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On the Zeros of Exponential Polynomials

CERINO E. AVELLAR*

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912; and Universidade Federal de São Carlos, Departamento de Matematica, 13560-São Carlos, São Paulo, Brazil

AND

JACK K. HALE†

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

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Suppose $r = (r_1, ..., r_M)$, $r_j \geqslant 0$, $\gamma_{kj} \geqslant 0$ integers, $k = 1, 2, ..., N$, $j = 1, 2, ..., M$, $\gamma_k \cdot r = \sum_j \gamma_{kj} r_j$. The purpose of this paper is to study the behavior of the zeros of the function $h(\lambda, a, r) = 1 + \sum_{j=1}^{N} a_j e^{-\lambda \gamma_j \cdot r}$, where each a_j is a nonzero real number. More specifically, if $\bar{Z}(a, r) = \text{closure}\{\operatorname{Re} \lambda : h(\lambda, a, r) = 0\}$, we study the dependence of $\bar{Z}(a, r)$ on a, r. This set is continuous in a but generally not in r. However, it is continuous in r if the components of r are rationally independent. Specific criterion to determine when $0 \notin \bar{Z}(a, r)$ are given. Several examples illustrate the complicated nature of $\bar{Z}(a, r)$. The results have immediate implication to the theory of stability for difference equations $x(t) - \sum_{k=1}^{M} A_k x(t - r_k) = 0$, where x is an n-vector, since the characteristic equation has the form given by $h(\lambda, a, r)$. The results give information about the preservation of stability with respect to variations in the delays. The results also are fundamental for a discussion of the dependence of solutions of neutral differential difference equations on the delays. These implications will appear elsewhere.

1. INTRODUCTION

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_*^+ = (0, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $a = (a_1, ..., a_N) \in \mathbb{R}^N$, $r = (r_1, ..., r_M) \in (\mathbb{R}^+)^M$, $\gamma_j = (\gamma_{j1}, ..., \gamma_{jM})$, γ_{jk} nonnegative integers, $j = 1, 2, ..., N$, $k = 1, 2, ..., M$, $\gamma_j \cdot r = \sum_{k=1}^{M} \gamma_{jk} r_k$. Let $a_0 = 1$, $\gamma_0 = (0, ..., 0)$,

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$a_j \neq 0$, $j = 1, 2, \ldots, N$. Our purpose in this paper is to study the behavior of the real parts of the zeros of the function

$$h(\lambda, a, r) = \sum_{j=0}^{N} a_j e^{-\lambda \gamma_j \cdot r}. \tag{1.1}$$

More specifically, if

$$Z(a, r) = \{\operatorname{Re} \lambda : h(\lambda, a, r) = 0\} \tag{1.2}$$

and $\bar{Z}(a, r) = \operatorname{cl} Z(a, r)$, the closure of $Z(a, r)$, we study the dependence in the Hausforff metric of $\bar{Z}(a, r)$ on a, r. It is shown that $\bar{Z}(a, r)$ is continuous in a with a certain type of uniformity in r. It has been known for some time (see Melvin [6] or Henry [4]) that $\bar{Z}(a, r)$ is not continuous in r. However, we show that it is continuous in r if the components of r are rationally independent.

We also give a characterization of $\bar{Z}(a, r)$ in a way which is amenable to computation. For the case in which $N = M$ and the function $h(\lambda, a, r)$ is given as

$$h(\lambda, a, r) = 1 + \sum_{j=1}^{N} a_j e^{-\lambda r_j}, \tag{1.3}$$

the characterization of $\bar{Z}(a, r)$ is more complete and the computation of $\bar{Z}(a, r)$ can be given rather explicitly.

Finally, we give several characterizations of the property that $\bar{Z}(a, r) \cap [-\delta, \delta] = \emptyset$, $\delta > 0$; that is, the polynomial $h(\lambda, a, r)$ is hyperbolic. The case $\bar{Z}(a, r) \subseteq (-\infty, -\delta]$, $\delta > 0$ is also discussed in detail. This corresponds to uniform asymptotic stability.

The implications of the results for difference equations are immediate. In fact, consider the equation

$$x(t) - \sum_{k=1}^{M} A_k x(t - r_k) = 0 \tag{1.4}$$

where $x \in \mathbb{R}^n$ and each A_k is an $n \times n$ matrix. For any $\phi \in C = C([-h, 0], \mathbb{R}^n)$, $h \geqslant \max\{r_k\}$, there is a unique solution $x = x(\phi)$ of (1.4) for $t \geqslant -h$ which satisfies $x(\phi)(t) = \phi(t)$, $t \in [-h, 0]$. If we let $x(\phi)(t + \theta) = (S(t)\phi)(\theta)$, $-h \leqslant \theta \leqslant 0$, then $S(t)\colon C \to C$, $t \geqslant 0$, is a strongly continuous semigroup of bounded linear operators. Furthermore, if

$$\alpha(a, r) = \inf\{b : \exists k \text{ with } |S(t)| \leqslant k e^{bt}\}$$

then it is known (see Henry [4], Hale [2]) that

$$\alpha(a, r) = \sup\{\operatorname{Re} \lambda : h(\lambda, a, r) = 0\}$$

$$h(\lambda, a, r) = \det\left[I - \sum_{k=1}^{N} A_k \exp(-\lambda r_k)\right].$$

Therefore, the above results give information about the behavior of the order $\alpha(a, r)$ of the semigroup $S(t)$ as a function of a, r.

The results also have implications for neutral functional differential equations of the type

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} A_k x(t - r_k)\right] = f(x_t) \tag{1.5}$$

where $f: C \to \mathbb{R}^n$ and $x_t(\theta) = x(t + \theta)$, $-h \leqslant \theta \leqslant 0$. The solution operator for Equation (1.5) can be written as a sum of a completely continuous operator and the operator $S(t)$ above (see Hale [2]). If f is linear, this gives information about the spectrum of the solution operator. One can then prove certain theorems on the continuous dependence in the delays. Results of this type will appear in Avellar and Hale [1].

2. Continuous Dependence

In this section, we present some results on the dependence of the set $\bar{Z}(a, r)$ on a, r. We need the Hausdorff metric which is defined as follows:

For any sets $E, F \subset \mathbb{R}$ and any point $\rho \in \mathbb{R}$, let

(i) $d(\rho, E) = \inf_{t\in E} | \rho - t |$

(ii) $\delta(E, F) = \sup_{\rho\in E} d(\rho, F)$

(iii) $D(E, F) = \max\{\delta(E, F), \delta(F, E)\}$.

The number $D(E, F)$ is called the Hausdorff distance between the sets E, F in $\mathbb{R}$.

We need the following result from Levin [5, p. 268], the proof of which is omitted.

LEMMA 2.1. *For a given $\alpha < \beta$, the following conclusions hold:*

(i) *There is an integer p such that, for all real t, there are no more than p zeros of h in the box*

$$\{\lambda: \alpha \leqslant \operatorname{Re} \lambda \leqslant \beta;\ t \leqslant \operatorname{Im} \lambda \leqslant t + 1\}.$$

(ii) *For any $\delta > 0$, there is an $m(\delta) > 0$ such that whenever $\alpha \leqslant \operatorname{Re} \lambda \leqslant \beta$ and λ is at a distance $\geqslant \delta$ from every zero of h, one has $| h(\lambda)| \geqslant m(\delta)$.*

Our first objective is to obtain an interval which contains $\bar{Z}(a, r)$. Observe that $\lambda = \mu + i\nu$ satisfies $h(\lambda, a, r) = 0$ if and only if

$$0 = \sum_{k=0}^{N} a_k e^{-\mu\gamma_k\cdot r} e^{-i\nu\gamma_k\cdot r} = \sum_{k=0}^{N} | a_k | e^{-\mu\gamma_k\cdot r} e^{i(\phi_k - \nu\gamma_k\cdot r)}$$

where $\phi_k = 0$ if $a_k > 0$, $\phi_k = \pi$ if $a_k < 0$.

For further reference, let us state this result as

LEMMA 2.2. *If the equation $h(\mu + i\nu, a, r) = 0$ is satisfied for some real μ, ν, then the lengths $\{| a_k | e^{-\mu\gamma_k \cdot r}, k = 0, 1, \ldots, N\}$ can form a closed polygon; that is, no one of these terms is greater than the sum of the others:*

$$| a_j | e^{-\mu\gamma_j \cdot r} \leqslant \sum_{k \neq j} | a_k | e^{-\mu\gamma_k \cdot r}, \qquad j = 0, 1, \ldots, N. \tag{2.1}$$

Following Henry [4], define $\rho_j = \rho_j(a, r)$, $j = 0, 1, 2, \ldots, N$, if they exist, by the relations

$$| a_j | e^{-\rho_j\gamma_j \cdot r} = \sum_{k \neq j} | a_k | e^{-\rho_j\gamma_k \cdot r}, \qquad j = 0, 1, \ldots, N. \tag{2.2}$$

If $\gamma_N \cdot r > \gamma_j \cdot r > 0$ for $j = 1, 2, \ldots, N - 1$, it is easy to verify that ρ_N and ρ_0 are uniquely defined by Relation (2.2) and

$$\rho_N = \rho_0 \quad \text{if} \quad N = 1, \rho_N < \rho_0 \quad \text{if} \quad N \geqslant 2. \tag{2.3}$$

LEMMA 2.3. *If $0 < \gamma_1 \cdot r < \cdots < \gamma_N \cdot r$, then*

$$\bar{Z}(a, r) \subseteq [\rho_N(a, r), \rho_0(a, r)].$$

Proof. Let $w_k = \gamma_k \cdot r$. From Relations (2.2) we have

$$| a_N | = \sum_{k=0}^{N-1} | a_k | e^{\rho_N(w_N - w_k)}; \qquad | a_0 | = \sum_{k=1}^{N} | a_k | e^{-\rho_0 w_k}.$$

We also have $w_N - w_k > 0$, $k = 0, 1, \ldots, N - 1$; $w_k > 0$, $k = 1, \ldots, N$. So,

$$\text{(i)} \quad \mu < \rho_N \Rightarrow | a_N | e^{-\mu w_N} > \sum_{k=0}^{N-1} | a_k | e^{-\mu w_k}$$

$$\text{(ii)} \quad \mu < \rho_0 \Rightarrow | a_0 | > \sum_{k=1}^{N} | a_k | e^{-\mu w_k}.$$

Lemma 2.2 implies $h(\mu + i\nu, a, r) \neq 0$ in either case, which proves the result.

The complete structure of $\bar{Z}(a, r)$ is known for the case when the components of r are commensurable. This will be stated as

LEMMA 2.4. *If $r_1, r_2, \ldots, r_M$ are commensurable, that is, $r_k = n_k\beta$ for some $\beta > 0$ and integers n_k, $k = 1, \ldots, M$, then $h(\lambda, a, r)$ is a polynomial of some degree p in $e^{-\beta\lambda}$,*

$$h(\lambda, a, r) = a_N \prod_{\nu=1}^{p} (e^{-\lambda\beta} - s_\nu)$$

and

$$\bar{Z}(a, r) = Z(a, r) = \left\{-\frac{1}{\beta} \ln |s_\nu|, \nu = 1, 2, \ldots, p\right\}.$$

Proof. Obvious.

THEOREM 2.1. *$\bar{Z}(a, r)$ is continuous in a in the Hausdorff metric. Also, if $S \subset (\mathbb{R}_*^+)^N$ is a given set and there exist $\alpha < \beta$ such that $\bar{Z}(a, r) \subset (\alpha, \beta)$ for $r \in S$, then there exist a $\delta > 0$ such that $\bar{Z}(b, r) \subset (\alpha, \beta)$ for $|b - a| < \delta$.*

Proof. From the relation

$$|h(\lambda, b, r) - h(\lambda, a, r)| \leqslant \sum_{k=0}^{N} |b_k - a_k| e^{(-\gamma_k \cdot r)} \operatorname{Re} \lambda$$

for any $\epsilon > 0$, there is a $\delta > 0$ such that

$|h(\lambda, b, r) - h(\lambda, a, r)| < \epsilon$ for $\operatorname{Re} \lambda \in [\rho_N(a, r) - \epsilon, \rho_0(a, r) + \epsilon]$, $|b - a| < \delta$

that is, $h(\lambda, b, r) - h(\lambda, a, r) \to 0$ as $b \to a$ uniformly for $\operatorname{Re} \lambda \in [\rho_N(a, r) - \epsilon, \rho_0(a, r) + \epsilon]$.

If $\rho \in Z(b, r)$ then $h(\rho + i\nu, b, r) = 0$ for some $\nu = \nu(b)$. If, in addition, $b \to a$, then every limit point z of the set $Z(b, r)$ as $b \to a$ satisfies $z \in \bar{Z}(a, r)$ from Lemma 2.1. This shows that $\delta(\bar{Z}(b, r), \bar{Z}(a, r)) \to 0$ as $b \to a$. Conversely, if $\rho \in Z(a, r)$, then there is a $\zeta = \zeta(a)$ such that $h(\rho + i\zeta(a), a, r) = 0$. Therefore, $h(\rho + i\zeta(a), b, r) \to 0$ as $b \to a$ and Lemma 2.1 implies $\rho \in \bar{Z}(b, r)$. Thus $\delta(\bar{Z}(a, r), \bar{Z}(b, r)) \to 0$ as $b \to a$ and the continuity of $\bar{Z}(a, r)$ is proved.

The last statement of the theorem is also a consequence of an argument similar to the above.

Our next objective is to discuss the dependence of $\bar{Z}(a, r)$ on r. The following example given by Silkowski [8], shows this problem is much more difficult.

EXAMPLE 2.1. Let

$$h(\lambda, r) = h(\lambda, r_1, r_2) = 1 + \tfrac{1}{2} e^{-\lambda r_1} + \tfrac{1}{2} e^{-\lambda r_2}.$$

For $r = (1, 2)$, that is,

$$h(\lambda, 1, 2) = 1 + \tfrac{1}{2} e^{-\lambda} + \tfrac{1}{2} e^{-2\lambda},$$

it is easy to see that the zeros of $h(\lambda, 1, 2)$ satisfy $\operatorname{Re} \lambda = -(\ln 2)/2 < 0$. Therefore, $\bar{Z}(r) = \{-(\ln 2)/2\}$ if $r = (1, 2)$.

Now let us consider $\tilde{r} = (\tilde{r}_1, \tilde{r}_2)$ close to $(1, 2)$. In particular, take $\tilde{r} = (1 - 4/(4n + 3), 2)$ where n is any nonnegative integer.

It is easy to verify that

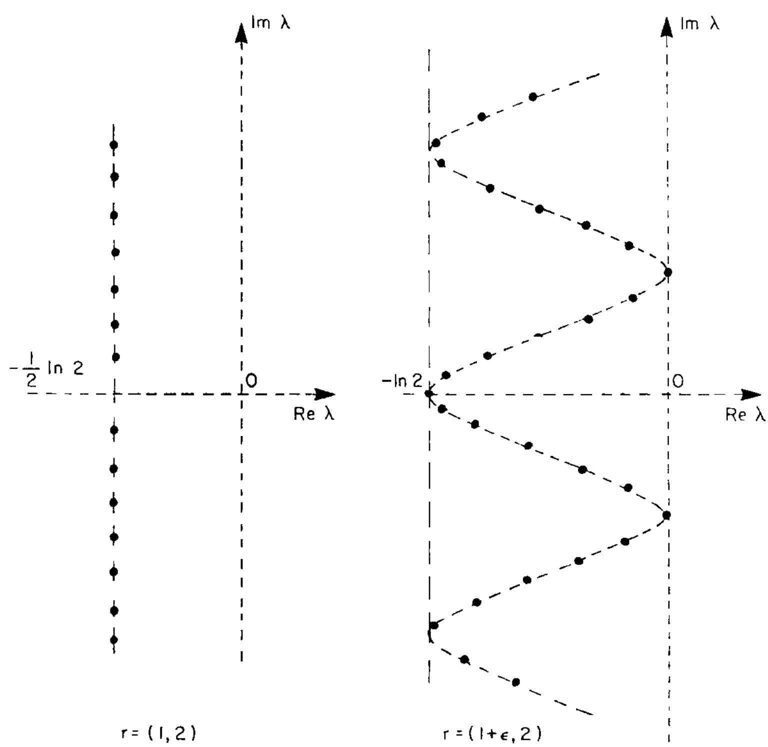
$$h(i(4n + 3)\pi/2, 1 - 4/(4n + 3), 2) = 0.$$

Therefore, $\bar{Z}(\tilde{r}) \supseteq \{0\}$ if $\tilde{r} = (1 - 4/(4n + 3, 2)$ and so, $\bar{Z}(r)$ is not continuous in r.

The numbers ρ_0, ρ_2 for this example are $\rho_0 = 0$, $\rho_2 = -\ln 2$ for $r = (1, 2)$. Also, $\rho_0(r) = 0$ for all r. Therefore, $\bar{Z}(r) \subseteq [\rho_2(r), 0]$ where $\rho_2(r) \to -\ln 2$ as $r \to (1, 2)$.

What is happening to the zeros of h, in Example 2.1, as r varies? By Rouchés's theorem, for any given r_0 and any compact set K in $\mathbb{C}$ for which no zeros of $h(\lambda, r_0)$ lie on ∂K, there is an $\epsilon > 0$ such that $| r - r_0 | < \epsilon$ implies $h(\lambda, r)$ has the same number of zeros as $h(\lambda, r_0)$ in K. However, a small change in r does not necessarily give a small change in h uniformly in a strip as was the case when the coefficients were varied as in Theorem 2.1. The noncompactness of the strip plays an essential role when r is varied.

For the purpose of intuition, it is worthwhile to note the following fact about Example 2.1. For $r = (1, 2)$, the zeros of h belonged to a vertical line $\operatorname{Re} \lambda = -(\ln 2)/2$ and were given by $\lambda = -(\ln 2)/2 + i(\tan^{-1}\sqrt{7} + 2k\pi)$, $k = 0, 1, 2, \ldots$. For a small change in r, this vertical line of zeros is moved a large distance. In fact, it may include $\operatorname{Re} \lambda = 0$. The figure below is suggestive of the way the line on which the zeros of h lie could vary with ϵ.



We shall see below that it is actually possible for the real parts of the zeros of h to fill an interval.

The above example shows that $\bar{Z}(a, r)$ is not necessarily continuous in r. However, it is always lower semicontinuous as shown in the next lemma. We write $Z(r) = Z(a, r)$ since a is fixed.

LEMMA 2.5. *$\bar{Z}(r)$ is lower semicontinuous in r, that is,*

$$\lim_{r \to r^0} \delta(\bar{Z}(r_0), \bar{Z}(r)) = 0.$$

Proof. Suppose $\rho \in Z(r_0)$. Then there exists a σ such that $h(\rho + i\sigma, r_0) = 0$. We also have

$$h(\rho + i\sigma, r) = \sum_{k=0}^{N} a_k e^{-(\rho+i\sigma)\gamma_k \cdot r_0} e^{-(\rho+i\sigma)\gamma_k \cdot (r-r_0)} \to 0 \qquad \text{as} \qquad r \to r_0 .$$

Therefore, $\delta(\bar{Z}(r_0), \bar{Z}(r)) \to 0$ as $r \to r_0$. This proves the lemma.

If the components of r are rationally independent, the next theorem states that $Z(a, r)$ is continuous.

THEOREM 2.2. *If $r_0 \in (\mathbb{R}_*^+)^M$ is fixed and the components of r_0 are rationally independent, then $\bar{Z}(r) \to \bar{Z}(r_0)$ in the Hausdorff metric as $r \to r_0$.*

Proof. Suppose $\rho(r) \in Z(r)$, $h(\rho(r) + i\sigma(r), r) = 0$ for some real $\sigma(r)$. For any sequence $r_j \to r_0$ we may assume $\rho(r_j) \to \rho_0$.

Consider $h(\rho_0 + i\nu, r_0)$. For any r,

$$\begin{aligned} h(\rho_0 + i\nu, r_0) &= \sum_{k=0}^{N} a_k e^{-\rho_0 \gamma_k \cdot r_0} e^{-i\nu\gamma_k \cdot r_0} \\ &= \sum_{k=0}^{N} a_k e^{-\rho_0 \gamma_k \cdot r_0} e^{-i\sigma(r)\gamma_k \cdot r} e^{i\gamma_k \cdot (\sigma(r) r - \nu r_0)} \\ &= \sum_{k=0}^{N} a_k e^{-(\rho_0 + i\sigma(r))\gamma_k \cdot r} e^{\rho_0 \gamma_k \cdot (r-r_0)} e^{i\gamma_k \cdot (\sigma(r) r - \nu r_0)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &h(\rho_0 + i\nu, r_0) - h(\rho(r) + i\sigma(r), r) \\ &= \sum_{i=0}^{N} a_k e^{-(\rho(r)+i\sigma(r))\gamma_k \cdot r} [e^{[\rho(r)-\rho_0]\gamma_k \cdot r} e^{\rho_0 \gamma_k \cdot (r-r_0)} e^{i\gamma_k \cdot (\sigma(r) r - \nu r_0)} - 1]. \end{aligned}$$

By Kronecker's Theorem, for any sequence $r_j \to r_0$, choose $\{\nu_{j,l}\}$, $\nu_{j,l} \to \infty$ as $l \to \infty$, such that

$$e^{i\gamma_k \cdot (\sigma(r_j) r_j - \nu_{j,l} r_0)} \to 1 \qquad \text{as} \qquad l \to \infty.$$

By the diagonalization procedure, we can choose a subsequence $\{\tilde{\nu}_j\}$, $\tilde{\nu}_j \to \infty$ as $j \to \infty$, such that

$$e^{i\gamma_k \cdot (\sigma(r_j) r_j - \tilde{\nu}_j r_0)} \to 1 \qquad \text{as} \qquad j \to \infty.$$

Thus, $h(\rho_0 + i\tilde{\nu}_j, r_0) \to 0$ as $j \to \infty$ and every limit point ρ_0 of $Z(r)$ as $r \to r_0$ satisfies $\rho_0 \in \bar{Z}(r_0)$. This shows that $\delta(\bar{Z}(r), \bar{Z}(r_0)) \to 0$ as $r \to r_0$. Lemma 2.5 completes the proof.

3. Characterization of $\bar{Z}(a, r)$

The following characterization of $\bar{Z}(r) = \bar{Z}(a, r)$ was stated without proof by Henry [4].

THEOREM 3.1. *If*

$$h(\lambda, r) = a_0 + \sum_{k=1}^{N} a_k e^{-\lambda \gamma_k \cdot r}, \qquad r = (r_1, r_2, ..., r_M)$$

$$H(\rho, \theta, r) = a_0 + \sum_{k=1}^{N} a_k e^{-\rho \gamma_k \cdot r} e^{i\gamma_k \cdot \theta}, \tag{3.1}$$

$$\theta = (\theta_1, \theta_2, ..., \theta_M), \qquad 0 \leqslant \theta_j \leqslant 2\pi$$

and the components of r are rationally independent, then $\rho \in \bar{Z}(r)$ if and only if there is a θ such that $H(\rho, \theta, r) = 0$.

Proof. If $h(\rho + i\nu, r) = 0$, then $\exists \theta = \nu r$ such that $H(\rho, \theta, r) = 0$.

Conversely, suppose there exist $\theta = (\theta_1, ..., \theta_M)$, $\theta_j \in [0, 2\pi]$, $j = 1, ..., M$, such that

$$a_0 + \sum_{k=1}^{N} a_k e^{-\rho \gamma_k \cdot r} e^{i\gamma_k \cdot \theta} = 0.$$

By Kronecker's Theorem, there exists a sequence $\{\nu^n\}$, such that

$$e^{i\gamma_k \cdot (\theta - \nu^n r)} \to 1 \qquad \text{as} \qquad n \to \infty.$$

Therefore,

$$h(\rho + i\nu^n, r) = a_0 + \sum_{k=1}^{N} a_k e^{-\rho \gamma_k \cdot r} e^{-i\nu^n \gamma_k \cdot r}$$

$$= a_0 + \sum_{k=1}^{N} a_k e^{-\rho \gamma_k \cdot r} e^{-i\gamma_k \theta} e^{i\gamma_k \cdot (\theta - \nu^n r)} \to 0$$

as $n \to \infty$. But this implies that $\rho \in \bar{Z}(r)$.

Theorem 2.2 states that $\bar{Z}(r)$ is continuous at those vectors r with rationally independent components and Theorem 3.1 gives a way for computing $\bar{Z}(r)$ at such vectors r.

An important consequence of Theorem 3.1 is the following result.

COROLLARY 3.1. *The following statements are equivalent*

(i) $0 \in \bar{Z}(r^0)$ *for some* r^0 *with rationally independent components.*

(ii) $0 \in \bar{Z}(r)$ *for all* r *with rationally independent components.*

Proof. Since $H(0, \theta, r)$ in Relation (3.1) is independent of r, it is clear from Theorem 3.1 that (i) $\Rightarrow$ (ii). The other way is obvious.

Another easy consequence of Theorem 3.1 is

COROLLARY 3.2. *For any* $r \in (\mathbb{R}_*^+)^M$, $\bar{Z}(r)$ *is the union of a finite number of intervals.*

Proof. If the components of r are rationally independent, then $\bar{Z}(r)$ is characterized by the solutions of $H(\rho, \theta, r) = 0$. Since these solutions are analytic varieties, it is impossible to have the following property: there exists a $\rho \in \bar{Z}(r)$ $\{\rho_j\}_{j=1}^{\infty} \subseteq \bar{Z}(r)$, $\rho_j \to \rho$ as $j \to \infty$, $(\rho_{j+1}, \rho_j) \cap \bar{Z}(r) = \varnothing$. This proves the corollary when the components are rationally independent.

For any $r \in (\mathbb{R}_*^+)^M$ there exists a $\beta \in (\mathbb{R}_*^+)^q$ for some integer q such that the components of β are rationally independent. Apply the previous result to β to complete the proof.

Another easy consequence of Theorem 3.1 and Theorem 2.2 is

COROLLARY 3.3. *If*

$$\begin{aligned}
\rho(r) &= \min\{\rho\colon \exists\theta \in \mathbb{R}^M \text{ with } H(\rho. \theta, r) = 0\} \\
\sigma(r) &= \max\{\rho\colon \exists\theta \in \mathbb{R}^M \text{ with } H(\rho, \theta, r) = 0\} \\
\tau_-(r) &= \max\{\rho(r) \leqslant \rho \leqslant 0\colon \exists\theta \in \mathbb{R}^M \text{ with } H(\rho, \theta, r) = 0\} \\
&\quad\ \text{if } \rho(r) \leqslant 0,\ \tau_-(r) = \rho(r) \text{ if } \rho(r) > 0 \\
\tau_+(r) &= \min\{\sigma(r) \geqslant \rho \geqslant 0\colon \exists\theta \in \mathbb{R}^M \text{ with } H(\rho, \theta, r) = 0\} \\
&\quad\ \text{if } \sigma(r) \geqslant 0,\ \tau_+(r) = \sigma(r) \text{ if } \sigma(r) < 0.
\end{aligned} \tag{3.2}$$

Then $\rho(r)$, $\sigma(r)$, $\tau_-(r)$, $\tau_+(r)$ *are continuous in* r, *and either* $\tau_-(r) < 0 < \tau_+(r)$ *for all* r *or* $\tau_-(r) = \tau_+(r)$ *for all* r, *and*

$$\bar{Z}(r) \subseteq [\rho(r), \tau_-(r)] \cup [\tau_+(r), \sigma(r)].$$

Furthermore, $\rho(r)$, $\tau_-(r)$, $\tau_+(r)$, $\sigma(r) \in \bar{Z}(r)$ *if the components of* r *are rationally independent.*

We remark that a finer structure theorem for $\bar{Z}(r)$ than Corollary 3.3 could be given by specifying a finite number of intervals which vary continuously with r and which coincide with $\bar{Z}(r)$ when the components of r are rationally independent. However, the number of disjoint intervals would not be constant in r (examples will be given later). On the other hand, the structure theorem in Corollary 3.3 is independent of r. In fact, for any r^0 with rationally independent components, there is a neighborhood $U(r^0)$ of r^0 such that only one of the following situation occur:

(i) $\tau_-(r) = \tau_+(r) = 0$ for all r; that is, $\bar{Z}(r)$ contains zero for all $r \in U(r^0)$.

(ii) $\tau_-(r) < 0 < \tau_+(r)$ for all $r \in U(r^0)$; that is, $\bar{Z}(r)$ contains elements < 0 and > 0 for all values of $r \in U(r^0)$.

(iii) $\tau_-(r) = \sigma(r) = \tau_+(r)$; that is, either $\bar{Z}(r) \cap [0, \infty) = \varnothing$ for all $r \in U(r^0)$ or $\bar{Z}(r) \cap [0, \infty) = \{0\}$ for all $r \in U(r^0)$.

(iv) $\tau_+(r) = \rho(r) = \tau_-(r)$; that is, either $\bar{Z}(r) \cap (-\infty, 0] = \varnothing$ or $\bar{Z}(r) \cap (-\infty, 0] = \{0\}$ for all $r \in U(r^0)$.

These remarks will be related to stability in a later section.

4. A Special Case

When the function $h(\lambda, a, r)$ has the special form

$$h(\lambda, a, r) = 1 + \sum_{j=1}^{N} a_j e^{-\lambda r_j} \tag{4.1}$$

one can give a more precise description of the set $\bar{Z}(r) = \bar{Z}(a\ r)$. This corresponds to the case where $N = M$ and $\gamma_{jk} = 0$ if $j \neq k$, $\gamma_{jj} = 1, j = 1, 2,..., N$. It is the purpose of this section to discuss the zeros of the function h in Relation (4.1). The following result is essentially in Henry [4], Moreno [7].

THEOREM 4.1. *Suppose* $0 < r_1 < \cdots < r_N$ *and define* $\rho_0 ,..., \rho_N$ *by relation* (2.2). *If the set* $\{r_k , k = 1, 2,..., N\}$ *is rationally independent, then* $\rho \in \bar{Z}(a, r)$ *if and only if* $\{| a_0 |, | a_k | e^{-\rho r_k}, k = 1, 2,..., N\}$ *can form a closed polygon. Also,* $[\rho_N(a, r), \rho_0(a, r)]$ *is the smallest closed interval containing* $\bar{Z}(a, r)$ *and* $\bar{Z}(a, r)$ *is a finite union of closed intervals. In fact, if* $I_j \subset [\rho_N(a, r), \rho_0(a, r)], j = 1, 2,..., N - 1$ *is the set (it may be empty such) that* $| a_j | e^{-\rho r_j} > \sum_{k \neq j} | a_k | e^{-\rho r_k}$ *for* $\rho \in I_j$, *then* $\bar{Z}(a, r) = [\rho_N , \rho_0] \backslash \bigcup_{j=1}^{N-1} I_j$.

Proof. If $N = 1$, the theorem is trivial. Thus, assume $N \geqslant 2$ and define $a_0 = 1$, $r_0 = 0$,

$$f_j(\rho) = | a_j | e^{-\rho r_j} - \sum_{k \neq j} | a_k | e^{-\rho r_k}, \qquad j = 0, 1, 2,..., N.$$

The set $\{|a_0|, |a_k| e^{-\rho r_k}, k = 1, 2, \ldots, N\}$ can form a closed polygon if and only if $f_j(\rho) \leqslant 0$ for all $j = 0, 1, 2, \ldots, N$. The function H in Theorem 3.1 for (4.1) is

$$H(\rho, \theta, r) = a_0 + \sum_{j=1}^{N} a_j e^{-\rho r_j} e^{i\theta_j}.$$

It is clear that "$f_j(\rho) \leqslant 0$ for all j and some ρ" is equivalent to "there exist a $\theta \in \mathbb{R}^N$ such that $H(\rho, \theta, r) = 0$". Thus, the first part of the lemma is proved. The second part is Corollary 3.2. The last part is simply writing down explicitly what it means to have $f_j(\rho) > 0$ for some j. This proves the theorem.

COROLLARY 4.1. *Suppose* $0 < r_1 < \cdots < r_N$ *and define* $\rho_0, \ldots, \rho_N$ *by Relation* (2.2). *If*

$$\begin{aligned}
\rho_{\alpha(r)} &= \max\{\rho_j(r) \colon \rho_N(r) \leqslant \rho_j(r) \leqslant 0, j = 0, 1, 2, \ldots, N\},\\
&= \rho_N(r) \text{ if } \rho_N(r) > 0.\\
\rho_{\beta(r)} &= \min\{\rho_j(r) \colon \rho_0(r) \geqslant \rho_j(r) \geqslant 0, j = 0, 1, 2, \ldots, N\},\\
&= \rho_0(r) \text{ if } \rho_0(r) < 0.
\end{aligned}$$

Then $\rho_{\alpha(r)}$, $\rho_{\beta(r)}$ *are continuous in* r *and either* $\rho_{\alpha(r)} < 0 < \rho_{\beta(r)}$ *for all* r *or* $\rho_{\alpha(r)} = \rho_{\beta(r)}$ *for all* r. *Furthermore,*

$$\bar{Z}(r) \subseteq [\rho_N(r), \rho_{\alpha(r)}] \cup [\rho_{\beta(r)}, \rho_0(r)]$$

and the end points of these intervals belong to $\bar{Z}(r)$ *if the components of* r *are rationally independent.*

Proof. This is a consequence of Corollary 3.3.

5. STABILITY AND HYPERBOLICITY

In this section $h(\lambda, a, r)$ is the function defined in Relation (1.1); that is

$$h(\lambda, a, r) = 1 + \sum_{j=1}^{N} a_k e^{-\lambda \gamma_k \cdot r}. \tag{5.1}$$

We need the following definitions.

DEFINITION 5.1. The function $h(\lambda, a, r)$ is said to be *hyperbolic at* r^0 if $0 \notin \bar{Z}(a, r^0)$. The function $h(\lambda, a, r)$ *is hyperbolic locally at* r^0 if there is a neighborhood U of r^0 and $\delta > 0$ such that $\bar{Z}(a, r) \cap [-\delta, \delta] = \varnothing$ for all $r \in U$. The function $h(\lambda, a, r)$ is *hyperbolic globally in* r if $0 \notin \bar{Z}(a, r)$ for each $r \in (\mathbb{R}^+)^M$.

DEFINITION 5.2. The function $h(\lambda, a, r)$ is said to be *uniformly asymptotically stable at* r^0 if $h(\lambda, a, r^0)$ is hyperbolic and $\bar{Z}(a, r^0) \cap [0, \infty) = \varnothing$. It is *uniformly asymptotically stable locally* at r^0, if it is hyperbolic locally at r^0 and $\bar{Z}(a, r) \cap [0, \infty) = \varnothing$ for $r \in U$. It is *uniformly asymptotically stable globally in* r if it is hyperbolic globally in r and $\bar{Z}(a, r) \cap [0, \infty) = \varnothing$ for all r.

We now prove the following fundamental result. In the statement of the theorem, $\rho(r)$, $\sigma(r)$, $\tau_-(r)$, $\tau_+(r)$ are defined in Relations (3.2).

THEOREM 5.1. *The following statements are equivalent.*

(i) *There is an* $r \in (\mathbb{R}^+)^M$, $r = (r_1, \ldots, r_M)$, *with the set* $\{r_j\}_{j=1}^M$ *rationally independent, such that the function* $h(\lambda, a, r)$ *is hyperbolic at* r^0.

(ii) $h(\lambda, a, r)$ *is hyperbolic locally at some* r^0.

(iii) $h(\lambda, a, r)$ *is hyperbolic globally in* r.

(iv) $\tau_-(r^0)\tau_+(r^0) \neq 0$ *for some* $r^0 \in (\mathbb{R}_+)^M$.

(v) *If*

$$h(\lambda, a, r) = \det\left[I - \sum_{j=1}^{M} A_j e^{-\lambda r_j}\right] \tag{5.2}$$

then

$$\left\{\mu(\theta) \colon \det\left[\mu(\theta) I - \sum_{j=1}^{M} A_j e^{-i\theta_j}\right] = 0, \theta \in \mathbb{R}^M\right\} \cap \{|\mu| = 1\} = \phi. \tag{5.3}$$

Proof. Let us first prove (iv) $\Leftrightarrow$ (v). If $h(\lambda, a, r)$ is given by Relation (5.2), then the function $H(\lambda, \theta, r)$ in Relation (3.1) is given by

$$H(\rho, \theta, r) = \det\left[I - \sum_{j=1}^{M} A_j e^{-\rho r_j} e^{i\theta_j}\right].$$

The equivalence of statements (iv) and (v) is now immediate.

From the definitions of $\tau_-(r)$, $\tau_+(r)$ and the remarks following Corollary 3.3, we have (iv) $\Leftrightarrow$ (i), (i) $\Leftrightarrow$ (iii). The fact that (i) $\Leftrightarrow$ (ii) is a consequence of Theorem 2.2. This proves the theorem.

Since stability is so important in the applications, we restate Theorem 5.1 for this case.

THEOREM 5.2. *The following statements are equivalent*

(i) *There is an* $r \in (\mathbb{R}^+)^M$, $r = (r_1, \ldots, r_M)$ *with the set* $\{r_j\}_{j=1}^M$ *rationally independent, such that the function* $h(\lambda, a, r)$ *is uniformly asymptotically stable.*

(ii) $h(\lambda, a, r)$ *is uniformly asymptotically stable locally at some* $r_0 \in (\mathbb{R}^+)^M$.

(iii) $h(\lambda, a, r)$ *is uniformly asymptotically stable globally in* r.

(iv) $\sigma(r) < 0$ *for some* $r \in (R^+)^M$.

(v) *If*

$$h(\lambda, a, r) = \det\left[I - \sum_{j=1}^{M} A_j e^{-\lambda r_j}\right]$$

then

$$\sup\left\{|\mu(\theta)|: \det\left[\mu(\theta) I - \sum_{j=1}^{M} A_j e^{i\theta_j}\right] = 0, \theta \in \mathbb{R}^M\right\} < 1.$$

Historically, Melvin [6] proved Theorem 5.2 for the scalar equation where Condition (v) becomes the simple condition

$$\sum_{j=1}^{N} |A_j| < 1, \qquad A_j \in \mathbb{R}.$$

Hale [3] proved (ii) $\Leftrightarrow$ (iii) in the general case. Silkowski [8] introduced the equivalent conditions (i) and (v).

6. Examples

In this section we collect some examples to illustrate the above results. Throughout the section, the numbers $\rho_j(r)$ are defined in Relations (2.2), the numbers $\rho(r)$, $\sigma(r)$, $\tau_-(r)$, $\tau_+(r)$ in Relations (3.2).

EXAMPLE 6.1. Let us reconsider Example 2.1; that is, the function

$$h(\lambda, r) = 1 + \tfrac{1}{2}e^{-\lambda r_1} + \tfrac{1}{2}e^{-\lambda r_2}.$$

We have seen that $\rho_0(r) = 0$ for all r and, for $r_0 = (1, 2)$, $\bar{Z}(r_0) = \{-(\ln 2)/2\}$. Now, Theorem 4.1 implies that, for any $r = (r_1, r_2)$ with r_1, r_2 rationally independent, $[\rho_2(r), \rho_0(r)] = [\rho_2(r), 0]$ is the smallest closed interval containing $\bar{Z}(r)$ and $\rho_2(r)$ is continuous in r, $\rho_2(r_0) = -\ln 2$. Furthermore,

$$\tfrac{1}{2}(e^{-\rho r_2} - e^{-\rho r_1}) + 1 > 0, \qquad \text{for } \rho < 0, \quad r_1 < r_2.$$

Therefore I_1 of Theorem 4.1 is the empty set and $\bar{Z}(r) = [\rho_2(r), 0]$. Thus, for any neighborhood U of r_0, there is an $r \in U$ such that $\bar{Z}(r)$ is a complete interval of length approximately $\ln 2$ whereas for $r = r_0$, $\bar{Z}(r_0)$ is a single point.

This example shows very clearly how dramatically the set $\bar{Z}(r)$ can change with r if the two vector $r = (r_1, r_2)$ is permitted to assume all values in $(\mathbb{R}^+)^2$. If r is restricted to lie along a ray, say $r = \alpha r_0$, $\alpha > 0$, $r_0 = (r_1^0, r_2^0)$ fixed, then the set

$$\bar{Z}_\alpha = \{\operatorname{Re}\lambda: 1 + \tfrac{1}{2}e^{-\lambda\alpha r_1^0} + \tfrac{1}{2}e^{-\lambda\alpha r_2^0} = 0\}$$

need not be close to $\bar{Z}(r)$. In fact, for $r_0 = (1, 2)$, that is, the delay $r = \alpha r_0$ has the property that the second coordinate is always twice the first, we have seen that $\bar{Z}_\alpha = \{-(\ln 2)/2\alpha\}$. A similar phenomena will occur with several delays. Example 6.5 below gives some details for three delays.

EXAMPLE 6.2. As for Example 6.1, one shows that $\bar{Z}(r^0) = [\rho_2, \rho_0] \sim [-.27, .37]$ for the function

$$h(\lambda, r^0) = 1 + e^{-\lambda} + e^{-\pi\lambda}, \; r^0 = (1, \pi).$$

EXAMPLE 6.3. Consider the equation

$$h(\lambda, a, r) = 1 + a_1 e^{-\lambda r_1} + a_2 e^{-\lambda r_2} = 0 \tag{6.1}$$

where $0 < r_1 < r_2$ and a_1, a_2 are real constants. The numbers ρ_j, $j = 0, 1, 2$, are defined by

$$\begin{aligned} |a_2| e^{-\rho_0 r_2} &= 1 - |a_1| e^{-\rho_0 r_1} \\ |a_2| e^{-\rho_1 r_2} &= |a_1| e^{-\rho_1 r_1} - 1 \\ |a_2| e^{-\rho_2 r_2} &= 1 + |a_1| e^{-\rho_2 r_1}. \end{aligned} \tag{6.2}$$

As remarked earlier, $\rho_2 < \rho_0$. The constant ρ_1 may or may not exist. From (6.2), it is clear that $\rho_0 < 0$ if and only if $|a_1| + |a_2| < 1$. Thus, $h(\lambda, a, r)$ is uniformly asymtotically stable globally in r if and only if $|a_1| + |a_2| < 1$. Also, $\rho_2 > 0$ if and only if $|a_2| > 1 + |a_1|$. This means $h(\lambda, a, r)$ is hyperbolic globally in r and has $\bar{Z}(a, r) \cap (-\infty, 0] = \emptyset$ if and only if $|a_2| > 1 + |a_1|$.

Let us now analyze the other regions in the (a_1, a_2) parameter space. The relation $|a_2| < 1 + |a_1|$ implies $\rho_2(r) < 0$ and $|a_1| + |a_2| > 1$ implies $\rho_0(r) > 0$. It follows from Theorem 5.1 and the definition of $\tau_-(r)$, $\tau_+(r)$ that the equation is hyperbolic globally in r in this region if and only if $\tau_-(r) < 0 < \tau_+(r)$.

It remains to be seen when $\tau_-(r) < 0$. The number $\tau_-(r)$ can be related to the solutions ρ_1 of Equation (6.2). In fact, from Theorem 4.1, $\tau_-(r) < 0$ if and only if there is a solution $\rho_{11}(r)$ of Equation (6.2) satisfying $\rho_2(r) < \rho_{11}(r) < 0$ and

$$|a_2| e^{-\rho_1 r_2} < |a_1| e^{-\rho_1 r_1} - 1 \qquad \text{for } \rho_{11}(r) < \rho_1 < 0.$$

If $|a_2| < |a_1| - 1$, then there is a $\rho_{11}(r)$ satisfying the above properties and $h(\lambda, a, r)$ is hyperbolic with $\bar{Z}(a, r) \cap (-\infty, 0) \neq \emptyset$, $\bar{Z}(a, r) \cap (0, \infty) \neq \emptyset$. The inequality $|a_2| < |a_1| - 1$ implies $|a_1| + |a_2| > 1$, $|a_1| > |a_2| - 1$.

If $|a_2| > |a_1| - 1$ then

$$|a_2| e^{-\rho_1 r_2} > |a_1| e^{-\rho_1 r_1} - 1 \qquad \text{for } |\rho_1| < \delta$$

for some $\delta > 0$. If, in addition, $|a_1| + |a_2| > 1$, $|a_2| < 1 + |a_1|$ then $0 \in \bar{Z}(a, r)$ by Theorem 4.1 if the components of r are rationally independent. Thus, the function $h(\lambda, a, r)$ is not hyperbolic globally in r.

In summary,

(i) $h(\lambda, a, r)$ is uniformly asymptotically stable globally in r if and only if $|a_1| + |a_2| < 1$.

(ii) $h(\lambda, a, r)$ is hyperbolic globally in r with $\bar{Z}(a, r) \cap (-\infty, 0) \neq \varnothing$ if and only if $|a_2| > 1 + |a_1|$.

(iii) $h(\lambda, a, r)$ is hyperbolic globally in r with $\bar{Z}(a, r) \cap (-\infty, 0) \neq \varnothing$, $\bar{Z}(a, r) \cap (0, \infty) \neq \varnothing$ if and only if $|a_1| > 1 + |a_2|$.

(iv) $h(\lambda, a, r)$ is not hyperbolic globally in r if the coefficients a_1, a_2 do not satisfy one of the conditions in (i)-(iii).

The structure of the set $\bar{Z}(a, r)$ obviously changes as the parameter a varies from the region in case (iii) to the region in case (ii) above since two intervals had to merge as $\bar{Z}(a, r)$ moved to positive axis. This structure can also change even when the parameters always stay in a region corresponding to one case. In fact, suppose $|a_1| + |a_2| < 1$; that is, uniform asymptotic stability globally in r. Since $|a_1| - 1 < 0$, there is an a_2 sufficiently small so that the equation

$$|a_2|\, e^{-\rho r_2} = |a_1|\, e^{-\rho r_1} - 1$$

has two distinct negative solutions $\rho_{11}(r) < \rho_{12}(r)$ in $[\rho_2(r), \rho_0(r)]$. Theorem 4.1 allows one to conclude that $\bar{Z}(a, r)$ consists of two intervals.

Let us make one other remark about this example. The number of intervals in $\bar{Z}(a, r)$ may also change with r. In fact, suppose $|a_2| = |a_1| - 1$. The function $h(\lambda, a, r)$ is not hyperbolic globally in r in this case. The equation

$$f(\rho, a, r) \stackrel{\text{def}}{=} |a_2|\, e^{-\rho r_2} - |a_1|\, e^{-\rho r_1} + 1 = 0$$

has the solution $\rho = 0$. Since

$$\frac{\partial f(0, a, r)}{\partial \rho} = -|a_2|\, r_2 + |a_1|\, r_1$$

and $f(\rho, a, r) \to 1$ as $\rho \to \infty$, there will be a positive zero of f if $|a_2|\, r_2 > |a_1|\, r_1$. Since $f(\rho, a, r) \to +\infty$ as $\rho \to -\infty$ there will be a negative zero of f if $|a_2|\, r_2 < |a_1|\, r_1$.

Therefore, if $|a_2|\, r_2 \neq |a_1|\, r_1$, that is, $r_1 \neq (|a_2|/(1 + |a_2|))r_2$, r_1, r_2 rationally independent, the set $\bar{Z}(a, r)$ will consist of two intervals. When $r_1 = (|a_2|/(1 + |a_2|))r_2$ the set $\bar{Z}(a, r)$ will consist of one interval.

EXAMPLE 6.4. Consider the equation

$$h(\lambda, \epsilon) = 1 - 2ce^{-\lambda\epsilon_1} + c^2 e^{-\lambda\epsilon_2} = 0. \tag{6.3}$$

Let us study $\bar{Z}(\epsilon)$ as $\epsilon \to 0$ and always assume that $\epsilon_2 > \epsilon_1 > 0$.

As a first case, if $\epsilon_2 = 2\epsilon_1$ then $h(\lambda, \epsilon) = 0$ if and only if $1 - ce^{-\lambda\epsilon_1} = 0$, $\operatorname{Re}\lambda = (1/\epsilon_1) \ln |c|$. Thus, if $|c| > 1$, $\operatorname{Re}\lambda \to +\infty$ as $\epsilon_1 \to 0$; if $|c| = 1$, $\operatorname{Re}\lambda = 0$ for all ϵ_1; if $|c| < 1$, then $\operatorname{Re}\lambda \to -\infty$ as $\epsilon_1 \to 0$.

If $\epsilon_2 > \epsilon_1 > 0$, we know that $\bar{Z}(\epsilon) \subset [\rho_2(\epsilon), \rho_0(\epsilon)]$ where $\rho_2 = \rho_2(\epsilon)$, $\rho_0 = \rho_0(\epsilon)$, satisfy the equations

$$\begin{aligned} &\text{(a)} \qquad 1 = 2\,|c|\,e^{-\epsilon_1\rho_0} + c^2 e^{-\epsilon_2\rho_0} \\ &\text{(b)} \qquad c^2 e^{-\epsilon_2\rho_2} = 1 + 2\,|c|\,e^{-\epsilon_1\rho_2}. \end{aligned} \tag{6.4}$$

Now suppose

$$-|c| < \frac{1-c^2}{2} < |c|\,. \tag{6.5}$$

If relation (6.5) is satisfied, then $2\,|c| + c^2 > 1$ and $\rho_0 = \rho_0(\epsilon) > 0$, $\rho_0(\epsilon) \to +\infty$ as $\epsilon \to 0$. Furthermore, if $\rho_2 \geqslant 0$, then

$$\begin{aligned} 1 + 2\,|c|\,e^{-\epsilon_1\rho_2} = c^2 e^{-\epsilon_2\rho_2} \leqslant c^2 e^{-\epsilon_1\rho_2} \qquad &\Rightarrow \\ 1 \leqslant (c^2 - 2\,|c|)\, e^{-\epsilon_1\rho} \leqslant c^2 - 2\,|c| \Rightarrow& \\ \frac{1-c^2}{2} \leqslant -|c|\,.& \end{aligned}$$

Thus, if Relation (6.5) is satisfied, then $\epsilon_2 = \epsilon_2(\epsilon) < 0$, $\rho_2(\epsilon) \to -\infty$ as $\epsilon \to 0$.

Also, if $\epsilon_2 > \epsilon_1 > 0$ are rationally independent, then $[\rho_2(\epsilon), \rho_0(\epsilon)]$ is the smallest interval containing $\bar{Z}(\epsilon)$. Thus, if Relation (6.5) holds, the smallest closed interval containing $\bar{Z}(\epsilon)$ approaches $(-\infty, +\infty)$ as $\epsilon \to 0$.

To determine when $\bar{Z}(\epsilon)$ is a single interval, we should find $\rho_1(\epsilon)$. The number $\rho_1(\epsilon)$, if it exists, must be a zero of the function

$$f(\rho, \epsilon) = c^2 e^{-\rho\epsilon_2} - 2\,|c|\,e^{-\rho\epsilon_1} + 1$$

this function has a unique minimum at a point given by

$$\alpha = \frac{1}{\epsilon_2 - \epsilon_1} \ln \frac{|c|\,\epsilon_2}{2\epsilon_1}.$$

If $|c| < 1$, then we can choose $\epsilon_2 > \epsilon_1$ such that $|c|\,\epsilon_2/2\epsilon_1 = 1$ and thus $\alpha = 0$. Since $f(0, \epsilon) = (|c| - 1)^2 > 0$ if $|c| < 1$, it follows that $f(\rho, \epsilon) > 0$ for all ρ and ρ_1 does not exist. This means that $\bar{Z}(a, \epsilon) = [\rho_2(\epsilon), \rho_0(\epsilon)]$. We can thus choose $\epsilon_1, \epsilon_2 \to 0$, $\epsilon_1 < \epsilon_2$, so that $|c| \epsilon_2/2\epsilon_1 = 1$ and $\bar{Z}(a, \epsilon) \to (-\infty, \infty)$. If

$$|c| < \frac{1 - c^2}{2} \tag{6.6}$$

then $\rho_0(\epsilon) < 0$ and $\rho_0(\epsilon) \to -\infty$ as $\epsilon \to 0$. If

$$|c| = \frac{1 - c^2}{2} \tag{6.7}$$

then $\rho_0(\epsilon) = 0$ for all $\epsilon_2 > \epsilon_1 > 0$, $\rho_2(\epsilon) < 0$, $\rho_2(\epsilon) \to -\infty$ as $\epsilon \to 0$ and the smallest closed interval $[\rho_2(\epsilon), 0]$, containing $\bar{Z}(\epsilon)$ approaches $(-\infty, 0]$ as $\epsilon \to 0$.

EXAMPLE 6.5. Let $r_1 < r_2 < r_3$, $h(\lambda, r) = 1 + e^{-\lambda r_1} + e^{-\lambda r_2} + e^{-\lambda r_3}$. If (r_1, r_2, r_3) are rationally independent, then the smallest closed interval containing $\bar{Z}(r)$ is $[\rho_3, \rho_0]$ where

$$e^{-\rho_3 r_3} = 1 + e^{-\rho_3 r_1} + e^{-\rho_3 r_2}$$
$$1 = e^{-\rho_0 r_1} + e^{-\rho_0 r_2} + e^{-\rho_0 r_3}.$$

The numbers ρ_1, ρ_2 are defined by

$$e^{-\rho_1 r_1} = 1 + e^{-\rho_1 r_2} + e^{-\rho_1 r_3}$$
$$e^{-\rho_2 r_2} = 1 + e^{-\rho_2 r_1} + e^{-\rho_2 r_3}$$

if they exist. This implies necessarily that $\rho_1 < 0$, $\rho_2 < 0$. On the other hand, for $\rho < 0$, the functions

$$f(\rho) = e^{-\rho r_3} + 1 - e^{-\rho r_1} + e^{-\rho r_2}$$
$$g(\rho) = e^{-\rho r_3} + 1 + e^{-\rho r_1} - e^{-\rho r_2}$$

are decreasing and positive for $\rho = 0$. Thus, ρ_1, ρ_2 do not exist and $\bar{Z}(r) = [\rho_3, \rho_0]$ if (r_1, r_2, r_3) are rationally independent.

Now let us consider the zeros of the function

$$h_1(\lambda, \omega) = 1 + e^{-\lambda\omega_1} + e^{-2\lambda\omega_1} + e^{-\lambda\omega_2}$$

were $\omega = (\omega_1, \omega_2) \in (\mathbb{R}^+)^2$. There are still three delays $\omega_1, 2\omega_1, \omega_2$ but they are not linear independent. What is the smallest interval containing $\bar{Z}_1(\omega) = \{\mathrm{Re}\,\lambda : h_1(\lambda, \omega) = 0\}$? If ω_1, ω_2 are rationally independent, Theorem 3.1

implies that $\rho \in \bar{Z}_1(\omega)$ if and only if there is a $\theta = (\theta_1, \theta_2)$ in $\mathbb{R}^2$ such that

$$H_1(\rho, \theta, \omega) = 1 + e^{-\rho\omega_1}e^{i\theta_1} + e^{-2\rho\omega_1}e^{2i\theta_1} + e^{-\rho\omega_2}e^{i\theta_2} = 0.$$

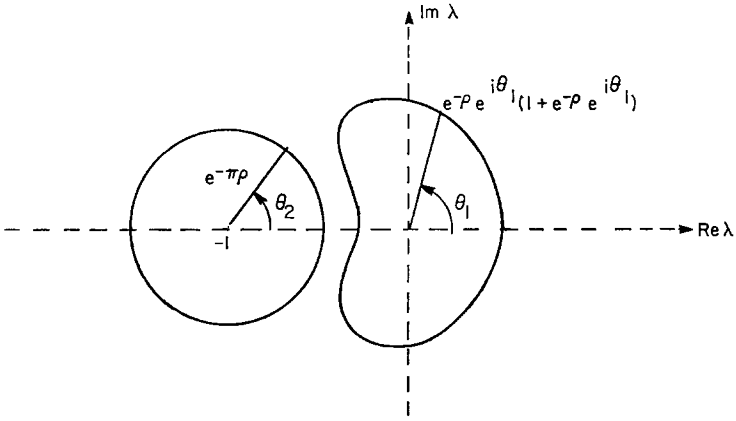
For the special case $\omega_0 = (1, \pi)$; that is, the function

$$h_1(\lambda, \omega_0) = 1 + e^{-\lambda} + e^{-2\lambda} + e^{-\pi\lambda},$$

we have $H_1(\rho, \theta, \omega_0) = 0$ is equivalent to

$$e^{-\rho}e^{i\theta_1}(1 + e^{-\rho}e^{i\theta_1}) = -1 - e^{-\pi\rho}e^{i\theta_2}.$$

Geometrically, this says these two curves in the complex plane must intersect



These curves intersect if and only if $\rho \in [\rho_3, \sigma]$ where ρ_3 is as above and σ satisfies

$$e^{-\pi\sigma} = \frac{3^{1/2}}{2}(1 - e^{-2\sigma}).$$

Thus, $\bar{Z}_1(\omega_0) = [\rho_3, \sigma] \cong [-.56, .30]$.

In the case where the three delays $r = (r_1, r_2, r_3)$ vary independently, we saw above that $\bar{Z}(r) = [\rho_3, \rho_0]$, that is, $\rho(r) = \rho_3$, $\sigma(r) = \rho_0$. For $r = r_0 = (1, 2, \pi)$ one sees that $[\rho(r_0), \sigma(r_0)] \sim [-.56, .60]$. When the delays were not allowed to vary independently and were of the form $(\rho_1, 2\omega_1, \omega_2)$, we also saw that the smallest interval $[\rho_1(\omega), \sigma_1(\omega)] \supseteq \bar{Z}_1(\omega)$ was approximately $[-.56, .30]$ at $\omega_0 = (1, \pi)$; that is, the three delays were $(1, 2, \pi)$. This interval is properly contained in the interval $[-.56, 60]$ which corresponds to varying the delays independently.

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