g., that the subspace $g(X)$ is complete (the proof when $f(X)$ is complete is similar). Then $\{y_n\}$ tends to some $\omega \in g(X)$, where $\omega = gz$ for some $z \in X$. To prove that

$fz = gz$, we have

$$\begin{aligned}\frac{1}{s}d(fz,gz) &\leq d(fz,fx_n) + d(gx_{n+1},gz)\\ &\leq \beta(d(gz,gx_n))\,d(gz,gx_n) + d(gx_{n+1},gz)\\ &< \frac{1}{s}d(gz,gx_n) + d(gx_{n+1},gz)\\ &\to \frac{1}{s}\cdot 0 + 0 = 0.\end{aligned}\tag{2.12}$$

Hence, $fz = gz = \omega$ is a unique point of coincidence of f, g.

If f, g are weakly compatible, well-known Jungck's result implies that f and g have a unique fixed point (here it is ω). □

Taking $g = I_X$ (identity mapping of X) in Theorem 2.1 we obtain the following variant of Geraghty-theorem in b-metric spaces (with correct proof).

Corollary 2.2 ([29], Theorem 3.8)**.** *Let $s > 1$, and let (X,d) be a complete b-metric space. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$d(fx,fy) \leq \beta(d(x,y))\,d(x,y),\tag{2.13}$$

for all $x, y \in X$ and some $\beta \in \mathcal{B}_s$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X,d).

The following theorem can be proved in a very similar way as ([15], Theorem 3.11.).

Theorem 2.3. *Let (X,d) be a b-metric space with $s > 1$, and let $f, g : X \to X$ be two mappings such that $f(X) \subseteq g(X)$ and one of these subsets of X is complete. Suppose that there exists $\lambda \in [0, \frac{1}{s})$ such that for all $x, y \in X$*

$$d(fx,fy) \leq \lambda M(f,g;x,y)\tag{2.14}$$

where

$$\begin{aligned}&M(f,g;x,y)\\ &\quad = \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2s}\right\}.\end{aligned}\tag{2.15}$$

Then f and g have a unique point of coincidence. If, moreover, the pair (f,g) is weakly compatible, then f and g have a unique common fixed point.

In the sequel we announce two lemmas which are useful for the proofs of some things in the setting of rectangular and b-rectangular metric spaces.

Lemma 2.4. *Let (X,d) be a b-rectangular metric space with $s \geq 1$, and let $f : X \to X$ be a self maps. If Picard's sequence $\{f^n x\}, x \in X$ and $f^n x \neq f^{n+1}x$ for all $n \in \mathbb{N}$ satisfies*

$$d\left(f^{n+1}x, f^n x\right) \leq \lambda d\left(f^n x, f^{n-1}x\right),\tag{2.16}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0,1)$, then $f^n x \neq f^m x$ whenever $n \neq m$.

Proof. Suppose that $f^n x = f^m x$ for some $n > m$. Now, we have $f^{n+1}x = f(f^n x) = f(f^m x) = f^{m+1}x$. Then (2.16) implies that

$$d\left(f^{n+1}x, f^n x\right) < d\left(f^n x, f^{n-1}x\right) < \cdots < d\left(f^{m+1}x, f^m x\right) = d\left(f^{n+1}x, f^n x\right). \tag{2.17}$$

A contradiction. Hence, $n \neq m$ implies $f^n x \neq f^m x$. □

Lemma 2.5. *Let (X, d) be a b-rectangular metric space with $s \geq 1$, and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$. If Jungck sequence $y_n = fx_n = gx_{n+1}, x_0 \in X$ and $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ satisfies*

$$d(y_{n+1}, y_n) \leq \lambda d(y_n, y_{n-1}), \tag{2.18}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0,1)$, then $y_n \neq y_m$ whenever $n \neq m$.

Proof. Suppose that $y_n = y_m$ for some $n > m$ then we choose $x_{n+1} = x_{m+1}$ (which is obviously possible by the definition of Jungck's sequence y_n) and hence also $y_{n+1} = y_{m+1}$. Then (2.18) implies

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}) < \cdots < d(y_{m+1}, y_m) = d(y_{n+1}, y_n). \tag{2.19}$$

A contradiction. We obtain that $n \neq m$ implies $y_n \neq y_m$. □

Let Ψ denote set of all continuous functions $\psi : [0,\infty) \to [0,\infty)$ for which $\psi(t) = 0$ if and only if $t = 0$. In the following result we generalize, complement and improve main results from ([9], Theorem 4) and ([12], Theorem 3.1) with much shorter proofs and without using Hausdorff assumption.

Theorem 2.6. *Let (X, d) be a rectangular metric space and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If, for some $\psi, \phi \in \Psi, L \geq 0$, the function ψ is non-decreasing,*

$$\psi(d(fx, fy)) \leq \psi(M(x,y)) - \phi(M(x,y)) + L\psi(N(x,y)) \tag{2.20}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy)\} \tag{2.21}$$

$$N(x,y) = \min\{d(gx, fx) + d(gy, fy), d(gx, fy), d(gy, fx)\}, \tag{2.22}$$

then f and g have a unique point of coincidence. If, moreover, f, g are weakly compatible, then they have a unique common fixed point.

Proof. First of all, it is easy to check that the conditions (2.20), (2.21) and (2.22) imply that the point of coincidence of f and g is unique (if it exists). In order to prove that f and g have a point of coincidence, similarly as in Theorem 2.1, take an arbitrary point $x_0 \in X$ and, using that $f(X) \subseteq g(X)$, choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \qquad \text{for } n = 0, 1, 2, \ldots. \tag{2.23}$$

If $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence. Suppose, further, that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_{n+1}, y = x_n$ in (2.20) we obtain that

$$\begin{aligned}&\psi\left(d\left(y_{n+1}, y_n\right)\right) = \psi\left(d\left(fx_{n+1}, fx_n\right)\right)\\&\leq \psi\left(M\left(x_{n+1}, x_n\right)\right) - \phi\left(M\left(x_{n+1}, x_n\right)\right) + L\psi\left(N\left(x_{n+1}, x_n\right)\right)\end{aligned} \tag{2.24}$$

where

$$M\left(x_{n+1}, x_n\right) = \max\left\{d\left(y_n, y_{n-1}\right), d\left(y_n, y_{n+1}\right)\right\} \tag{2.25}$$

and

$$N\left(x, y\right) = \min\left\{d\left(y_n, y_{n+1}\right) + d\left(y_{n-1}, y_n\right), d\left(y_n, y_n\right), d\left(y_{n-1}, y_{n+1}\right)\right\} = 0. \tag{2.26}$$

Further from (2.24), (2.25) and (2.26) follows that

$$\begin{aligned}\psi\left(d\left(y_{n+1}, y_n\right)\right) \leq\ &\psi\left(\max\left\{d\left(y_n, y_{n-1}\right), d\left(y_{n+1}, y_n\right)\right\}\right)\\&- \phi\left(\max\left\{d\left(y_n, y_{n-1}\right), d\left(y_{n+1}, y_n\right)\right\}\right).\end{aligned} \tag{2.27}$$

If $d\left(y_n, y_{n-1}\right) < d\left(y_{n+1}, y_n\right)$ for some, then from (2.27) follows

$$\psi\left(d\left(y_{n+1}, y_n\right)\right) \leq \psi\left(d\left(y_{n+1}, y_n\right)\right) - \phi\left(d\left(y_{n+1}, y_n\right)\right) < \psi\left(d\left(y_{n+1}, y_n\right)\right), \tag{2.28}$$

which is a contradiction. Hence, we have that

$$\psi\left(d\left(y_{n+1}, y_n\right)\right) \leq \psi\left(d\left(y_n, y_{n-1}\right)\right) - \phi\left(d\left(y_n, y_{n-1}\right)\right) < \psi\left(d\left(y_n, y_{n-1}\right)\right), \tag{2.29}$$

or $d\left(y_{n+1}, y_n\right) < d\left(y_n, y_{n-1}\right)$ for all $n \in \mathbb{N}$ (because ψ is non-decreasing). Hence, there exists $\lim_{n\to\infty} d\left(y_{n+1}, y_n\right) = d^* \geq 0$, as $n \to \infty$. From (2.28) follows that $\psi\left(d^*\right) \leq \psi\left(d^*\right) - \phi\left(d^*\right) \leq \psi\left(d^*\right)$, that is, $d^* = 0$.

Now, we easy get that $y_n \neq y_m$ whenever $n \neq m$. Indeed, if $y_n = y_m$ for some $n > m$, then we choose $x_{n+1} = x_{m+1}$ (and hence also $y_{n+1} = y_{m+1}$). Then we have

$$d\left(y_{n+1}, y_n\right) < d\left(y_n, y_{n-1}\right) < \cdots < d\left(y_{m+1}, y_m\right) = d\left(y_{n+1}, y_n\right). \tag{2.30}$$

A contradiction.

The rest of the proof that $\{y_n\}$ is a Cauchy sequence is further as in ([12], page 46).

Suppose, e.g., that the subspace $g\left(X\right)$ is complete (the proof when $f\left(X\right)$ is complete is similar). Then y_n tends to some gz for some $z \in X$. In order to prove that $fz = gz$, suppose that $fz \neq gz$. By (2.24), we have

$$\psi\left(d\left(fx_n, fz\right)\right) \leq \psi\left(M\left(x_n, z\right)\right) - \phi\left(M\left(x_n, z\right)\right) + L\psi\left(N\left(x_n, z\right)\right), \tag{2.31}$$

where

$$\begin{aligned}&M\left(x_n, z\right) = \max\left\{d\left(gx_n, gz\right), d\left(gx_n, fx_n\right), d\left(gz, fz\right)\right\} \to d\left(gz, fz\right),\\&\quad \text{as } n \to \infty\end{aligned} \tag{2.32}$$

and

$$N(x_n, z) = \min\{d(gx_n, fx_n) + d(gu, fu), d(gx_n, fz), d(gz, fx_n)\} \to 0, \text{ as } n \to \infty. \tag{2.33}$$

Taking upper limit as $n \to \infty$ in (2.31), we obtain

$$\psi\left(\limsup_{n\to\infty} d(fx_n, fz)\right) \leq \psi(d(gz, fz)) - \phi(d(gz, fz)) < \psi(d(gz, fz)), \tag{2.34}$$

and using the non-decreasing of function ψ, we get

$$\limsup_{n\to\infty} d(fx_n, fz) < d(gz, fz). \tag{2.35}$$

On the other hand, by Lemma 1.3, it follows that y_n differs from both fz and gz for n sufficiently large. Hence, we can apply the rectangular inequality to obtain

$$d(fz, gz) \leq d(fz, fx_n) + d(fx_n, fx_{n+1}) + d(fx_{n+1}, gz), \tag{2.36}$$

from which it follows

$$d(fz, gz) \leq \limsup_{n\to\infty} d(fx_n, fz), \tag{2.37}$$

because $d(fx_n, fx_{n+1}) \to 0$ and $d(fx_{n+1}, gz) \to 0$, as $n \to \infty$. Now, (2.34) by (2.35) and (2.37) becomes

$$\psi(d(gz, fz)) \leq \psi(d(gz, fz)) - \phi(d(gz, fz)), \tag{2.38}$$

or $\phi(d(gz, fz)) = 0$, that is, $fz = gz$, a contradiction with the assumption that $fz \neq gz$.

In the case when f and g are weakly compatible, well-known Jungck's result implies that f and g have a unique common fixed point. □

Let Φ denote the set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:

(a) $\phi(0) = 0$;
(b) $\phi(t) < t$ for all $t > 0$;
(c) ϕ is upper semi-continuous from the right (that is, for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \to t$ as $n \to \infty$, we have $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$).

It is worth to notice that for every $\phi \in \Phi$, we have that $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

In the following new result we consider Boyd–Wong type theorem for b-rectangular metric spaces. Namely, we announce the following result:

Theorem 2.7. *Let (X, d) be a b-rectangular metric space with $s > 1$ and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If, for some function $\phi \in \Phi$,*

$$sd(fx, fy) \leq \phi(d(gx, gy)) \tag{2.39}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence say $\omega \in X$. Moreover, for each $x_0 \in X$, the corresponding Jungck sequence $\{y_n\}$ can be chosen such *that* $\lim_{n\to\infty} y_n = \omega$. *In addition, if* f *and* g *are weakly compatible, then they have a unique common fixed point.*

Proof. Let us prove first that the point of coincidence of f and g is unique (if it exists). Suppose that ω_1 and ω_2 are two distinct points of coincidence of f and g. Then there exist two points u_1 and u_2 such that $fu_1 = gu_1 = \omega_1$ and $fu_2 = gu_2 = \omega_2$. Then (2.39) implies that

$$\begin{aligned} d(\omega_1, \omega_2) &= d(fu_1, fu_2) \le sd(fu_1, fu_2) \le \phi(d(gu_1, gu_2)) \\ &= \phi(d(\omega_1, \omega_2)) < d(\omega_1, \omega_2), \end{aligned} \tag{2.40}$$

which is a contradiction.

In order to prove that f and g have a point of coincidence, take an arbitrary point $x_0 \in X$ and, using that $f(X) \subseteq g(X)$, choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.41}$$

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $gx_{n_0+1} = y_{n_0} = y_{n_0+1} = fx_{n_0+1}$ and f and g have a (unique) point of coincidence. Suppose, further, that $y_n \ne y_{n+1}$ for all $n \in \mathbb{N}$. For the rest, assume that $y_n \ne y_{n+1}$ for all $n \in \mathbb{N}$. In this case we can show that $y_n \ne y_m$ if $n \ne m$. Indeed, let $y_n = y_m$ for some $n > m$. Then (2.39) implies that

$$\begin{aligned} d(y_m, y_{m+1}) &= d(y_n, y_{n+1}) \\ &= d(fx_n, fx_{n+1}) \le sd(fx_n, fx_{n+1}) \le \phi(d(gx_n, gx_{n+1})) \\ &= \phi(d(y_{n-1}, y_n)) < d(y_{n-1}, y_n) < \cdots < d(y_m, y_{m+1}), \end{aligned} \tag{2.42}$$

a contradiction. Thus, in the sequel, we will assume that $y_n \ne y_m$ for $n \ne m$.

Further, we shall prove that the sequence $\{d(y_n, y_{n+1})\}$ tends to 0 as $n \to \infty$. In order to prove this we first have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \le sd(fx_n, fx_{n+1}) \le \phi(d(gx_n, gx_{n+1})) \\ &= \phi(d(y_{n-1}, y_n)) < d(y_{n-1}, y_n), \end{aligned} \tag{2.43}$$

that is, the sequence $\{d(y_n, y_{n+1})\}$ is strictly decreasing. Therefore, there exists $d^* \ge 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = d^*. \tag{2.44}$$

Letting $n \to \infty$ in (2.43) and using the upper semi-continuity from the right of ϕ, we get that $d^* = 0$. Hence, $d(y_n, y_{n+1})$ tends to 0 as $n \to \infty$.

We next prove that $\{y_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{y_n\}$ is not a Cauchy sequence (for metric space see [2], Lemma 1). Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k \quad \text{and} \quad d(y_{m(k)}, y_{n(k)}) \ge \varepsilon. \tag{2.45}$$

This means that

$$d(y_{m(k)}, y_{n(k)-2}) < \varepsilon. \tag{2.46}$$

Using (2.46) and taking the upper limit as $k \to \infty$, we get

$$\limsup_{n\to\infty} d\left(y_{m(k)}, y_{n(k)-2}\right) \leq \varepsilon. \tag{2.47}$$

On the other hand, we have

$$\begin{aligned} d\left(y_{m(k)}, y_{n(k)}\right) \leq\ & sd\left(y_{m(k)}, y_{m(k)+1}\right) + sd\left(y_{m(k)+1}, y_{n(k)-1}\right) \\ & + sd\left(y_{n(k)-1}, y_{n(k)}\right). \end{aligned} \tag{2.48}$$

Using (2.44), (2.45) and taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{n\to\infty} d\left(y_{m(k)+1}, y_{n(k)-1}\right). \tag{2.49}$$

Using the b-rectangular inequality once again we have the following inequalities:

$$\begin{aligned} d\left(y_{m(k)}, y_{n(k)}\right) \leq\ & sd\left(y_{m(k)}, y_{n(k)-2}\right) \\ & + sd\left(y_{n(k)-2}, y_{n(k)-1}\right) + sd\left(y_{n(k)-1}, y_{n(k)}\right). \end{aligned} \tag{2.50}$$

Using (2.44), (2.45) and taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{n\to\infty} d\left(y_{m(k)}, y_{n(k)-2}\right). \tag{2.51}$$

Now, putting $x = x_{m(k)+1}, y = x_{n(k)-1}$ in (2.39), we obtain

$$\begin{aligned} sd\left(y_{m(k)+1}, y_{n(k)-1}\right) &= sd\left(fx_{m(k)+1}, fx_{n(k)-1}\right) \\ &\leq \phi\left(d\left(gx_{m(k)+1}, gx_{n(k)-1}\right)\right) \\ &= \phi\left(d\left(y_{m(k)}, y_{n(k)-2}\right)\right). \end{aligned} \tag{2.52}$$

Further, taking the upper limit as $k \to \infty$ in (2.52) and using (2.49) and (2.47) we obtain

$$\varepsilon = s \cdot \frac{\varepsilon}{s} \leq \phi(\varepsilon) < \varepsilon, \tag{2.53}$$

a contradiction to $\phi(t) < t$ for $t > 0$. Thus, $\{y_n\} = \{fx_n\} = \{gx_{n+1}\}$ is a b-rectangular (b-g.m.s.)-Cauchy sequence in (X, d). It follows from the completeness of $g(X)$ (the proof when $f(X)$ is complete is similar) that there exists $z \in X$ such that $\omega = gz \in g(X)$ and $y_n \to \omega$. We shall show that $fz = gz$. In view of Lemma 1.3 we can assume that y_n differs from both fz and gz for sufficient large n. Hence,

$$\begin{aligned} \frac{1}{s} d(fz, gz) &\leq d(fz, fx_n) + d(fx_n, fx_{n+1}) + d(fx_{n+1}, gz) \\ &\leq \phi(d(gz, gx_n)) + d(y_n, y_{n+1}) + d(y_{n+1}, gz) \\ &\leq d(gz, y_{n-1}) + d(y_n, y_{n+1}) + d(y_{n+1}, gz) \to 0, \end{aligned} \tag{2.54}$$

as $n \to \infty$. It follows that $fz = gz = \omega$ is a point of coincidence of f and g. Thus, ω is the unique point of coincidence of f and g.

If f and g are weakly compatible, then by known Jungck's result ω is the unique common fixed point of f and g. □

Putting $g = I_X$ in (2.39) we obtain Boyd–Wong type theorem for b-rectangular metric spaces.

Corollary 2.8. *Let (X, d) be a complete b-rectangular metric space with $s > 1$ and $f : X \to X$ satisfies*

$$sd\left(fx.fy\right) \leq \phi\left(d\left(x,y\right)\right), \tag{2.55}$$

for all $x, y \in X$, where $\phi \in \Phi$. Then f has a unique fixed point, say $z \in X$, and $f^n x \to z$ as $n \to \infty$ for all $x \in X$.

The following example supports Theorem 2.3.

Example 2.9. Let $X = \{0, 1, 3\}$ be equipped with the b-metric d given by $d(x, y) = (x - y)^2$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f0 = f1 = 1, f3 = 0$. Let $\lambda \in [0, \frac{1}{9}]$. Then we obtain

$$d\left(f0, f3\right) = d\left(1, 0\right) = 1 \leq \lambda M\left(0, 3\right) = \lambda \max\left\{9, 0, 9, \frac{5}{4}\right\} = 9\lambda$$

and

$$d\left(f1, f3\right) = d\left(1, 0\right) = 1 \leq \lambda M\left(1, 3\right) = \lambda \max\left\{4, 0, 9, \frac{5}{4}\right\} = 9\lambda.$$

Hence, f satisfies all assumptions of Theorem 2.3 and thus it has a unique fixed point (here it is 1).

The following example supports Corollary 2.8.

Example 2.10. Let $X = A \cup B$, where $A = \left\{\frac{1}{n} : n \in \{2, 3, 4, 5\}\right\}$ and $B = [1, 2]$. Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = 0.03; \quad d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = 0.02;$$
$$d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = 0.6; \quad d\left(x, y\right) = |x - y|^2 \quad \text{otherwise.}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 4 > 1$. But (X, d) is neither a metric space nor a rectangular metric space. Let $f : X \to X$ be defined as:

$$f\left(x\right) = \begin{cases} \frac{1}{4} & \text{if } x \in A \\ \frac{1}{5} & \text{if } x \in B. \end{cases}$$

Then f satisfies all conditions of Corollary 2.8 with $\phi(t) = \frac{12}{25}t$ and has a unique fixed point $x = \frac{1}{4}$.

3. Competing interests

The authors declare that they have no competing interests.

4. Author's contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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