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Existence of Nonoscillatory Solutions of Higher-Order Neutral Delay Difference Equations with Variable Coefficients

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Abstract—In this paper, we consider the following higher-order neutral delay difference equations with positive and negative coefficients:

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \qquad n \geq n_0,$$

where $c \in \mathbb{R}$, $m \geq 1$, $k \geq 1$, $r, l \geq 0$ are integers, and $\{p_n\}_{n=n_0}^{\infty}$ and $\{q_n\}_{n=n_0}^{\infty}$ are sequences of nonnegative real numbers. We obtain the global results (with respect to c) which are some sufficient conditions for the existences of nonoscillatory solutions. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Neutral difference equations, Nonoscillatory solutions, Existence.

1. INTRODUCTION

In this paper, we shall consider the following higher-order neutral delay difference equations with positive and negative coefficients:

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \qquad n \geq n_0 \in \{0, 1, 2, \dots\}, \tag{1}$$

where $c \in \mathbb{R}$, $m \geq 1$, $k \geq 1$, $r, l \geq 0$ are integers, and $\{p_n\}_{n=n_0}^{\infty}$ and $\{q_n\}_{n=n_0}^{\infty}$ are sequences of nonnegative real numbers. The forward difference Δ is defined as usual; i.e., $\Delta x_n = x_{n+1} - x_n$.

Let $\sigma = \max\{k, r, l\}$ and $N_0 \geq n_0$ be a fixed nonnegative integer. By a solution of (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq N_0 - \sigma$ and satisfies (1) for $n \geq N_0$.

Recently, there have been a lot of activities concerning the oscillations and nonoscillations of delay difference equations; see, for example, [1–11]. Agarwal and Wong [2], Agarwal *et al.* [3], Agarwal and Grace [4], and Zhang and Yang [9] investigate the oscillatory behavior of solutions of nonlinear neutral difference equations of order $m (\geq 1)$ of the following form:

$$\Delta^m(x_n + cx_{n-k}) + p_n f(x_{n-r}) = 0, \qquad n \geq n_0. \tag{2}$$

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The oscillation and nonoscillation of solution of the first-order neutral delay difference equation with positive and negative coefficients

$$\Delta(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \qquad n \geq n_0, \tag{3}$$

have been investigated by Chen and Zhang [5], Lalli and Zhang [7], Zhang and Wang [8], and Zhou [11]. The higher-order neutral difference equation with positive and negative coefficients received much less attention, which is due mainly to the technical difficulties arising in its analysis. In particular, there is no nonoscillation result for (1).

In this paper, we obtain the global results (with respect to c) in the nonconstant coefficient case, which are some sufficient conditions for the existence of a nonoscillatory solution of (1) for all values of $c \neq \pm 1$.

As is customary, solution $\{x_n\}$ of (1) is said to oscillate about zero or simply to oscillate if the terms x_n of the sequence $\{x_n\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory. For $t \in \mathbb{R}$, we define the usual factorial expression $(t)^{(m)} = \prod_{i=0}^{m-1}(t-i)$ with $(t)^{(0)} = 1$.

2. NONOSCILLATION OF ODD ORDER EQUATIONS

In this section, we assume that $m \geq 1$ is an odd integer.

THEOREM 1. *Assume that* $0 \leq c < 1$ *and that*

$$\sum_{i=n_0}^{\infty} i^{m-1} p_i < \infty, \qquad \sum_{i=n_0}^{\infty} i^{m-1} q_i < \infty. \tag{4}$$

Further, assume that there exist a constant $\alpha > 1/(1-c)$ *and a sufficiently large* $N_1 \geq n_0$ *such that*

$$p_n \geq \alpha q_n, \qquad \text{for } n \geq N_1. \tag{5}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (4) and (5), there exists a $n_1 > \max\{N_1, n_0 + \sigma\}$ sufficiently large such that

$$c + \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} (p_i + q_i) \leq \theta_1 < 1, \qquad \text{for } n \geq n_1, \tag{6}$$

where θ_1 is a constant, and

$$0 \leq \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} (\alpha M p_i - M q_i) \leq c - 1 + \alpha M, \qquad \text{for } n \geq n_1, \tag{7}$$

hold, where M is a positive constant such that

$$\frac{1-c}{\alpha} < M \leq \frac{1-c}{1+c\alpha} \tag{8}$$

holds.

Consider the Banach space $l_\infty^{n_0}$ of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ with the norm $\|x\| = \sup_{n \geq n_0} |x_n|$. We define a closed bounded subset Ω of $l_\infty^{n_0}$ as follows:

$$\Omega = \left\{x = \{x_n\} \in l_\infty^{n_0} : M \leq x_n \leq \alpha M,\ n \geq n_0\right\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 - c - cx_{n-k} + \dfrac{1}{(m-1)!} \displaystyle\sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} (p_i x_{i-r} - q_i x_{i-l}), & n \geq n_1, \\ Tx_{n_1}, & n_0 \leq n \leq n_1. \end{cases}$$

We shall show that $T\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $n \geq n_1$, using (7) and (8), we get

$$\begin{aligned} Tx_n &= 1 - c - cx_{n-k} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}) \\ &\leq 1 - c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(\alpha M p_i - M q_i) \\ &\leq \alpha M. \end{aligned}$$

Furthermore, in view of (5) and (8), we have

$$\begin{aligned} Tx_n &= 1 - c - cx_{n-k} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}) \\ &\geq 1 - c - c\alpha M + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(M p_i - \alpha M q_i) \\ &\geq 1 - c - c\alpha M \\ &\geq M. \end{aligned}$$

Thus, we proved that $T\Omega \subset \Omega$.

Now we shall show that operator T is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $n \geq n_1$, we have

$$\begin{aligned} |Tx_n - Ty_n| &\leq c|x_{n-k} - y_{n-k}| + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}p_i|x_{i-r} - y_{i-r}| \\ &\quad + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i|x_{i-l} - y_{i-l}| \\ &\leq \left[c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i + q_i)\right]\|x - y\| \\ &\leq \theta_1\|x - y\|. \end{aligned}$$

This implies that

$$\|Tx - Ty\| \leq \theta_1\|x - y\|,$$

where in view of (6), $\theta_1 < 1$, which proves that T is a contraction operator on Ω. Therefore, T has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1). This completes the proof of Theorem 1.

THEOREM 2. *Assume that $1 < c < +\infty$ and that (4) holds. Further, assume that there exist a constant $\beta > c/(c-1)$ and a sufficiently large $N_1 \geq n_0$ such that*

$$p_n \geq \beta q_n, \qquad \text{for } n \geq N_1. \tag{9}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (4) and (9), there exists a $n_1 \geq \max\{N_1, \sigma\}$ sufficiently large such that

$$\frac{1}{c} + \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_i + q_i) \leq \theta_2 < 1, \qquad \text{for } n \geq n_1, \tag{10}$$

where θ_2 is a constant, and

$$0 \leq \frac{1}{(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(\beta H p_i - H q_i) \leq 1 - c + c\beta H, \qquad \text{for } n \geq n_1, \tag{11}$$

where H is a positive constant such that

$$\frac{c-1}{\beta c} < H \le \frac{c-1}{c+\beta} \tag{12}$$

holds.

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \{x = \{x_n\} \in l_\infty^{n_0} : H \le x_n \le \beta H,\ n \ge n_0\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 - \frac{1}{c} - \frac{1}{c} x_{n+k} \\ + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (p_i x_{i-r} - q_i x_{i-l}), & n \ge n_1, \\ Tx_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

We shall show that $T\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $n \ge n_1$, using (11) and (12), we get

$$\begin{aligned} Tx_n &= 1 - \frac{1}{c} - \frac{1}{c} x_{n+k} + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (p_i x_{i-r} - q_i x_{i-l}) \\ &\le 1 - \frac{1}{c} + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (\beta H p_i - H q_i) \\ &\le \beta H. \end{aligned}$$

Furthermore, in view of (9) and (12), we get

$$\begin{aligned} Tx_n &= 1 - \frac{1}{c} - \frac{1}{c} x_{n+k} + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (p_i x_{i-r} - q_i x_{i-l}) \\ &\ge 1 - \frac{1}{c} - \frac{\beta H}{c} + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (H p_i - \beta H q_i) \\ &\ge 1 - \frac{1}{c} - \frac{\beta H}{c} \\ &\ge H. \end{aligned}$$

Thus, we proved that $T\Omega \subset \Omega$.

Now we shall show that operator T is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $n \ge n_1$, we have

$$\begin{aligned} |Tx_n - Ty_n| &\le \frac{1}{c} |x_{n+k} - y_{n+k}| + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} p_i |x_{i-r} - y_{i-r}| \\ &\quad + \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} q_i |x_{i-l} - y_{i-l}| \\ &\le \frac{1}{c} \left[1 + \frac{1}{(m-1)!} \sum_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (p_i + q_i) \right] \|x - y\| \\ &\le \theta_2 \|x - y\|. \end{aligned}$$

This implies that

$$\|Tx - Ty\| \le \theta_2 \|x - y\|,$$

where in view of (10), $\theta_2 < 1$, which proves that T is a contraction operator. Consequently, T has the unique fixed point x, which is obviously a bounded positive solution of equation (1). This completes the proof of Theorem 2.

THEOREM 3. *Assume that* $-1 < c < 0$ *and that (4) holds. Further, assume that there exist a constant* $\gamma > 1$ *and a sufficiently large* $N_1 \geq n_0$ *such that*

$$p_n \geq \gamma q_n, \qquad \text{for } n \geq N_1. \tag{13}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (4) and (13), there exists an $n_1 \geq N_1$ sufficiently large such that the inequalities

$$-c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i+q_i) \leq \theta_3 < 1, \qquad \text{for } n \geq n_1, \tag{14}$$

where θ_3 is a constant, and

$$0 \leq \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(\gamma M_1 p_i - M_1 q_i) \leq (c+1)(\gamma M_1 - 1), \qquad \text{for } n \geq n_1, \tag{15}$$

hold, where the constant M_1 satisfies

$$\frac{1}{\gamma} < M_1 \leq 1. \tag{16}$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \left\{x = \{x_n\} \in l_\infty^{n_0} : M_1 \leq x_n \leq \gamma M_1,\ n \geq n_0\right\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 + c - cx_{n-k} + \dfrac{1}{(m-1)!}\sum\limits_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}), & n \geq n_1, \\ Tx_{n_1}, & n_0 \leq n \leq n_1. \end{cases}$$

For every $x \in \Omega$ and $n \geq n_1$, using (15) and (16), we get

$$\begin{aligned} Tx_n &= 1 + c - cx_{n-k} + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}) \\ &\leq 1 + c - c\gamma M_1 + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(\gamma M_1 p_i - M_1 q_i) \\ &\leq 1 + c - c\gamma M_1 + (c+1)(\gamma M_1 - 1) \\ &= \gamma M_1. \end{aligned}$$

Further, in view of (13) and (16), we have

$$\begin{aligned} Tx_n &= 1 + c - cx_{n-k} + \frac{1}{(m-1)}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}) \\ &\geq 1 + c - cM_1 + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(M_1 p_i - \gamma M_1 q_i) \\ &\geq 1 + c - cM_1 \\ &\geq M_1. \end{aligned}$$

Thus, we proved that $T\Omega \subset \Omega$.

For $x, y \in \Omega$ and $n \geq n_1$, we have

$$\begin{aligned}|Tx_n - Ty_n| &\leq -c|x_{n-k} - y_{n-k}| + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}p_i|x_{i-r} - y_{i-r}| \\ &\quad + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}q_i|x_{i-l} - y_{i-l}| \\ &\leq \left[-c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(p_i+q_i)\right]\|x-y\| \\ &\leq \theta_3\|x-y\|.\end{aligned}$$

This implies that

$$\|Tx - Ty\| \leq \theta_3\|x-y\|,$$

where in view of (14), $\theta_3 < 1$. This proves that T is a contraction operator. Consequently, T has the unique fixed point x, which is obviously a bounded positive solution of equation (1). This completes the proof of Theorem 3.

THEOREM 4. *Assume that $-\infty < c < -1$ and that (4) holds. Further, assume that there exists a constant $\delta > 1$ and a sufficiently large $N_1 \geq n_0$ such that*

$$p_n \geq \delta q_n, \qquad \text{for } n \geq N_1. \tag{17}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (4) and (17), there exists a $n_1 \geq n_0$ sufficiently large such that the inequalities

$$-\frac{1}{c} - \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_i+q_i) \leq \theta_4 < 1, \qquad \text{for } n \geq n_1, \tag{18}$$

where θ_4 is a constant, and

$$0 \leq \frac{1}{(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(\delta H_1 p_i - H_1 q_i) \leq (c+1)(H_1-1), \qquad \text{for } n \geq n_1, \tag{19}$$

hold, where the positive constant H_1 satisfies

$$\frac{1}{\delta} \leq H_1 < 1. \tag{20}$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \{x = \{x_n\} \in l_\infty^{n_0} : H_1 \leq x_n \leq \delta H_1,\ n \geq n_0\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 + \frac{1}{c} - \frac{1}{c}x_{n+k} \\ \quad + \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_i x_{i-r} - q_i x_{i-l}), & n \geq n_1, \\ Tx_{n_1}, & n_0 \leq n \leq n_1. \end{cases}$$

For every $x \in \Omega$ and $n \geq n_1$, using (17) and (20), we get

$$\begin{aligned}Tx_n &= 1 + \frac{1}{c} - \frac{1}{c}x_{n+k} + \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_i x_{i-m} - q_i x_{i-l}) \\ &\leq 1 + \frac{1}{c} - \frac{\delta H_1}{c} + \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(H_1 p_i - \delta H_1 q_i) \\ &\leq 1 + \frac{1}{c} - \frac{\delta H_1}{c} \\ &\leq \delta H_1.\end{aligned}$$

Furthermore, in view of (19) and (20), we have

$$\begin{aligned}Tx_n &= 1+\frac{1}{c}-\frac{1}{c}x_{n+k}+\frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_ix_{i-r}-q_ix_{i-l})\\ &\ge 1+\frac{1}{c}-\frac{H_1}{c}+\frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(\delta H_1p_i-H_1q_i)\\ &\ge 1+\frac{1}{c}-\frac{H_1}{c}+\frac{1}{c}(c+1)(H_1-1)\\ &= H_1.\end{aligned}$$

Thus, we proved that $T\Omega\subset\Omega$.

For $x,y\in\Omega$ and $n\ge n_1$, we have

$$\begin{aligned}|Tx_n-Ty_n| &\le -\frac{1}{c}|x_{n+k}-y_{n+k}|-\frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}p_i|x_{i-r}-y_{i-r}|\\ &\quad-\frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}q_i|x_{i-l}-y_{i-l}|\\ &\le -\frac{1}{c}\left[1+\frac{1}{(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(p_i+q_i)\right]\|x-y\|\\ &\le \theta_4\|x-y\|.\end{aligned}$$

This immediately implies that

$$\|Tx-Ty\|\le\theta_4\|x-y\|.$$

In view of (18), $\theta_4<1$. This proves that T is a contraction operator. Consequently, T has the unique fixed point x, which is obviously a bounded positive solution of equation (1). This completes the proof of Theorem 4.

In the special case where $q_n\equiv 0$, conditions (5), (9), (13), and (17) are redundant. By Theorems 1–4, we have the following result.

COROLLARY 1. *Assume that* $-\infty<c<+\infty$, $c\ne -1$ *and that*

$$\sum_{i=n_0}^{\infty} i^{m-1}p_i<\infty.$$

Then the neutral difference equation

$$\Delta^m(x_n+cx_{n-k})+p_nx_{n-r}=0 \tag{21}$$

has a bounded nonoscillatory solution.

3. NONOSCILLATION OF EVEN ORDER EQUATIONS

In this section, we assume that $m\ge 2$ is an even integer.

THEOREM 5. *Assume that* $0\le c<1$ *and that*

$$\sum_{i=n_0}^{\infty} i^{m-1}q_i<\infty,\qquad \sum_{i=n_0}^{\infty} i^{m-1}p_i<\infty. \tag{22}$$

Further, assume that there exist a constant $\alpha>1/(1-c)$ *and a sufficiently large* $N_1\ge n_0$ *such that*

$$q_n\ge\alpha p_n,\qquad \text{for } n\ge N_1. \tag{23}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (22) and (23), there exists an $n_1 > \max\{N_1, n_0 + \sigma\}$ sufficiently large such that

$$c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(q_i+p_i) \le \theta_1 < 1, \qquad \text{for } n \ge n_1,$$

where θ_1 is a constant, and

$$0 \le \frac{1}{(m-1)!}\sum_{i=n_1}^{\infty}(i-n+m-1)^{(m-1)}(\alpha M q_i - M p_i) \le c - 1 + \alpha M, \qquad \text{for } n \ge n_1,$$

hold, where M is a positive constant such that

$$\frac{1-c}{\alpha} < M \le \frac{1-c}{1+c\alpha}$$

holds.

Consider the Banach space $l_\infty^{n_0}$ of all real sequence $x = \{x_n\}_{n=n_0}^{\infty}$ with the norm $\|x\| = \sup_{n\ge n_0}|x_n|$. We define a closed bounded subset Ω of $l_\infty^{n_0}$ as follows:

$$\Omega = \{x = \{x_n\} \in l_\infty^{n_0} : M \le x_n \le \alpha M,\ n \ge n_0\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 - c - cx_{n-k} + \dfrac{1}{(m-1)!}\sum\limits_{i=n}^{\infty}(i-n+m-1)^{(m-1)}\left(q_i x_{i-l} - p_i x_{i-r}\right), & n \ge n_1, \\ Tx_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

It can be proved that T has the unique fixed point x, which is a bounded positive solution of equation (1). The rest of the proof is similar to that of Theorem 1, and thus, is omitted.

THEOREM 6. *Assume that $1 < c < +\infty$ and that (22) holds. Further, assume that there exist a constant $\beta > c/(c-1)$ and a sufficiently large $N_1 \ge n_0$ such that*

$$q_n \ge \beta p_n, \qquad \text{for } n \ge N_1. \tag{24}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (22) and (24), there exists a $n_1 \ge \max\{N_1, \sigma\}$ sufficiently large such that

$$\frac{1}{c} + \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(q_i+p_i) \le \theta_2 < 1, \qquad \text{for } n \ge n_1,$$

where θ_2 is a constant, and

$$0 \le \sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(\beta H q_i - H p_i) \le 1 - c + c\beta H, \qquad \text{for } n \ge n_1,$$

where H is a positive constant such that

$$\frac{c-1}{\beta c} < H \le \frac{c-1}{c+\beta}$$

holds.

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \{x = \{x_n\} \in l_\infty^{n_0} : H \le x_n \le \beta H,\ n \ge n_0\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 - \dfrac{1}{c} - \dfrac{1}{c}x_{n+k} \\ \quad + \dfrac{1}{c(m-1)!}\sum\limits_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(q_i x_{i-l} - p_i x_{i-r}), & n \ge n_1, \\ Tx_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

It can be proved that T has the unique fixed point x, which is a bounded positive solution of equation (1). The rest of the proof is similar to that of Theorem 2, and thus, is omitted.

THEOREM 7. *Assume that $-1 < c < 0$ and that (22) holds. Further, assume that there exist a constant $\gamma > 1$ and a sufficiently large $N_1 \geq n_0$ such that*

$$q_n \geq \gamma p_n, \qquad \text{for } n \geq N_1. \tag{25}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (22) and (25), there exists a $n_1 \geq N_1$ sufficiently large such that the inequalities

$$-c + \frac{1}{(m-1)!}\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(q_i+p_i) \leq \theta_3 < 1, \qquad \text{for } n \geq n_1,$$

where θ_3 is a constant, and

$$0 \leq \frac{1}{(m-1)!}\sum_{i=n_1}^{\infty}(i-n+m-1)^{(m-1)}(\gamma M_1 q_i - M_1 p_i) \leq (c+1)(\gamma M_1 - 1), \qquad \text{for } n \geq n_1,$$

hold, where the constant M_1 satisfies

$$\frac{1}{\gamma} < M_1 \leq 1.$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \left\{x = \{x_n\} \in l_\infty^{n_0} : M_1 \leq x_n \leq \gamma M_1,\ n \geq n_0\right\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 + c - cx_{n-k} + \dfrac{1}{(m-1)!}\displaystyle\sum_{i=n}^{\infty}(i-n+m-1)^{(m-1)}(q_i x_{i-l} - p_i x_{i-r}), & n \geq n_1, \\ Tx_{n_1}, & n_0 \leq n \leq n_1. \end{cases}$$

It can be proved that T has the unique fixed point x, which is a bounded positive solution of equation (1). The rest of the proof is similar to that of Theorem 3, and thus, is omitted.

THEOREM 8. *Assume that $-\infty < c < -1$ and that (22) holds. Further, assume that there exists a constant $\delta > 1$ and a sufficiently large $N_1 \geq n_0$ such that*

$$q_n \geq \delta p_n, \qquad \text{for } n \geq N_1. \tag{26}$$

Then (1) has a bounded nonoscillatory solution.

PROOF. By (22) and (26), there exists a $n_1 \geq n_0$ sufficiently large such that the inequalities

$$-\frac{1}{c} - \frac{1}{c(m-1)!}\sum_{i=n+k}^{\infty}(i-n-k+m-1)^{(m-1)}(q_i+p_i) \leq \theta_4 < 1, \qquad \text{for } n \geq n_1,$$

where θ_4 is a constant, and

$$0 \leq \frac{1}{(m-1)!}\sum_{i=n+k}^{\infty}(i-n+m-1)^{(m-1)}(\delta H_1 q_i - H_1 p_i) \leq (c+1)(H_1 - 1), \qquad \text{for } n \geq n_1,$$

hold, where the positive constant H_1 satisfies

$$\frac{1}{\delta} \leq H_1 < 1.$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1. Set

$$\Omega = \{x = \{x_n\} \in l_\infty^{n_0} : H_1 \le x_n \le \delta H_1,\ n \ge n_0\}.$$

Define an operator $T : \Omega \to l_\infty^{n_0}$ as follows:

$$Tx_n = \begin{cases} 1 + \dfrac{1}{c} - \dfrac{1}{c} x_{n+k} & \\ + \dfrac{1}{c(m-1)!} \sum\limits_{i=n+k}^{\infty} (i-n-k+m-1)^{(m-1)} (q_i x_{i-l} - p_i x_{i-r}), & n \ge n_1, \\ Tx_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

It can be proved that T has the unique fixed point x, which is a bounded positive solution of equation (1). The rest of the proof is similar to that of Theorem 4, and thus, is omitted.

In the special case where $p_n \equiv 0$, conditions (23)–(26) are redundant. By Theorems 5–8, we have the following result.

COROLLARY 2. *Assume that* $-\infty < c < +\infty$ *and that*

$$\sum_{i=n_0}^{\infty} i^{m-1} q_i < \infty.$$

Then the neutral difference equation

$$\Delta^m (x_n + c x_{n-k}) = q_n x_{n-l} \tag{27}$$

has a bounded nonoscillatory solution.

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